

Uniform sublevel set estimates and applications

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- Sublevel set estimates generally attempt to quantify the heuristic idea that a function cannot be perpetually small if it has derivatives which are not small.
- At the very least, sublevel set estimates are important toy models for related questions concerning oscillatory integrals.
- More than that, they are an indispensable half of oscillatory integral estimates: after you apply TT^* / integration-by-parts to the “good” piece, the principal redeeming quality of what’s left is that it’s “small.”
- Cast in a sufficiently general framework, sublevel problems encompass all L^p -improving problems for linear and multilinear averaging operators.

Strongly uniform multilinear estimates

Typical Goal: For a function F on domain $D \subset \mathbb{R}^d$, make estimates

$$\int_{|F(x_1, \dots, x_d)| \leq \epsilon} \prod_{j=1}^d |f_j(x_j)| dx \leq C \epsilon^\alpha \prod_{j=1}^d \|f_j\|_{p_j},$$

$$\left| \int e^{i\lambda F(x)} \prod_{j=1}^d f_j(x_j) dx \right| \leq C \lambda^{-\alpha} \prod_{j=1}^d \|f_j\|_{p_j}$$

uniformly for all F and D in some broad classes and relevant exponents p_j . The classes are hopefully qualitatively simple.

Variations and interpretations of this problem may be found in: Varčenko (1976); Phong and Stein (1997); Phong, Stein, and Sturm (1999); Ikromov, Kempe, Müller (2010); Magyar (2009); Karpushkin (1986); Phong and Sturm (2006); Greenblatt (2005); Seeger (1998); Robert and Sargos (2006) and many other places

- Carbery, Christ, and Wright (1999): Initiated study of the problem in the form to be discussed. Proved a result for very general class of F 's (namely, that $\partial^\beta F \geq 1$ on the unit cube). Also achieved scale-invariant results under additional convexity assumptions on F .
- Phong, Stein, and Sturm (2001): Took F to be polynomial of fixed degree and D to be any algebraic domain on which $\partial^\beta F \geq 1$.
- Carbery and Wright (2002): Took F to have nonvanishing partial derivatives on a "type M domain" (namely, where axis-parallel line segments intersect the sublevel sets of $\partial^\gamma F$ in boundedly many segments).
- Bottom line: estimates are all governed by the Newton polyhedron (or reduced Newton polyhedron) of F in the multilinear case and by the smallest order of a nonvanishing derivative in the "scalar" cases.

Carbery (2009) observed that the Newton polyhedron cannot generate all possible sublevel set estimates by establishing the first nonlinear analogue of the previous works.

Carbery's Theorem

If F defined on \mathbb{R}^d is nonnegative, C^2 and strictly convex and the determinant of its Hessian is bounded below by one on the (convex) domain of F , then

$$|D| \leq C_d \|F\|_{L^\infty(D)}^{\frac{d}{2}}$$

and consequently

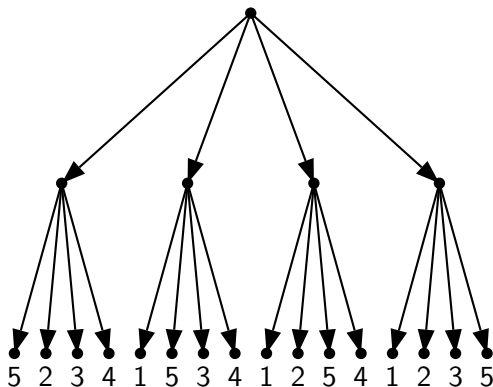
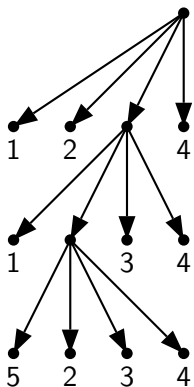
$$|\{x \in D \mid F(x) \leq \epsilon\}| \leq C_d \epsilon^{\frac{d}{2}}.$$

It turns out that the Hessian determinant is one of many nonlinear operators which arise naturally in connection with sublevel set estimates.

Fix a d -dimensional real analytic manifold \mathcal{M} with a nonvanishing d -form ω . We can generate nonlinear operators inductively by the following construction: Let \mathcal{G}_0 be the set $\{1, \dots, m\}$. For all integers $k \geq 1$, let $\mathcal{G}_k := \mathcal{G}_{k-1} \cup (\mathcal{G}_{k-1})^d$ (that is, the union of \mathcal{G}_{k-1} and elements of the d -fold Cartesian product of \mathcal{G}_{k-1} 's). For any $G \in \bigcup_{k=0}^{\infty} \mathcal{G}_k$, define constants $\#G$, $G^{(1)}, \dots, G^{(m)}$, and the operator ∂^G as follows:

- 1 If $G \in \mathcal{G}_0$, define $\#G := 0$, and for each $i = 1, \dots, m$, define $G^{(i)} := 1$ if $G = i$ and $G^{(i)} := 0$ otherwise. Given functions π_1, \dots, π_m , define $\partial^G \pi := \pi_G$.
- 2 If $G \in \mathcal{G}_k \setminus \mathcal{G}_{k-1}$ for some $k \geq 1$, then $G = (G_1, \dots, G_d)$ for some $G_1, \dots, G_d \in \mathcal{G}_{k-1}$. In this case, define $\#G := \#G_1 + \dots + \#G_d + 1$ and $G^{(i)} := G_1^{(i)} + \dots + G_d^{(i)}$. Given (analytic) functions π_1, \dots, π_m , define $\partial^G \pi$ to satisfy

$$d(\partial^{G_1} \pi) \wedge \dots \wedge d(\partial^{G_d} \pi) = \left(\partial^G \pi \right) \omega.$$



- The graph on the left is a mixed third derivative in \mathbb{R}^4 ; on the right is a Hessian determinant in \mathbb{R}^4 .
- $\#G$ equals the number of unlabeled nodes
- $G^{(j)}$ equals the number of nodes labeled by j .

Theorem (G. 2010)

Let \mathcal{M}, ω , and π_1, \dots, π_m be as already defined. Let $D \subset \mathcal{M}$ be compact. For any K and $G \in \mathcal{G}_K \setminus \mathcal{G}_0$, there exists a finite constant C such that

$$\left| \int_D \left(\prod_{j=1}^m f_j \circ \pi_j \right) \omega \right| \leq C \left(\inf_{x \in D} |\partial^G \pi(x)| \right)^{-\frac{1}{\#G}} \prod_{j=1}^m \|f_j\|_{L^{p_j,1}}$$

for any locally integrable functions f_1, \dots, f_m , where $\frac{1}{p_j} = \frac{G^{(j)}}{\#G}$ for $j = 1, \dots, m$. If $\mathcal{M} \subset \mathbb{R}^d$, $\omega = dx_1 \wedge \dots \wedge dx_d$, and the functions π_j are Pfaffian on some connected, open set U containing \mathcal{M} , then the constant C depends only on d, K, m , and the formats of π_1, \dots, π_m .

Fixing an $f_j = \chi_{[-\epsilon, \epsilon]}$ relates back to sublevel sets.

Theorem (G. 2010)

Let \mathcal{M}, ω , and π_1, \dots, π_m be as defined above. Fix any K and any $G \in \mathcal{G}_K \setminus \mathcal{G}_0$. Let D be any semi-analytic set in \mathcal{M} . There exists a finite constant C such that

$$\left| \int_{E_\epsilon \cap D} e^{i\lambda \pi_n} \omega \right| \leq C \left(\inf_{x \in E_\epsilon \cap D} |\partial^G \pi(x)| \right)^{-\frac{1}{\#G}} |\lambda|^{-\frac{G(m)}{\#G}} \prod_{j=1}^{m-1} \epsilon_j^{\frac{G(j)}{\#G}},$$

for any real λ and any positive $\epsilon_1, \dots, \epsilon_{m-1}$, where

$$E_\epsilon := \{x \in \mathcal{M} \mid |\pi_j(x)| \leq \epsilon_j \text{ } j = 1, \dots, m-1\}.$$

If $\mathcal{M} \subset \mathbb{R}^d$, $\omega = dx_1 \wedge \dots \wedge dx_d$, and the functions π_j are Pfaffian on some connected, open set U containing \mathcal{M} , and D is semi-Pfaffian, then the constant C depends only on d, K, m , the formats of π_1, \dots, π_m , and the format of D .

Sketch of Proof

Fix measurable sets $E_1, \dots, E_m \subset \mathbb{R}$, any closed $D \subset \mathcal{M}$, and any $\delta > 0$. The set $E := \{x \in D \mid \pi_i(x) \in E_i, i = 1, \dots, m\}$ satisfies the inequality

$$|E|_\omega \leq C\delta + \left| E \cap \bigcap_{G \in \mathcal{G}_K(\mathcal{M}, \pi)} \left\{ x \in \mathcal{M} \mid |\partial^G \pi(x)| \leq \delta^{-\#G} \prod_{j=1}^m |E_j|^{G(j)} \right\} \right|_\omega$$

for some C independent of the E_i 's (where $|\cdot|_\omega$ indicates the ω -measure of the corresponding set). The proof is by induction, with the base case being the change of variables formula. If $G = (i_1, \dots, i_d)$ and the multiplicity of isolated preimages under the map $(\pi_{i_1}, \dots, \pi_{i_d})$ is bounded by N , then

$$\int_D \chi_E |\partial^G \pi| \omega \leq N \int_{\mathbb{R}^d} \prod_{j=1}^d \chi_{E_{i_j}}(t_j) dt = N \prod_{j=1}^d |E_{i_j}|.$$

Now let $E' := E \cap \left\{ x \in D \mid |\partial^G \pi(x)| \leq \frac{1}{2} \delta^{-1} \prod_{i=1}^m |E_i|^{G^{(i)}} \right\}$. On $E \setminus E'$, one has

$$\int_{E \setminus E'} \omega \leq \int_{E \setminus E'} 2\delta \prod_{j=1}^d |E_{j_j}|^{-1} |\partial^G \pi| \omega \leq 2N\delta.$$

Thus it follows that

$$\int_{\mathcal{M}} \chi_E \omega \leq 2N\delta + \int_{\mathcal{M}} \chi_{E'} \omega.$$

Now E' has the same form as E with an additional constraint: the function $\partial^G \pi$ is constrained to take values in an interval centered at the origin of width $\delta^{-1} \prod_{j=1}^m |E_j|^{G^{(j)}}$. Building trees inductively in this way establishes the main theorem by choosing

$$\delta := \left(\left(\inf_{x \in D} |\partial^G \pi(x)| \right)^{-1} \prod_{j=1}^m |E_j|^{G^{(j)}} \right)^{\frac{1}{\#G}}.$$

- In the Euclidean setting, nonlinear operators yield “sharper” estimates than the Newton polytope approach for all phases except flat ones (level sets are parallel hyperplanes in some small neighborhood). In some cases, you obtain sharpest-possible results:

$$\left| \int_{\mathbb{R}^2} \rho(xy) f(x) g(y) dx dy \right| \leq C_p \|\rho\|_{p,1} \|f\|_{p',1} \|g\|_{p',1} \quad \forall p \in (1, 2]$$

- For some phases it's still not “sharply” sharp (analogous construction on \mathbb{R}^3 with $\rho(xyz)$ gives only $p \in (2, 3]$)—there appears to be a connection to curvature...
- We obtain an analogue of Carbery's theorem for Pfaffian F

$$\left| B \cap \left\{ x \in \mathbb{R}^d \mid |F(x)| \leq \epsilon \right\} \right| \leq C \epsilon^{\frac{d}{d+1}} |B|^{\frac{d-1}{d+1}}$$

for any parallelepiped B on which $\det \text{Hess } F \geq 1$. Taking $\epsilon = \|F\|_{L^\infty(B)}$ recovers the earlier sublevel set estimate, since the left-hand side equals $|B|$. $\blacklozenge\blacklozenge$

Multilinear Determinant Functionals

$$T_{\mu,k}^{-\gamma}(f_1, \dots, f_{k+1}) := \int \prod_{j=1}^{k+1} f_j(y_j) \frac{d\mu(y_1) \cdots d\mu(y_{k+1})}{[\text{VOL}(y_1, \dots, y_{k+1})]^\gamma}$$

where VOL is the Euclidean volume of the k -simplex with vertices y_1, \dots, y_{k+1} and μ is some geometrically-inspired measure.

- MDFs go way back in restriction theory on curves: C. Fefferman (1970), Zygmund (1974), Carleson and Sjölin (1972), Prestini (1979), Christ (1985), Drury and Marshall (1985, 1987)
- Other measures and related inequalities: Christ (1984), Drury (1988), Beckner (1995), Valdimarsson (preprint)
- Closely related to combinatorial methods for averaging operators: Christ (1998), D. Oberlin (2000), Tao and Wright (2003), Erdoğan and R. Oberlin (2008), many many others

Consider an ellipsoid B in some Hilbert space \mathcal{H} with center x_0 and axes $\{\omega_j\}$ of lengths $\{\ell_j\}$. We define the k -content of B as

$$|B|_k := \sup_{i_1 < \dots < i_k} \ell_{i_1} \cdots \ell_{i_k}.$$

Let μ be a nonnegative Borel measure on \mathcal{H} . It will be called k -curved with exponent $\alpha > 0$ when there exists $C_\alpha < \infty$ for which

$$\mu(B) \leq C_\alpha |B|_k^\alpha \text{ for all ellipsoids } B.$$

- D. Oberlin (2000): If μ is supported on a smooth hypersurface in \mathbb{R}^d , μ is d -curved with exponent $\frac{d-1}{d+1}$ iff $\|\mu \star f\|_{d+1} \lesssim \|f\|_{\frac{d+1}{d}, 1}$ for all f (modulo a bounded nondegenerate multiplicity condition).
- Bak, Oberlin, Seeger (2008): If μ is supported on a curve in \mathbb{R}^d satisfying monotonicity conditions, $\|\hat{f}\|_{L^{1+\alpha}(\mu)} \lesssim \|f\|_{1+\alpha, 1}$ when μ is d -curved with exponent α .

- The Oberlin condition is implied by various curvature conditions appearing in the literature on L^p bounds for maximal averages and elsewhere. Two Examples:
 - Nagel, Seeger, Wainger (1993): L^p -integrability of $\sup_B |B|_k^{-\alpha} \mu(x + B)$ implies k -curvature with exponent $\frac{\alpha p}{p+1}$.
 - Iosevich and Sawyer (1997): $\mu(\{y \mid \text{dist}(y, H)\} \leq \delta) \lesssim \delta^{\alpha k}$ for all affine $(k - 1)$ -dimensional hyperspaces H implies k -curvature with exponent α .
- The Oberlin condition for ellipsoids has an equivalent formulation for testing μ on Gaussians, which has connections to Gaussian quasi-extremizability, Brascamp-Lieb inequalities, and heat-flow monotonicity: Lieb (1990); Bennett, Bez, Carbery, Hundertmark (2009); Bennett, Bez, Carbery (2009); Bennett and Bez (2010); Bennett, Carbery, Christ, and Tao (2008, 2010)

Theorem (G., Proc. AMS 2011)

For any $\alpha > 0$ and k -admissible μ , the following are equivalent:

- The measure μ is k -curved with exponent α .
- For any $\gamma \in (0, \alpha)$ and any exponents $p_i \in [1, \frac{k\alpha}{k\alpha - \gamma})$ satisfying $\sum_{i=1}^{k+1} (1 - \frac{1}{p_i}) = \frac{\gamma}{\alpha}$,

$$\left| T_{\mu, k}^{-\gamma}(f_1, \dots, f_{k+1}) \right| \lesssim \prod_{i=1}^{k+1} \|f_i\|_{L^{p_i}(\mu)}.$$

- For any $\gamma \in (0, \infty)$, and any exponents $p_i \in (\frac{k\alpha}{k\alpha + \gamma}, 1]$ satisfying $\sum_{i=1}^{k+1} (\frac{1}{p_i} - 1) = \frac{\gamma}{\alpha}$,

$$T_{\mu, k}^{\gamma}(|f_1|, \dots, |f_{k+1}|) \gtrsim \prod_{i=1}^{k+1} \|f_i\|_{L^{p_i}(\mu)}.$$

Sketch of Proof

Key Sublevel Set Estimate

Let μ be a Borel probability measure on \mathcal{H} and

$$I_k^\delta(\mu) := \mu^k \left(\left\{ (y_1, \dots, y_k) \in \mathcal{H}^k \mid 0 < \text{VOL}(0, y_1, \dots, y_k) < \delta \right\} \right).$$

For any positive integer k , there are constants c_k and C_k such that for any $\epsilon > 0$ and any Borel probability measure μ on \mathcal{H} ,

$$I_k^{c_k \delta}(\mu) \leq C_k \epsilon$$

for any δ which is less than or equal to the infimum of the k -content of all centered ellipsoids B satisfying $\mu(B) \geq \epsilon$.

The proof of this estimate is by induction on k . The case $k = 1$ is trivial, since $I_1^\delta(\mu)$ is exactly the mass of the ball $B_\delta(0) \setminus \{0\}$.

- Induction step: Treat y_k as fixed and lying outside a ball so large that the μ -measure of the complement is comparable to ϵ . Applying Fubini to I_k^δ reduces to the situation of estimating $I_{k-1}^{\delta^*}(\mu^*)$, where μ^* is the orthogonal projection of μ onto the hyperplane orthogonal to y_k . and $\delta^* = \delta/\|y_k\|$.
- Any ellipsoid in this hyperplane which has μ^* -mass at least ϵ extends (with length $\|y_k\|$ in the direction of y_k) to a ball with μ -mass at least ϵ . In this way we get a favorable pointwise estimate for this integral whenever $\|y_n\|$ is sufficiently large.
- The end result is

$$I_k^{c_k \delta}(\mu) \lesssim \epsilon^2 + I_k^{c_k \delta}(\mu_0)$$

where μ_0 equals μ restricted to the inside of the large ball. Iterating and bootstrapping gives the desired inequality.

- Now the Oberlin condition is exactly what one needs to make restricted weak-type estimates for the multilinear sublevel set operator with “phase” $\text{VOL}(0, y_1, \dots, y_k)$.

Restriction corollary.

Suppose $\gamma : [-A, A] \rightarrow \mathbb{R}^d$ is C^1 and satisfies

$$\sup_{Q \in \text{GL}(d, \mathbb{R})} |\det Q|^\alpha \int_{-A}^A \frac{\omega(t)}{|Q\gamma'(t)|^{d\alpha}} dt \leq K.$$

For nonnegative ω . Suppose further that for any $x \in \mathbb{R}^d$, there are at most N isolated d -tuples (t_1, \dots, t_d) such that $\gamma(t_1) + \dots + \gamma(t_d) = x$. Then for exponents p, q such that

$$\frac{1}{q} = \frac{1 + d\alpha}{\alpha} \frac{1}{p'}, \quad \frac{1}{p'} < \min \left\{ \frac{1}{d} \frac{\alpha}{1 + \alpha}, \frac{1}{2d} \right\},$$

$$\left(\int_{-A}^A |\hat{f}(\gamma(t))|^q |\omega(t)|^{\frac{1}{1+d\alpha}} dt \right)^{\frac{1}{q}} \lesssim \|f\|_p$$

for all f with implicit constant depending on p, d, N, K . ◆◆

- Uniform estimates for MDFs allow one to bridge the gap between one-dimensional multilinear sublevel set estimates and L^p -improving estimates for Radon-like operators.
- Related Work
 - Earlier work: Drury (1990); Phong and Stein (1991, 1994, 1998); Sogge and Stein (1990); Christ (1984); Littman (1971); D. Oberlin and Stein (1982); Bak (2000); Iosevich and Sawyer (1996); Lee (2003, 2004); Bak, Oberlin, and Seeger (2002)
 - Combinatorial: Christ (1998); D. Oberlin (2000); Schlag (2003)
 - FIOs: Greenleaf and Seeger (1994); Mockenhaupt, Seeger, and Sogge (1993)
- An alternate way of thinking of this is a multi-dimensional, multilinear sublevel set estimate.
- The remarkable thing is that there is a feedback loop: new Radon-like estimates allow for new tests for k -curvature of measures (specifically providing a machine to damp measures), which, in turn, imply new estimates for Radon-like operators.

Let ρ, φ be real, smooth functions on $\mathbb{R}^d \times \mathbb{R}^d$. For compact $E \subset \mathbb{R}^d \times \mathbb{R}^d$, let

$$|\rho(E)|_L := \sup_y |\rho(E \cap \mathbb{R}^d \times \{y\})| \text{ etc. for } |\rho(E)|_R.$$

Theorem (G. 2010)

There exist weights R_ρ (equal to rotational curvature) and $R_{\rho, \varphi}^2$ (affine-invariant) such that

$$\int_E |R_\rho(x, y)|^{\frac{1}{d+1}} |f(x)g(y)| dx dy \lesssim |\rho(E)|_L^{\frac{d}{d+1}} |\rho(E)|_R^{\frac{1}{d+1}} \|f\|_{\frac{d+1}{d}} \|g\|_{\frac{d+1}{d}}$$

$$\begin{aligned} \int_E |R_{\rho, \varphi}^2(x, y)|^{\frac{1}{d(d-1)}} |f(x)g(y)| dx dy \\ \lesssim |\rho(E)|_L^{\frac{d-2}{d-1}} |\rho(E)|_R^{\frac{1}{d-1}} |\varphi(E)|_R^{\frac{1}{d-1}} \|f\|_{\frac{d(d-1)}{d^2-2d+2}} \|g\|_{\frac{d-1}{d-2}} \end{aligned}$$

with implied constants that are “topological.”

- These are just the first two in a recursively-defined sequence of weights measuring notions of higher curvature.
- $R_{\rho, \varphi}^2$ is sensitive to the rotational curvature of ρ on level sets of φ and to the curvature of these same level sets. If $\mathcal{C} \subset T^*\mathbb{R}^d \setminus 0 \times T^*\mathbb{R}^d \setminus 0$ is given by

$$\begin{aligned} \mathcal{C}^c &:= \{((x, \lambda \partial_x \rho(x, y)), (y, -\lambda \partial_y \rho(x, y))) \\ &\quad \mid (x, y) \in U, \lambda \in \mathbb{R} \setminus 0, \rho(x, y) = c\}; \\ \mathcal{L}^b &:= \{((x, \xi), (y, \eta)) \in \mathcal{C}^c \mid \varphi(x, y) = b\} \end{aligned}$$

for fixed x , the projection of \mathcal{L}^b onto $T_x^*\mathbb{R}^d$ is locally a hypersurface and has $d - 2$ nonvanishing principal curvatures (the maximum number possible since the hypersurface is homogeneous).

- Corollary (a la Greenleaf and Seeger): If rotational curvature vanishes on a hypersurface maximally curved in the sense above, then the corresponding Radon-like operator maps $L^{d/(d-1), 1}$ to $L^{d^2/(d-1), \infty}$.

Sketch of Proof:

Key idea: Estimate a MDF above and below. A singular change of variables gives

$$\int_{(\mathbb{R}^d)^{d+1}} |\det(\partial_x \rho(x, y_1) \cdots \partial_x \rho(x, y_d))| \prod_{i=1}^d \chi_{|\rho(x, y_i)| \leq \epsilon} |f(y_i)| dy_1 \cdots dy_d dx$$

$$\leq N(2\epsilon)^d \|f\|_{L^1(\mathbb{R}^d)}^d$$

Depending on curvature properties of the measure μ supported on graph of $\partial_x \rho(x, y)$ for each fixed x , you get an estimate

$$\int_{(\mathbb{R}^d)^{d+1}} |\det(\partial_x \rho(x, y_1) \cdots \partial_x \rho(x, y_d))| \prod_{i=1}^d \chi_{|\rho(x, y_i)| \leq \epsilon} |f(y_i)| dy_1 \cdots dy_d dx$$

$$\gtrsim \int_{\mathbb{R}^d} \left[\int \chi_{|\rho(x, y)| \leq \epsilon} |f(x, y)|^s dy \right]^{\frac{d}{s}} dx \quad \exists s \in (0, 1).$$

The recursive generation of new estimates comes from the observation that L^p -improving estimates for certain classes of Radon-like operators imply Oberlin-like curvature conditions.

$$\left| \int_{\mathbb{R}^n} \int_U f(x + \Phi(y))g(x)w(y)dydx \right| \leq C_{\Phi,w,r} \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}$$

for some p, q satisfying $\frac{1}{p} + \frac{1}{q} = \frac{2}{r}$ and $\Phi : U \rightarrow \mathbb{R}^n$. Let $(\Phi, 1) : U \rightarrow \mathbb{R}^{n+1}$ be given by $(\Phi(y), 1) := (\Phi_1(y), \dots, \Phi_n(y), 1)$. Then

$$\sup_{Q \in \text{GL}(n+1, \mathbb{R})} |Q|^{\frac{2}{r}-1} \int_U \frac{w(y)dy}{|Q(\Phi(y), 1)|^{(n+1)(\frac{2}{r}-1)}} \leq c_n C_{\Phi,w,r}$$

for some constant c_n depending only on n .

Concluding Remarks

- The sublevel set problem is far from resolved, even for multilinear one-dimensional objects.
- The interconnectedness of sublevel set estimates, MDFs, and Radon-like operators demonstrates that geometry (and specifically curvature) plays in understanding nonoscillatory problems and nonoscillatory pieces of oscillatory problems.
- Perhaps something like a nonlinear Newton polytope can completely describe these objects, but it will necessarily be quite complicated.
- Ultimately one would like to know if there is a simpler way of identifying what estimates a given operator satisfies (a la Tao and Wright (2003), for example). There seems to be mounting evidence that this may not be possible to do in a very general way (or is, at the moment, beyond reach).