# Title

On the local extension of Killing vector-fields in Ricci flat manifolds

Alexandru D. Ionescu

Question : Assume we are given a smooth connected pseudo-riemannian manifold $(\mathbf{M}, \mathbf{g})$, an open subset $O \subseteq \mathbf{M}$ and a smooth Killing vector-field $Z$ in $O$. Under what assumptions does $Z$ extend (uniquely) as a Killing vector-field in $\mathbf{M}$ ?

If $(\mathbf{M}, \mathbf{g})$ is a *real-analytic* manifold then the unique extension exists provided that $O$ is connected and $\mathbf{M}$ is simply connected (Nomizu 1960).

In the $C^\infty$ case we consider manifolds $(\mathbf{M}, \mathbf{g})$ which are solutions of the Einstein vacuum equations

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The **Minkowski spaces** :

$$(\mathbb{R}^4, -dt^2 + dx^2 + dy^2 + dz^2).$$

In polar coordinates

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + (\sin\theta)^2 d\phi^2).$$

The **Schwarzschild spaces** :

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + (\sin\theta)^2 d\phi^2),$$

where $(r, t, \theta, \phi) \in (2m, \infty) \times \mathbb{R} \times (0, \pi) \times \mathbb{S}^1$.

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The **Kerr spaces** : $m$ (the mass), $J$ (the angular momentum), $m > 0$, $a = J/m \in [0, m)$, and let $r_+ = m + (m^2 - a^2)^{1/2}$. In Boyer-Lindquist coordinates $(r, t, \theta, \phi) \in (r_+, \infty) \times \mathbb{R} \times (0, \pi) \times \mathbb{S}^1$,

$$-\frac{\rho^2 \Delta}{\Sigma^2}(dt)^2 + \frac{\Sigma^2 (\sin\theta)^2}{\rho^2}\left(d\phi - \frac{2amr}{\Sigma^2}dt\right)^2 + \frac{\rho^2}{\Delta}(dr)^2 + \rho^2 (d\theta)^2,$$

where

$$\left\{ \begin{array}{l} \Delta = r^2 + a^2 - 2mr; \\ \rho^2 = r^2 + a^2 (\cos\theta)^2; \\ \Sigma^2 = (r^2 + a^2)^2 - a^2 (\sin\theta)^2 \Delta. \end{array} \right.$$

The Kerr spaces are algebraically special : 18 of the 20 components of the Riemann curvature tensor vanish in a suitable frame (Type D).

**Theorem 1.** (I.-Klainerman) Assume that $(\mathbf{M}, \mathbf{g})$ is a smooth $d$-dimensional Ricci flat, pseudo-riemannian manifold, $f : \mathbf{M} \to \mathbb{R}$ is a smooth function, $O = \{x \in \mathbf{M} : f(x) < 0\}$, and $p \in \delta(O)$. Assume

- $Z$ is a smooth Killing vector-field in $O$ ;
- $L$ is a smooth geodesic vector-field in $\mathbf{M}$, commuting with $Z$ in $O$,

$$\mathbf{D}_L L = 0 \text{ in } \mathbf{M}, \qquad [L, Z] = 0 \text{ in } O;$$

- $L(f)(p) \neq 0$ and $f$ is strongly pseudo-convex at $p$, i.e.

$$\mathbf{D}^2 f(X, X)(p) < 0$$

for any vector $X \neq 0 \in T_p(\mathbf{M})$ for which $X(f)(p) = 0$ and $\mathbf{g}_p(X, X) = 0$.

Then $Z$ extends as a Killing vector-field to a neighborhood of $p$ in $\mathbf{M}$.

**Earlier Theorem.** (Alexakis–I–Klainerman) Assume $\mathcal{N}, \underline{\mathcal{N}}$ are smooth, null, nonexpanding hypersurfaces in an Einstein vacuum $(\mathbf{O}, \mathbf{g})$, which intersect transversally in a 2-sphere $S$. Then there is an open neighborhood $\mathbf{O}'$ of $S$ and a nontrivial Killing vector-field $\mathbf{K}$ in $\mathbf{O}'$, which is tangent to the null generators of $\mathcal{N}, \underline{\mathcal{N}}$.

This is a local version of Hawking's Rigidity Theorem, without assuming analyticity of the spacetime.

The issue is to extend the vector-field $\mathbf{K}$ (constructed by Friedrich–Racz–Wald) from the domain of dependence of $\mathcal{N} \cup \underline{\mathcal{N}}$ to a full neighborhood of $S$.

**Proof of the extension theorem :** Define $Z$ in a neighborhood of $p$ by

$$[L, Z] = 0.$$

In $\mathbf{M}$ we define

$$\begin{aligned}
\pi_{\alpha\beta} &= (\mathcal{L}_Z \mathbf{g})_{\alpha\beta} = \mathbf{D}_\alpha Z_\beta + \mathbf{D}_\beta Z_\alpha, \\
P_{\alpha\beta\mu} &= \mathbf{D}_\alpha \pi_{\beta\mu} - \mathbf{D}_\beta \pi_{\alpha\mu} - \mathbf{D}_\mu \omega_{\alpha\beta}, \\
B_{\alpha\beta} &= \frac{1}{2}(\pi_{\alpha\beta} + \omega_{\alpha\beta}), \\
\dot{B}_{\alpha\beta} &= \mathcal{L}_L B_{\alpha\beta}, \\
W_{\alpha\beta\gamma\delta} &= (\mathcal{L}_Z \mathbf{R})_{\alpha\beta\gamma\delta} - (B \odot \mathbf{R})_{\alpha\beta\gamma\delta},
\end{aligned}$$

where

$$\begin{aligned}
&\mathbf{D}_L \omega_{\alpha\beta} = \pi_{\alpha\rho} \mathbf{D}_\beta L^\rho - \pi_{\beta\rho} \mathbf{D}_\alpha L^\rho, \qquad \omega = 0 \text{ in } O, \\
&(B \odot \mathbf{R})_{\alpha\beta\gamma\delta} := B_\alpha{}^\lambda \mathbf{R}_{\lambda\beta\gamma\delta} + B_\beta{}^\lambda \mathbf{R}_{\alpha\lambda\gamma\delta} + B_\gamma{}^\lambda \mathbf{R}_{\alpha\beta\lambda\delta} + B_\delta{}^\lambda \mathbf{R}_{\alpha\beta\gamma\lambda}.
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Then

$$\begin{aligned}
\mathbf{D}_L B &= \mathcal{M}(B, \dot{B}, P, W),\\
\mathbf{D}_L \dot{B} &= \mathcal{M}(B, \dot{B}, P, W),\\
\mathbf{D}_L P &= \mathcal{M}(B, \dot{B}, P, W),\\
\Box W &= \mathcal{M}(B, \mathbf{D}B, \dot{B}, \mathbf{D}\dot{B}, P, \mathbf{D}P, W, \mathbf{D}W).
\end{aligned}$$

**Lemma :** Assume that $G_i, H_j : B_{\delta_0}(p) \to \mathbb{C}$ are smooth functions, $i = 1, \ldots, I$, $j = 1, \ldots, J$, such that

$$\begin{cases} |\Box_{\mathbf{g}} G_i| \leq M \sum_{l=1}^{I}(|G_l| + |\partial^1 G_l|) + M \sum_{m=1}^{J} |H_m|; \\ |L(H_j)| \leq M \sum_{l=1}^{I}(|G_l| + |\partial^1 G_l|) + M \sum_{m=1}^{J} |H_m|. \end{cases}$$

Assume that $G_i = 0$ and $H_j = 0$ in $B_{\delta_0}(p) \cap O$. Assume also that $f$ is strongly pseudo-convex at $p$ and $L(f)(p) \neq 0$. Then $G_i = 0$ and $H_j = 0$ in $B_{\delta_1}(p)$, for some $\delta_1 \in (0, \delta_0)$ sufficiently small.

**Carleman estimates :**

$$\begin{aligned} &\lambda \|e^{-\lambda f_\epsilon} \phi\|_{L^2} \lesssim \|e^{-\lambda f_\epsilon} L(\phi)\|_{L^2}, \\ &\lambda \|e^{-\lambda f_\epsilon} \phi\|_{L^2} + \|e^{-\lambda f_\epsilon} |\partial^1 \phi| \,\|_{L^2} \lesssim \lambda^{-1/2} \|e^{-\lambda f_\epsilon} \Box_{\mathbf{g}} \phi\|_{L^2}, \end{aligned} \tag{1}$$

for all smooth $\phi$ supported in a small neighborhood of $p$, and $\lambda$ sufficiently large.

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for all smooth $\phi$ supported in a small neighborhood of $p$, and $\lambda$ sufficiently large.

**Uniqueness :** Assume $u_1, u_2$ are smooth solutions of the equation $T(u) = 0$ in some connected open domain $A$. Assume $B \subseteq A$ is an open set.

1. Complete lack of uniqueness : $u_1 \equiv u_2$ in $B$ but $u_1$ and $u_2$ completely different outside $B$.

2. Well-posedness : if $u_1$ and $u_2$ are "close" in $B$ then they have to stay "close" in $A$.

3. Unique continuation : if $u_1 \equiv u_2$ in $B$ then $u_1 \equiv u_2$ in $A$. However, $u_1$ could be "close" to $u_2$ in $B$, but completely different outside $B$.

Example : $A = B(2)$, $B = B(1/2)$, $u_n = (x + iy)^n$ are solutions of the Laplace equation $\Delta u_n = 0$ which are "small" in $B$ and "large" in A.

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**Counterexamples :**

**Theorem 2.** (I.-Klainerman) Assume that $(\mathbf{M}, \mathbf{g})$ is a Ricci-flat connected 4-dimensional Lorentz manifold, $Z$ is a nontrivial Killing vector-field in $\mathbf{M}$ and $f : \mathbf{M} \to \mathbb{R}$ satisfies

$$\mathbf{D}^\alpha f \mathbf{D}_\alpha f = 0, \qquad Z(f) = 0, \qquad \text{in } \mathbf{M}.$$

Let $O = \{x \in \mathbf{M} : f(x) < 0\}$ and $p \in \delta(O)$ and assume that $\nabla f(p) \neq 0$. Then there exists a neighborhood $U$ of $p$, diffeomorphic to the open unit ball of $\mathbb{R}^4$, and a smooth, Ricci flat, Lorentz metric $\mathbf{h}$ in $U$, such that $\mathbf{h} = \mathbf{g}$ in $O \cap U$, but $Z$ does not admit an extension as a smooth Killing vector-field for the metric $\mathbf{h}$ in $U$.

Recall the Kerr spacetime $\mathcal{K}(m,a) : (r,t,\theta,\phi) \in (r_+,\infty) \times \mathbb{R} \times (0,\pi) \times \mathbb{S}^1$,

$$-\frac{\rho^2\Delta}{\Sigma^2}(dt)^2 + \frac{\Sigma^2(\sin\theta)^2}{\rho^2}\left(d\phi - \frac{2amr}{\Sigma^2}dt\right)^2 + \frac{\rho^2}{\Delta}(dr)^2 + \rho^2(d\theta)^2,$$

where

$$\begin{cases} \Delta = r^2 + a^2 - 2mr; \\ \rho^2 = r^2 + a^2(\cos\theta)^2; \\ \Sigma^2 = (r^2+a^2)^2 - a^2(\sin\theta)^2\Delta. \end{cases}$$

We make the change of variables (Kruskal coordinates)

$$du_- = dt - (r^2+a^2)\Delta^{-1}dr, \qquad d\phi_- = d\phi - a\Delta^{-1}dr.$$

In the new coordinates $(\theta, r, \phi_-, u_-)$ the spacetime metric becomes

$$\begin{aligned} \mathbf{g} = {} & \rho^2 d\theta^2 - 2du_-dr + 2a(\sin\theta)^2 d\phi_- dr \\ & - \frac{4amr(\sin\theta)^2}{\rho^2}d\phi_-du_- + \frac{\Sigma^2(\sin\theta)^2}{\rho^2}d\phi_-^2 + \frac{2mr-\rho^2}{\rho^2}du_-^2, \end{aligned} \tag{2}$$

and the Killing vector-field $\mathbf{T} = d/dt$ becomes $\mathbf{T} = d/du_-$.

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The hypersurface $\mathcal{H}^- = \{x : r(x) = r_+\}$ defines one sheet of the **event horizon** of the Kerr spacetime.

**Theorem 3.** (I.-Klainerman) Assume $0 < a < m$ and $p \in \mathcal{H}^-$. Then there is a neighborhood $U$ of the point $p$ and a smooth metric $\mathbf{h}$ in $U$ such that

$$
\begin{aligned}
&{}^{\mathbf{h}}\mathbf{Ric} = 0 \quad \text{and} \quad \mathcal{L}_{\mathbf{T}}\mathbf{h} = 0 \quad \text{in } U,\\
&\mathbf{h} = \mathbf{g} \text{ in } U \cap \mathcal{W} = \{x \in U : r(x) < r_+\},\\
&Z = d/d\phi_- \text{ does not extend as a Killing vector-field of } \mathbf{h} \text{ in } U.
\end{aligned}
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**Theorem (Alinhac-Baouendi) :** Let $B_1 = \{x^2 + y^2 + t^2 < 1\}$ and $B_1^+ = \{(x, y, t) \in B_1 : x \geq 0\}$. Then there are smooth functions $u, A : B_1 \to \mathbb{C}$ such that

$$(-\partial_t^2 + \partial_x^2 + \partial_y^2)u = Au \qquad \text{in } B_1, \tag{3}$$

and

$$\operatorname{supp} u = B_1^+, \qquad \operatorname{supp} A \subseteq B_1^+.$$

**Open question :** With the notation above, assume $u \in C^\infty(B_1)$ satisfies the nonlinear equation $\Box u = u^2$ and $u$ vanishes in $B_1^-$. Does it follow that $u$ vanishes in a neighborhood of 0 ?

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