Oscillatory integrals

Let $f(x_1, ..., x_n)$ be a real-analytic function on a neighborhood of the origin in $\mathbf{R}^n$. For a cutoff function $\phi(x_1, ..., x_n)$ supported near the origin, define

$$I_\phi(\lambda) = \int_{\mathbf{R}^n} e^{i\lambda f(x_1,...,x_n)} \phi(x_1, ..., x_n) dx_1...dx_n$$

Here λ is a real parameter. Such integrals come up frequently in various capacities in analysis and mathematical physics. The goal is to understand the behavior of $I_\phi(\lambda)$ as $|\lambda| \to \infty$.

By resolution of singularities, one can write $\phi(x) = \sum_{j=1}^m \phi_j(x)$, thereby breaking the integral into finitely many terms, in such a way that on the jth term one may apply a coordinate change $g_j(x)$ such that the above becomes

$$I_\phi(\lambda) = \sum_{j=1}^m \int_{\mathbf{R}^n} e^{i\lambda(f(g_j(x))} J_j(x)\phi_j(g_j(x)) dx_1...dx_n$$

Here $J_j(x)$ denotes the Jacobian of the coordinate change $g_j(x)$, the function $\phi_j(g_j(x))$ is smooth, and most significantly, each new phase $f(g_j(x))$ is of the form $a_j(x)m_j(x)$ where $m_j(x)$ is a monomial and $a_j(x)$ never vanishes.

Thus the new phases are effectively monomials. Furthermore, the Jacobian factors $J_j(x)$ can be also made to be of this form. As a result, it is not hard to show that asymptotics exist. One obtains:

$$I_\phi(\lambda) = e^{i\lambda f(0)}\lambda^{-\epsilon}(\ln\lambda)^k + o(\lambda^{-\epsilon}(\ln\lambda)^k)$$

Here $\epsilon > 0$ and k is an integer with $0 \leq k \leq n-1$.

Clearly, it is useful to determine ϵ and k.

But resolution of singularities in itself just shows they exist.

It turns out there is a large class of phase functions where the coordinate changes $g_j(x)$ can be taken to be functions whose components are monomials, and for which ϵ and k can be explicitly determined in terms of easily describable aspects of the phase $f(x_1, ..., x_n)$.

This was discovered by Varchenko, a student of Arnold, in the 1970s.

It uses the notion of Newton polyhedra from algebraic geometry, and the proofs only need resolution of singularities for toric varieties, substantially easier than results such as the deep theorem of Hironaka.

Some relevant definitions.

Let $f(x) = \sum_\alpha f_\alpha x^\alpha$ denote the Taylor expansion of $f(x)$ at the origin, which we assume has at least one nonvanishing term.

Definition. For any α for which $f_\alpha \neq 0$, let Q_α be the octant $\{x \in \mathbf{R}^n : x_i \geq \alpha_i \text{ for all } i\}$. The **Newton polyhedron** $N(f)$ of f is defined to be the convex hull of all Q_α.

Newton polyhedron in 2 dimensions for $f(x) = xy^5 + 3x^2y^4 - 2x^4y^3 + x^7y^2 + 12x^6y^6$.

In general, a Newton polyhedron can contain faces of various dimensions in various configurations. These faces can be either compact or unbounded.

An important role is played by the following functions.

Definition. Suppose F is a compact face of the $N(f)$. Then define $f_F(x) = \sum_{\alpha \in F} f_\alpha x^\alpha$, the sum of the terms of the Taylor expansion of $f(x)$ whose exponents are on the face F.

Also useful is the following terminology.

Definition. The **Newton distance** of $f(x)$ is defined to be $\inf\{t : (t, t, ..., t, t) \in N(f)\}$.

Varchenko's theorem is as follows.

Theorem (Varchenko) Suppose for each compact face F of $N(f)$, the function $\nabla f_F(x)$ is nonvanishing on $(\mathbf{R} - \{0\})^n$. Further suppose that the Newton distance of f is equal to some $d > 1$.

Then if k denotes the dimension of the face of $N(f)$ of minimal dimension containing $(d, d, ..., d, d)$, as long as $\phi(x)$ is supported on a sufficiently small neighborhood of 0 and $\phi(0) \neq 0$, then the leading term of the asymptotic expansion for the oscillatory integral $I_\phi(\lambda)$ is given by

$$C_\phi \lambda^{-\frac{1}{d}} (\ln \lambda)^{n-k-1}$$

Here $C_\phi \neq 0$ and the exponents are sharp.

There are various senses in which the conditions of Varchenko's theorem "generically" hold.

Varchenko's proof proceeds by associating a toric variety to the Newton polyhedron $N(f)$ and then applying toric resolution of singularities to this variety. This resolution of singularities induces a decomposition

$$I_\phi(\lambda) = \sum_{j=1}^{m} \int_{\mathbf{R}^n} e^{i\lambda l_j(x)} \phi_j(x)\, dx_1 ... dx_n$$

Although $l_j(x)$ are not monomials, for each j there is a monomial $m_j(x)$ and an index k depending on j such that one has

$$|l_j(x)| < C|m_j(x)|$$

$$|\partial_k l_j(x)| > C\frac{1}{|x_k|}|m_j(x)|$$

These are enough to obtain sharp estimates using the method of stationary phase.

The estimates can then be related to the Newton polyhedron of f as in the statement of Varchenko's theorem.

Natural question: Can resolution of singularities methods beyond those for toric varieties extend Varchenko's theorem?

I developed a resolution of singularities method which provides such theorems.

Theorem (G). Varchenko's bound $|I_\phi(\lambda)| < C\lambda^{-\frac{1}{d}}(\ln\lambda)^{n-k-1}$ still holds as long as the zeroes of each $f_F(x)$ on $(\mathbf{R}-\{0\})^n$ are of order $< d = d(f)$, and one loses an additional factor of at most $\ln\lambda$ if the maximum order is actually equal to d. Again the exponents are optimal.

Furthermore, the hypotheses of this theorem are best possible in the sense that if a $f_F(x)$ corresponding to a compact face F containing $(d, d, ..., d, d)$ has a zero in $(\mathbf{R}-\{0\})^n$ of order greater than d, then Varchenko's estimates will not hold for any cutoff function $\phi(x)$ with $\phi(0) \neq 0$.

When the maximal order of the zeroes of the $f_F(x)$ on $(\mathbf{R} - \{0\})^n$ is equal to some $e > d$, I show that one has a weaker estimate
$$|I_\phi(\lambda)| < C|\lambda|^{-\frac{1}{e}}$$

The latter estimates are only occasionally sharp. To get sharper estimates in this setting one needs more information about the zero set of $f(x)$ than the Newton polyhedron of $f(x)$ can provide

The proof of Varchenko's theorem holds for functions with convergent power series over other fields, when oscillatory integrals are defined appropriately. In particular, there is an analogue for functions over any local field of characteristic zero. This raises the question if an analogue of my theorem also holds for functions over a general local field of characteristic zero.

What is a local field?

We won't give the official definition here. Instead, we use the characterization that any local field K of characteristic zero is either $\mathbf{R}$, $\mathbf{C}$, the p-adic numbers $\mathbf{Q}_p$, or a finite field extension of $\mathbf{Q}_p$. (We'll define these soon.)

These are all locally compact abelian groups with a natural metric. We may define a Haar measure, characters, and then, oscillatory integrals.

Specifically, if $\chi(y)$ is an additive character on K, $f(x_1, ..., x_n)$ is a function equal to the sum of a convergent power series on a neighborhood of the origin in K^n, and $\phi(x_1, ..., x_n)$ is a smooth compactly supported function, we examine

$$I_\phi(\lambda) = \int_{K^n} \chi(\lambda f(x_1, ..., x_n))\phi(x_1, ..., x_n)\, dx_1 ... \, dx_n$$

Here we integrate over the n-fold product measure of Haar measure on K, and the parameter λ is in K. The goal is to find upper bounds on $|I_\phi(\lambda)|$ as $|\lambda| \to \infty$.

The p-adic numbers $\mathbf{Q}_p$.

Let p be prime, and write an element $x \in \mathbf{Z}_{p^m}$ in the form

$$x = \sum_{i=0}^{m-1} a_i p^i, \quad a_i \in \{0, ..., p-1\}$$

One adds elements of $\mathbf{Z}_{p^m}$ written in this form by adding the corresponding a_i, carrying to the right when needed. One has an analogous way of multiplying elements termswise.

One can extend these notions of addition and multiplication to infinite sums formally written as

$$x = \sum_{i=0}^{\infty} a_i p^i$$

Such infinite sums are called "p-adic integers". One can extend this a bit further by considering infinite sums of the form

$$x = \sum_{i=k}^{\infty} a_i p^i$$

Here k is allowed to be negative. The p-adic numbers $\mathbf{Q}_p$ are defined to be the set of all such x, with addition and multiplication defined componentwise with carrying as above.

Some facts about the p-adic numbers.

The p-adic numbers are a field (of characteristic zero); given any $x \in \mathbf{Q}_p$ one can find the multiplicative inverse of x by iteratively solving for the coefficients of the various p^i for x^{-1} using the equation $xx^{-1} = 1$.

There is a natural metric on $\mathbf{Q}_p$; if $x = \sum_{i=k}^{\infty} a_i p^i$ with $a_k \neq 0$, one sets $|x| = p^{-k}$.

The resulting topology has a basis of clopen sets; for any $x \in \mathbf{Q}_p$ and any k the set $\{y \in \mathbf{Q}_p : |y - x| \leq p^{-k}\}$ is both open and closed. The resulting topology is locally compact and totally disconnected.

Since the topology is locally compact, there is a Haar measure μ on $\mathbf{Q}_p$ which one uses to define integrals.

The continuous additive characters on $\mathbf{Q}_p$, used to define oscillatory integrals, can be explicitly written down as follows.

Define $\chi(\sum_{i=k}^{\infty} a_i p^i) = \exp(2\pi i \sum_{i=k}^{0} a_i p^i)$ for $k < 0$ and then define $\chi(\sum_{i=k}^{\infty} a_i p^i) = 1$ for $k \geq 0$.

Then any continuous additive character on $\mathbf{Q}_p$ is of the form $a(x) = \chi(yx)$ for some $y \in \mathbf{Q}_p$.

Thus the analogue here of the real oscillatory integrals considered before are integrals of the form

$$J_B(y) = \int_{\mathbf{Q}_p^n} \chi(yf(x_1, ..., x_n))\phi(x_1, ..., x_n)\, dx_1 ... dx_n$$

Here $y \in \mathbf{Q}_p$, $f(x_1, ..., x_n)$ is a function equal to a convergent power series on a neighborhood of the origin in $\mathbf{Q}_p^n$ and $\phi(x_1, ..., x_n)$ is a smooth compactly supported function supported on a neighborhood of the origin.

The goal is now to find the best estimate of the form $|J_B(y)| < C|y|^{-\epsilon}(\ln|y|)^k$ as $|y| \to \infty$.

It turns out that any smooth function on $\mathbf{Q}_p^n$ is locally constant. Thus one does not lose generality if one considers integrals

$$J_B(y) = \int_B \chi(yf(x_1, ..., x_n)))dx_1 ... dx_n$$

Here $B = B_1 \times ... \times B_n$ is a product of sufficiently small balls centered at the origin.

Varchenko's method immediately gives a p-adic analogue for $J_B(y)$ of his oscillatory integral theorem: If each polynomial $f_F(x)$ corresponding to a compact face F of the Newton polyhedron $N(f)$ has nonvanishing gradient on $(\mathbf{Q}_p - \{0\})^n$, then assuming B is a small enough product of balls centered at the origin one has the estimate (if $d > 1$)

$$|J_B(y)| < C|y|^{-\frac{1}{d}}(\ln|y|)^{n-k-1}$$

Like before, d is the minimal number such that $(d, d, ..., d, d) \in N(f)$, and k denotes the dimension of the face of $N(f)$ of minimal dimension containing $(d, d, ..., d, d)$.

In recent years there have been many papers analyzing p-adic oscillatory integrals of various kinds under nondegeneracy conditions like that of Varchenko's paper, such as by Cluckers, Denef, Lichtin, Loeser, Meuser, Veys, and Zuniga-Galindo. This is due to their number-theoretic connections, such as connections with Igusa zeta functions. Igusa zeta functions are also often studied directly, again often under such a nondegeneracy condition.

The p-adic analogue of my earlier oscillatory integral theorem is as follows.

Theorem ([G]). If B is a sufficiently small ball containing the origin, the following hold.

a) Suppose the zeroes of each $f_F(x)$ on $(\mathbf{Q}_p - \{0\})^n$ are of order $< d(f)$. Then one has the estimate

$$|J_B(y)| < C|y|^{-\frac{1}{d}}(\ln|y|)^{n-k-1}$$

If the maximal order is actually equal to $d(f)$, then again we lose an additional factor of $\ln|y|$:

$$|J_B(y)| < C|y|^{-\frac{1}{d}}(\ln|y|)^{n-k}$$

b) If the maximal order of the zeroes of the $f_F(x)$ on $(\mathbf{Q}_p - \{0\})^n$ is equal to some $e > d(f)$, then one can at least say

$$|J_B(y)| < C|y|^{-\frac{1}{e}}$$

The sharpness situation of this theorem is not clear, unlike in the case of real oscillatory integrals where the analogue of the first statement is sharp.

One can use this theorem to obtain estimates for exponential sums by judiciously choosing y and $f(x_1, ..., x_n)$.

As a concrete example, consider the case where $f(x_1, ..., x_n) = q(x_1, ..., x_n)$ for some polynomial q with integer coefficients. Let $y = p^{-m}$ for some $m > 0$ (recall $|y| = p^m$ here). Then the oscillatory integral becomes

$$J_B(y) = \int_B \chi\left(\frac{q(x_1, ..., x_n)}{p^m}\right) dx_1 ... dx_n$$

Because the coefficients of q are all integers, $\chi\left(\frac{q(x_1,...,x_n)}{p^m}\right)$ is constant on any set $B_1 \times ... \times B_n$ where each B_k is a ball of radius p^{-m}.

Specifically, it takes the value $exp\left(2\pi i \frac{q(x_1,...,x_n)}{p^m}\right)$, where the point $(x_1, ..., x_n)$ can be any point in $B_1 \times ... \times B_n$.

As a result, if we take $B = \{x : |x_i| < 1 \text{ for all } i\}$ then the oscillatory integral becomes a sum over elements of $(\mathbf{Z}_{p^m})^n$, namely the exponential sum

$$J_B(y) = \frac{1}{p^{mn}} \sum_{(x_1,...,x_n) \in \{0,...,p^m-1\}^n} exp\left(2\pi i \frac{q(x_1, ..., x_n)}{p^m}\right)$$

Thus if my theorem held for functions on the whole unit ball (rather than just sufficiently small neighborhoods of the origin), we would obtain bounds for this exponential sum.

But if $k_1, ..., k_n$ are large enough integers, the theorem will hold on the entire unit ball for $q(p^{k_1}x_1, ..., p^{k_n}x_n)$ in place of $q(x_1, ..., x_n)$. (Recall $|p^{k_i}x_i| = p^{-k_i}|x_i|$ in the p-adic world).

Thus we obtain bounds for the exponential sums

$$\frac{1}{p^{mn}} \sum_{(x_1,...,x_n)\in\{0,...,p^m-1\}^n} exp\left(2\pi i \frac{q(p^{k_1}x_1, ..., p^{k_n}x_n)}{p^m}\right)$$

Observe that in the case where $q(x_1^{j_1}, ..., x_n^{j_n})$ is homogeneous to some rational degree α for some positive integers $j_1, ..., j_n$ (so in particular if $q(x_1, ..., x_n)$ itself is homogeneous) one can select $k_1, ..., k_n$ so that $q(p^{k_1}x_1, ..., p^{k_n}x_n)$ is a power of p times $q(x_1, ..., x_n)$.

As a result, one recovers bounds for the original exponential sums $J_B(y)$. Specifically, we get:

Theorem. Suppose $q(x_1, .., x_n)$ is a polynomial and there are positive integers $j_1, ..., j_n$ such that $q(x_1^{j_1}, ..., x_n^{j_n})$ is homogeneous to any degree. Let d denote the Newton distance of the Newton polyhedron $N(q)$, and let k denote the minimal dimension of any face of $N(q)$ that contains $(d, d, ..., d, d)$.

If the zeroes of each polynomial $q_F(x)$ on $(\mathbf{Q}_p - \{0\})^n$ are of order $< d$, one has the bounds

$$\frac{1}{p^{mn}} \sum_{(x_1,...,x_n)\in\{0,...,p^m-1\}^n} exp\left(2\pi i \frac{q(x_1, ..., x_n)}{p^m}\right) < Cp^{-\frac{m}{d}} m^{n-k-1}$$

If the maximal order is equal to d, one loses a factor of m:

$$\frac{1}{p^{mn}} \sum_{(x_1,...,x_n)\in\{0,...,p^m-1\}^n} exp\left(2\pi i \frac{q(x_1, ..., x_n)}{p^m}\right) < Cp^{-\frac{m}{d}} m^{n-k}$$

If the maximal order is equal to some $e > f$, we have the upper bounds

$$\frac{1}{p^{mn}} \sum_{(x_1,...,x_n)\in\{0,...,p^m-1\}^n} exp\left(2\pi i \frac{q(x_1, ..., x_n)}{p^m}\right) < Cp^{-\frac{m}{e}}$$

When the gradient of each $q_F(x)$ is nonvanishing on $(\mathbf{Q}_p - \{0\})^n$, the first statement follows from Varchenko's method.

The sharpness of such estimates is not well-understood. There are even unresolved issues in one dimension.

Sublevel set measures.

Focusing on the $\mathbf{R}^n$ case again, we once again let $f(x_1, ..., x_n)$ be a real-analytic function on a neighborhood of the origin in $\mathbf{R}^n$, and let $\phi(x_1, ..., x_n)$ be a cutoff function supported near the origin. For a parameter $\eta > 0$, define

$$K_\phi(\eta) = \int_{\{x \in \mathbf{R}^n : |f(x_1,...,x_n)| < \eta\}} \phi(x_1, ..., x_n)\, dx_1 ... dx_n$$

Using resolution of singularities as in the oscillatory integral case, one can show that $K_\phi(\eta)$ has an asymptotic expansion as $\eta \to 0$:

$$K_\phi(\eta) = c_\phi \eta^\epsilon \ln(\eta)^j + o(\eta^\epsilon \ln(\eta)^j)$$

If $\phi(x)$ is nonnegative with $\phi(0) \neq 0$, then as long as the support of $\phi(x)$ is in a sufficiently small neighborhood of the origin, the exponents ϵ and j will be independent of what $\phi(x)$ is.

Furthermore, by reducing to the monomial case via resolution of singularities, one can show that if (ϵ, j) are as above, then as $|\lambda| \to \infty$ the oscillatory integral $I_\phi(\lambda)$ will satisfy

$$|I_\phi(\lambda)| < C|\lambda|^{-\epsilon}(\ln|\lambda|)^j$$

Furthermore, these exponents are generally best possible. They can only be nonoptimal if ϵ is an odd integer.

It is sometimes easier to find bounds for the sublevel set measures since cancellations do not play a role as in the oscillatory integral case.

Furthermore, the relations between sublevel set measures and oscillatory integrals for functions on $\mathbf{R}^n$ generally extend to functions over general local fields of characteristic zero.

Again focusing on the p-adic case, for a ball B in $\mathbf{Q}_p^n$ centered at the origin, we consider

$$L_B(\eta) = |\{x \in B : |f(x)| < \eta\}|$$

Again the measure denotes the product measure induced by the Haar measure on $\mathbf{Q}_p$. It turns out that p-adic resolution of singularities can be used to show that for small enough η there is actually a *finite* expansion of the form

$$L_B(\eta) = \sum_{j=0}^{N} \sum_{k=0}^{n-1} c_{jk} \eta^{\frac{j}{M}} \ln(\eta)^k$$

Here M is a positive integer. This was essentially first observed by Igusa.

Again, the smallest $\frac{j}{K}$ and the smallest k corresponding to this $\frac{j}{K}$ will be independent of the ball chosen if it is sufficiently small, and again this $\frac{j}{K}$ and k implies a corresponding estimate for the oscillatory integral $J_B(y)$ as $|y| \to \infty$, namely

$$|J_B(y)| < C|y|^{-\frac{j}{K}} \ln|y|^k$$

In general it is hard to determine when sharp estimates for the sublevel set measures imply sharp estimates for the oscillatory integrals in this way, however.

In the p-adic case, sublevel set measure estimates are often easier to find than oscillatory integral estimates, for the same reason as in the real case; one doesn't have to deal with the cancellations coming from the oscillations.

The p-adic sublevel set measure estimates have some number-theoretic consequences of note.

Again, let $q(x_1, ..., x_n)$ be a polynomial with integer coefficients. Let $\eta = p^{-m}$.

For any $B' = B_1 \times ... \times B_n$, where each B_i is a p-adic ball of radius p^{-m} contained in the unit ball centered at the origin, there is some ball B'' of radius p^{-m} such that $q(x_1, ..., x_n) \in B''$ for all $(x_1, ..., x_n) \in B'$.

Thus either $|q(x_1, ..., x_n)|$ is less than p^{-m} on all of B' or on none of B'.

Stated another way, whether $|q(x_1, ..., x_n)|$ is less than p^{-m} is determined by the residue classes of $x_1, ..., x_n$ modulo p^m.

Furthermore, $|q(x_1, ..., x_n)| < p^{-m}$ iff $q(x_1, ..., x_n) = 0 \text{ mod } p^m$, where $x_1, ..., x_n$ are viewed as their residue classes modulo p^m.

Consequently, if B denotes $\{(x_1, ..., x_n) : |x_i| < 1 \text{ for all } i\}$, then L_B, defined originally as $|\{x \in B : |q(x)| < p^{-m}\}|$, is equal to

$$\frac{1}{p^{mn}} \#\{(x_1, .., x_n) \in \{0, ..., p^m - 1\}^n : q(x_1, .., x_n) = 0 \ mod\ p^m\}$$

When we view the sublevel set measure estimates in this way, one obtains the following analogue to the exponential sum theorem from before.

Theorem ([G]) Suppose $q(x_1, .., x_n)$ is a polynomial and there are positive integers $j_1, ..., j_n$ such that $q(x_1^{j_1}, ..., x_n^{j_n})$ is homogeneous to any degree. Let d denote the Newton distance of the Newton polyhedron $N(q)$, and let k denote the minimal dimension of any face of $N(q)$ containing $(d, d, ..., d, d)$.

If the the zeroes of each polynomial $q_F(x)$ on $(\mathbf{Q}_p - \{0\})^n$ are of order $< d$, one has the bounds

$$\frac{1}{p^{mn}}\#\{(x_1, .., x_n) \in \{0, ..., p^m - 1\}^n : q(x_1, .., x_n) = 0 \ mod \ p^m\}$$
$$< Cp^{-\frac{m}{d}} m^{n-k-1}$$

If the maximal order is equal to d, one has

$$\frac{1}{p^{mn}}\#\{(x_1, .., x_n) \in \{0, ..., p^m - 1\}^n : q(x_1, .., x_n) = 0 \ mod \ p^m\}$$
$$< Cp^{-\frac{m}{d}} m^{n-k}$$

If the maximal order is equal to some $e > f$, we have the upper bounds

$$\frac{1}{p^{mn}}\#\{(x_1, .., x_n) \in \{0, ..., p^m - 1\}^n : q(x_1, .., x_n) = 0 \ mod \ p^m\}$$
$$< Cp^{-\frac{m}{e}}$$

Unlike in the case of the exponential sums, we have sharpness in the first part of this theorem. In fact, in all three situations the expression is bounded below by $C'p^{-\frac{m}{d}}m^{n-k-1}$ for some C'.

Other fields.

Although we have focused here on sublevel set measures and oscillatory integrals for functions over $\mathbf{R}^n$ or $\mathbf{Q}_p^n$, there are analogous results for any local field K of characteristic zero.

For example consider the case $K = \mathbf{C}$. One can show that the continuous additive characters on $\mathbf{C}$ are all of the form $e^{iRe(wz)}$, where w is some complex number. Thus a natural analogue to the oscillatory integrals considered before is

$$I_\phi(w) = \int_{\mathbf{C}^n} e^{iRe(wz)} \phi(z_1, ..., z_n)$$

The goal is now to obtain optimal estimates of the following form, as $|w| \to \infty$:

$$|I_\phi(w)| < C|w|^{-\epsilon}(\ln|w|)^j$$

Sure enough, if the orders of the zeroes of each $q_F(z)$ are all less than d for example, one obtains the estimate

$$|I_\phi(w)| < C|w|^{-\frac{2}{d}}(\ln|w|)^{n-k-1}$$

We have $-\frac{2}{d}$ instead of $-\frac{1}{d}$ here because $\mathbf{C}$ is a field extension of $\mathbf{R}$ of degree 2.

In a similar vein, if one considers oscillatory integrals and sub-level set measures for functions over a field extension of $\mathbf{Q}_p$ of degree b, one gets theorems analogous to the ones we had for $\mathbf{Q}_p$, except with an exponent of $-\frac{b}{d}$ rather than $-\frac{1}{d}$.