A multi-dimensional resolution of singularities with applications

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Analysis related to singularities of $F \in C^\omega(\mathbb{R}^n)$

- Critical integrability index and sublevel set growth rate
- Decay rate of oscillatory integrals
- L^2 decay for oscillatory integral operators ↔ Sobolev regularity of GRT/FIO
- Restriction and maximal operators for degenerate surfaces

Analyst-friendly resolution of singularities

- $\mathbb{R}^2$: Phong-Stein-(Sturm), Greenblatt, Carbery-Wainger-Wright, Magyar, Ikromov-(Kempe)-Müller,...

- $n \geq 3$: Bierstone-Milman, Sussman, Greenblatt, ...

- **Here:** adapt ideas of Parusiński $\longrightarrow$ algorithmic, inductive method, avoiding genericity assumptions

Invariants of an $F \in C^{\omega}(\mathbb{R}^n), \quad F(0) = 0$

- **Critical integrability index**

$$\mu_0(F) := \sup\left\{\mu > 0 : \exists \text{ nhood } U \text{ of } 0 \text{ s.t. } \int_U |F|^{-\mu} d\mathbf{x} < \infty\right\}$$

- **Sublevel set growth rate**

$$\nu_0(F) := \sup\left\{\nu > 0 : \sup_{\epsilon > 0} \epsilon^{-\nu}\big|\mathcal{E}_\epsilon(F)\big| < \infty\right\},$$

$$\mathcal{E}_\epsilon(F) := \left\{\mathbf{x} \in U : |F(\mathbf{x})| < \epsilon\right\}$$

- **Scalar oscillatory decay index**

$$\rho_0(F) = \ \sup\left\{\rho > 0 : \sup_{\lambda \longrightarrow \infty} \lambda^{\rho}|I_\lambda| < \infty\right\},$$

$$I_\lambda = \int e^{i\lambda F(\mathbf{x})} a(\mathbf{x}) d\mathbf{x}, \quad a \in C_0^\infty$$

- **Oscillatory integral operators:** $\mathbb{R}^n = \mathbb{R}^m_x \times \mathbb{R}^k_y$,

$$\sigma_0(F) = \sup\left\{\sigma > 0 : \sup_{\lambda \longrightarrow \infty} \lambda^\sigma ||T_\lambda||_{L^2 \longrightarrow L^2} < \infty\right\},$$

$$T_\lambda g(\mathbf{x}) = \int_{\mathbb{R}^k} e^{i\lambda F(\mathbf{x},\mathbf{y})} a(\mathbf{x},\mathbf{y}) g(\mathbf{y}) d\mathbf{y}, \quad a \in C_0^\infty$$

- $\mu_0 = \nu_0$ and $\rho_0 = \nu_0$ (unless $v_0(F)$ is odd)
- These are $Diff^\omega(\mathbb{R}^n, 0)$-invariant, σ_0 is $Diff^\omega(\mathbb{R}^m) \times Diff^\omega(\mathbb{R}^k)$-invariant

For $n = 2$, indices are controlled by the **Newton exponent** of F at 0

$$F(\mathbf{x}) = \sum_{|\alpha|>0} a_\alpha x^\alpha \longrightarrow \textbf{Newton polyhedron} \text{ of } F,$$

$$NP(F) = \text{ convex hull } \bigcup_{\{\alpha : a_\alpha \neq 0\}} (\alpha + \mathbb{R}^n_+)$$

Newton distance: $d_0(F) = \inf\{d > 0 : (d, d, \dots, d) \in NP(F)\}$

Newton exponent: $\delta_0(F) = 1/d_0(F)$

Good news: $\delta_0(F)$ is computable from power series of F

Bad news: $\delta_0(F)$ is coordinate-dependent

So: For $\Phi : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$, let $\delta_0(F; \Phi) = \delta_0(F \circ \Phi)$.

$$\delta(F) := \inf\{\delta_0(F; \Phi) : \Phi \in Diff^\omega\} \text{ is invariant}$$

Def. Say Φ is **adapted** if $\delta_0(F; \Phi) = \delta(F)$.

Varchenko: In $\mathbb{R}^2$, adapted coordinates exist and can be found in the class of analytic shifts, $\Phi(y_1, y_2) = (y_1, y_2 - r(y_1))$, $r \in C^\omega(\mathbb{R})$

Thm. (i) [Phong-Stein-Sturm] In $\mathbb{R}^2$, $\mu_0(F) = \delta(F)$. Furthermore, can tell from $NP(F)$ whether the coordinates are adapted.

(ii) [P-S] For oscillatory integral operators in 1+1 variables,

$$\sigma_0(F) = \frac{1}{2} \cdot \frac{\delta''}{1+\delta''}, \qquad \delta'' = \delta(F''_{xy})$$

Goal: to understand $\mu_0, \nu_0, \rho_0, \sigma_0$ in higher dimensions.

Are the decay rates expressible in terms of Newton(-like) data?

Obstacles for $n \geq 3$

- Can no longer restrict to global C^ω coordinate charts $\mathbf{x} = \Phi(\mathbf{y})$
- For $n = 2$, have Puiseux expansions

$$F(x_1, x_2) = x_1^{b_1} x_2^{b_2} \prod_{j=1}^{M} \left(x_2 - u_j\left(x_1\right) x_1^{a_j}\right), \quad u_j(0) \neq 0$$

…but these don't exist for $n \geq 3$.

Our approach (following Parusiński[1994,2000]):

- Allow **local** coordinates $\vec{\Phi} = \{(V_j, \Phi_j)\}_{j=1}^{M}$ with

$$0 \in \overline{V_j} \text{ and } \cup_{j=1}^{M} V_j = \mathbb{R}^n \setminus (\text{lower dim set})$$

- Allow functions to have **fractional power series** expansions

- Study $\delta_0(F; \vec{\Phi}) := \inf_j \delta_0(F; \Phi_j)$ and $\delta_0(F; \mathcal{C}) = \inf\{\delta_0(F; \vec{\Phi}) : (\vec{V}, \vec{\Phi}) \in \mathcal{C}\}$ for appropriate classes $\mathcal{C}$ of local coordinates

Ex. $F(\mathbf{x}) = x_3^2 - x_1^2 - x_2^2$ on $\mathbb{R}^3$. Then easy to see $\mu_0(F) = 1$**.**

- For **all** global Φ, again $F \circ \Phi$ has a nondeg critical pt at 0 $\quad\Longrightarrow$

$$NP(F \circ \Phi) = \Big[\Big[(2,0,0),(0,2,0),(0,0,2)\Big]\Big]$$

$$\Longrightarrow \quad d_0(F \circ \Phi) = 2/3 \implies \delta_0(F \circ \Phi) = 3/2 > 1.$$ **Too large.**

- So: introduce **local, fractional power series** coordinates

$$V = \big\{\mathbf{x} : 0 < x_2 < x_1, 0 < x_3 < c\big\},$$

$$\Phi_V(\mathbf{y}) = \left(\sqrt{y_1}, \sqrt{y_1} \cdot y_2, y_3 + \sqrt{y_1} \cdot \sqrt{1 + y_2^2}\right)$$

$F \circ \Phi_V(\mathbf{y}) = y_3^2 + 2y_3\sqrt{y_1} \cdot \sqrt{1 + y_2^2} \implies$ defining NP for frac. power series,

$NP(F \circ \Phi_V) = \Big[\Big[\,(0,0,2),(\tfrac{1}{2},0,1)\Big]\Big] \implies \delta_0(F \circ \Phi_V) = 1$

Cover $\mathbb{R}^3 \setminus$ (variety) with 8 such charts, all giving $\delta_0(F \circ \Phi_{V_j}) = 1$.

$\Longrightarrow \quad \delta_0(F; \mathcal{C}) = 1 = \mu_0(F)$

Fractional power series: On $\mathbb{R}^n$ (or $V \subset \mathbb{R}^n_+$, $0 \in \overline{V}$),

$$F(\mathbf{x}) = \sum_{\alpha \in \mathbb{Z}^n_+} a_\alpha \mathbf{x}^\alpha \quad \text{s.t.} \quad \exists N \in \mathbb{N} \quad \text{s.t.} \quad \sum a_\alpha \mathbf{x}^{N\alpha} \in C^\omega$$

Can still define $NP(F), d_0(F), \delta_0(F)$

Def. $f(\mathbf{x})$ is a **fractional normal crossing** if

$$f(\mathbf{x}) = u(\mathbf{x}) \prod_{j=1}^{M} x_j^{\alpha_j} := u(\mathbf{x}) \cdot \mathbf{x}^\alpha, \quad u(0) \neq 0,\ \alpha_j \in \mathbb{Q}_+$$

Jung-Abhyankar Thm (roughly) If $F(\mathbf{x}) = x_n^d + \sum_{j=1}^d c_j(\mathbf{x}')x_n^{d-j}$ is a Weierstrass polynomial, with discriminant $\Delta_F(\mathbf{x}') \not\equiv 0$ and having normal crossings at $0 \in \mathbb{R}^{n-1}$, then $\exists s \in \mathbb{N}$ s.t.

$$F(y_1^s, \ldots, y_{n-1}^s, y_n) = \prod \left(y_n - b_j\left(\mathbf{y}'\right)\right),$$

with each $b_j - b_{j'}$ having normal crossings.

- NC assumption on Δ_F is very strong, but can find almost covering $\{V_j\}$ s.t. holds on each V_j after suitable transformation

Replacement for Puiseux expansion: on $V \subset \mathbb{R}^n_+$, with frac. powers,

$$F(\mathbf{x}) = \mathbf{x}'^{\beta'} x_n^{\beta_n} \prod_{j=1}^{M} \left(x_n - u_i\left(\mathbf{x}'\right) \mathbf{x}'^{\gamma_j}\right), \quad u_j(0) \neq 0$$

with $\{\gamma_j : 1 \le j \le M\}$ totally ordered, $\gamma_j \le \gamma_{j+1}$ $\Longrightarrow$

$$\mathrm{NP}(F) = \text{ convex hull} \bigcup_{\ell=1}^{L} \left[(\mathbf{A}_\ell, B_\ell) + \mathbb{R}^n_{\ge 0}\right].$$

- $\{\boldsymbol{\alpha}_\ell : 1 \le \ell \le L\}$ the distinct multi-exponents in $\{\boldsymbol{\gamma}_i : 1 \le i \le M\}$, arranged so that

$$\boldsymbol{\alpha}_1 < \boldsymbol{\alpha}_2 < \cdots < \boldsymbol{\alpha}_L$$

- For $0 \le \ell \le L$, let

$$\mathcal{L}_\ell = \left\{1 \le i \le M : \boldsymbol{\gamma}_i = \boldsymbol{\alpha}_\ell\right\}, \qquad m_\ell = \#(\mathcal{L}_\ell) \text{ so that } \sum_{\ell=1}^{L} m_\ell = M,$$

$$\mathbf{A}_\ell = \boldsymbol{\beta} + \sum_{k \le \ell} m_k \boldsymbol{\alpha}_k, \qquad B_\ell = \beta_{n+1} + \sum_{k > \ell} m_k.$$

Then: $\mathbf{A}_\ell < \mathbf{A}_{\ell+1}$ and $B_\ell > B_{\ell+1}$, $0 \leq \ell \leq L-1$.

A Newton polyhedron is of **monotone edge type** if it is generated by such a set.

Many nice properties, e.g.,

- For any $c > 0$, $NP(F) \cap \{\alpha_n = c\}$ is empty or a translate of $\mathbb{R}^n_+$

- $\delta_0(F) = \min_{1 \leq j \leq n} \delta_j$, where δ_j is the 2-dim Newton exponent of the projection of $NP(F)$ into the (α_j, α_n) plane

Coordinate transformations

Definition 0.1 *Let* $V \subset \mathbb{R}^n_+$ *open,* $0 \in \overline{V}$ *. A* **generalized coordinate transformation** *on* V *is a vector-valued function* σ *s.t.*

- *each entry of* $\sigma(\mathbf{y})$ *is a fractional power series in* $\mathbf{y}$ *on* $(0,1)^n$*,*
- $\sigma(\mathbf{0}) = \mathbf{0}$, $\sigma \in C^1((0,1)^n)$,
- σ *is a bijection from* $(0,1)^n$ *onto* V*.*

Scaling by a unit: $\sigma(y_1, \cdots, y_n) = (y_1, \cdots, y_{n-1}, y_n u(\mathbf{y}'))$

A shift: $\sigma(\mathbf{y}) = (y_1, \cdots, y_{n-1}, y_n - f(\mathbf{y}'))$, f is a fractional power series.

Blowdown: $\sigma(\mathbf{y}) = (y_1, \cdots, y_k, y_{k+1}y_n, \cdots, y_{n-1}y_n, y_n)$

Power transformation: $\sigma(\mathbf{y}) = \Phi_{\mathbf{r}}(\mathbf{y}) = (y_1^{r_1}, \cdots, y_n^{r_n})$

σ is a **coordinate system** if the Jacobian is a unit ($\neq 0$)
First two are, second two aren't, but compositions can be.

Domains: Horns

Def. For $n = 1$, a **horn** in $(0, \infty)$ is $V = (0, 1)$. A horn in $\mathbb{R}^n_{>0}$ for $n \geq 2$ is a subset of the positive orthant with one of the following two possible structures: Let f be a fractional power series on $(0, 1)^{n-1}$.

- A horn $V \subseteq \mathbb{R}^n_{>0}$ is said to be **adjacent to** f and is an **adjacent horn** if

$$V = \left\{\mathbf{y} = (\mathbf{y}', y_n) \in \mathbb{R}^n_{>0} : 0 < \kappa \left(y_n - f(\mathbf{y}')\right) < g(\mathbf{y}'),\ \mathbf{y}' \in (0, 1)^{n-1}\right\},$$

where $\kappa = \pm 1$ and g is fractional normal crossings on $(0, 1)^{n-1}$.

- A horn $V \subseteq \mathbb{R}^n_{>0}$ is **separated from** f and is **distant horn** if it is of the form

$$V = \left\{(\mathbf{y}', y_n) \in \mathbb{R}^n_{>0} : g_1(\mathbf{y}') < y_n - f(\mathbf{y}') < g_2(\mathbf{y}'),\ \mathbf{y}' \in (0, 1)^{n-1}\right\},$$

where g_1 is a fractional monomial, and g_2 is either a nonzero constant or a fractional monomial with $g_1 g_2 > 0$ and

$$g_1(\mathbf{y}') = a y_k^{\mu} g_2(\mathbf{y}'), \quad a \neq 0,$$

for some $1 \leq k \leq n - 1$ and $\mu > 0$.

Inductively define **towers of horns** in $\mathbb{R}^n$ ...

Main Result. (roughly) Let $F \in C^{\omega}(\mathbb{R}^{n+1})$, $F(0) = 0$, $F \not\equiv 0$. There exists a class $\mathcal{C} = \mathcal{C}(F)$ of coordinate transformations $\Phi(\phi, V, r)$ of the form

$$(\mathbf{y}, y_{n+1}) \longrightarrow (\mathbf{x}, x_{n+1}) = \Phi(\mathbf{y}, y_{n+1}) = \big(\phi(\mathbf{y}), y_{n+1} + r(\mathbf{y})\big)$$

where

- $V \subseteq (-\epsilon, \epsilon)^n \setminus \cup_{j=1}^n \{x_j = 0\}$ is open, connected, $\mathbf{0} \in \overline{V}$

- $\phi : (0,1)^n \longrightarrow V$ is a C^1 bijection with nonvanishing Jacobian and given by a fractional power series

- r is a fractional power series

- If $\{r_i(\mathbf{y}) : 1 \leq i \leq N\}$ are the roots (in y_{n+1}) of $F \circ \Phi(\mathbf{y}, y_{n+1})$ with $r_i(\mathbf{0}) = 0$, then every element of $\mathcal{A} = \{r_i, \mathrm{Re}(r_i) : 1 \leq i \leq N\}$ and the difference set $\mathcal{A} - \mathcal{A}$ is either identically zero or fractional normal crossings.

- The critical integrability exponent $\mu_0(F) = \inf\{\delta_0(F; \Phi) : \Phi(\phi, V, r) \in \mathcal{C}\}$

- We can replace $\mathcal{C}$ with an algorithmically defined **finite subset** $\mathcal{C}^* \subset \mathcal{C}$.

The algorithm. General procedure, worked out for previous example:
(Some steps vacuous)

- $F(x_1, x_2, x_3) = F(x', x_3) = x_3^2 - x_1^2 - x_2^2$
- After rotation, express as (unit) × Weierstrass poly $G(x', x_3)$ - already is
- Factor as a product of powers of irreducible W.-polys: $G = \prod_{l=1}^{L} G_l^{m_l}$
  Here, $L = 1, G_1 = F, m_1 = 1$.
- Define $c(x') = \prod G_l^{m_l}(x', 0) = F(x', 0) = -(x_1^2 + x_2^2)$ and
  $\Delta_F(x') =$ **discriminant** of F w.r.t. $x_3 = -4(x_1^2 + x_2^2)$
- Finally, set $\Lambda(x') := c(x') \cdot \Delta_F(x') = -4(x_1^2 + x_2^2)^2$
- Subdivide $\mathbb{R}^2 \setminus \{x_1 \cdot x_2 \cdot (x_2 - x_1) \cdot (x_1 + x_2) = 0\}$ into a union of (towers of) horns where one of x_1, x_2 is dominant
- Coordinatize each tower of horns as in the Theorem ... here, just horns:
  sectors $\{V_j\}_{j=1}^{8}$ as in example.

Full details at http://arxiv.org/abs/1007.0519