Uniform L^p estimates for local Fourier restriction to curves

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joint work with Detlef Müller

Edinburgh, 8 June 2011

Statement of main result

Curve $\Gamma : I \to \mathbb{R}^d$, $I \subseteq \mathbb{R}$, affine arclength measure $d\sigma$:

$$d\sigma(\phi) = \int_I \phi(\Gamma(t)) |L_\Gamma(t)|^{\frac{2}{d(d+1)}} dt,$$

where

$$L_\Gamma(t) = \det(\Gamma'(t), \ldots, \Gamma^{(d)}(t)).$$

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$$\int_I |\widehat{f}(\Gamma(t))|^q |L_\Gamma(t)|^{\frac{2}{d(d+1)}}\,dt \leq C\|f\|_p^q?$$

for p, q, C uniform over a large class of curves Γ.

Non-degenerate curve $\Gamma(t) = (t, t^2, \dots, t^d)$:

$$p' = \frac{d(d+1)}{2} q, \quad 1 \le p < \frac{d^2+d+2}{d^2+d} \tag{1}$$

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OK $\Gamma(t) = (t^{a_1}, \ldots, t^{a_d})$, $t \geq 0$, restricted p range.

Theorem

Let $\Gamma(t) = (t^{a_1}\theta_1(t), \dots, t^{a_d}\theta_d(t))$, $t \geq 0$, where, for each $1 \leq i \leq d$, $a_i \in \mathbb{R}\backslash\{0\}$, a_i are distinct, θ_i are real-valued and in $C^d(\mathbb{R})$, the limits $\lim_{t\to 0}\theta_i(t)$ exist and are not equal to 0 and for all $1 \leq m \leq d$,

$$\lim_{t\to 0} t^m \theta_i^{(m)}(t) = 0.$$

Then there exist C, only depending on d and p, and δ, depending on the a_i, the θ_i, d and p, such that

$$\int_0^\delta |\widehat{f}(\Gamma(t))|^q |L_\Gamma(t)|^{\frac{2}{d(d+1)}} dt \leq C\|f\|_p^q, \qquad (2)$$

for all Schwartz functions f, where p and q satisfy (1). In the particular case where, for all $1 \leq i \leq d$, the θ_i are identically constant, δ can be taken to be equal to ∞.

Remark 1. Inequality (2) is invariant under reparametrisations of Γ and this is exploited in the proof by using an exponential parametrisation.

Remark 2. Inequality (2) is invariant under replacement of Γ by $A\Gamma$ for any invertible affine transformation A.

Remark 3. Curves of finite type at the origin, i.e. those whose derivatives at the origin do not vanish to infinite order, satisfy the conditions of the theorem with $a_i \in \mathbb{N}$ after a suitable affine transformation.

Remark 4. The global result ($\delta = \infty$), for the particular case where the θ_i are identically constant, follows from the local result by nonisotropic scaling using the dilations $x \mapsto (\lambda^{a_1} x_1, \ldots, \lambda^{a_d} x_d)$.

Offspring curves and geometric inequalities

Use reparametrisation $t \mapsto e^{-t}$. We may assume

$$\Gamma(t) = (\Gamma_1(t), \ldots, \Gamma_d(t)) = (e^{a_1 t}\psi_1(t), \ldots, e^{a_d t}\psi_d(t)), \qquad (3)$$

with new a_i's such that $a_1 < \ldots < a_d$. For each $1 \leq i \leq d$, $\psi_i(\infty) := \lim_{t\to\infty} \psi_i(t)$ exists and is nonzero and $\lim_{t\to\infty} \psi_i^{(m)}(t) = 0$ for all $1 \leq m \leq d$.

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Offspring curves of Γ: $\Gamma_h(t) = \sum_{j=1}^{d} \Gamma(t + h_j)$,

where the parameter $h = (h_1, \ldots, h_d) \in \mathbb{R}^d$ satisfies $0 = h_1 \leq h_2 \leq \ldots \leq h_d$.

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For J_{Φ_Γ} the Jacobian of the mapping

$$\Phi_\Gamma(t_1,\ldots,t_d) = \sum_{i=1}^{d} \Gamma(t_i),$$

we have

$$J_{\Phi_\Gamma}(t_1,\ldots,t_d) \gtrsim \prod_{i=1}^{d} |L_\Gamma(t_i)|^{\frac{1}{d}} \prod_{1\le i<j\le d} (t_j - t_i).$$

Let

$$L_{\Gamma_1\cdots\Gamma_m}(t) = \begin{vmatrix} \Gamma_1'(t) & \cdots & \Gamma_1^{(m)}(t) \\ \vdots & & \vdots \\ \Gamma_m'(t) & \cdots & \Gamma_m^{(m)}(t) \end{vmatrix}$$

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Let $1 \le m \le d$. Then there exists T, depending on d, the a_i, and the ψ_i, such that

$$L_{\Gamma_1\cdots\Gamma_m}(t) \sim e^{t\sum_{i=1}^m a_i} \prod_{i=1}^m (a_i\psi_i(\infty)) \prod_{1\le i<j\le m} (a_j - a_i) \qquad (4)$$

for all $t > T$.

Let Γ_h be an offspring curve of Γ. Then there exists T, depending on d, the a_i, and the ψ_i, such that

$$L_{\Gamma_h}(t) \sim e^{t\sum_{i=1}^{d} a_i} \prod_{i=1}^{d} \left(a_i \psi_i(\infty) \sum_{j=1}^{d} e^{a_i h_j} \right) \prod_{1 \leq i < j \leq d} (a_j - a_i) \quad (5)$$

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for all $t > T$.

1st Geometric Inequality:

Let T be such that the estimates (4) and (5) hold. Then

$$L_{\Gamma_h}(t) \gtrsim \prod_{j=1}^{d} |L_\Gamma(t + h_j)|^{\frac{1}{d}}$$

for all $t > T$.

Proof. *Let $t > T$. Then*

$$\begin{aligned}
L_{\Gamma_h}(t) &\sim e^{t\sum_{i=1}^d a_i} \prod_{i=1}^d \left(a_i \psi_i(\infty) \sum_{j=1}^d e^{a_i h_j} \right) \prod_{1\le i<j\le d} (a_j - a_i) \\
&\gtrsim e^{t\sum_{i=1}^d a_i} \prod_{i=1}^d \left(a_i \psi_i(\infty) e^{\frac{a_i}{d}\sum_{j=1}^d h_j} \right) \prod_{1\le i<j\le d} (a_j - a_i) \\
&= e^{\frac{1}{d}(\sum_{i=1}^d a_i)(\sum_{j=1}^d t+h_j)} \prod_{i=1}^d (a_i \psi_i(\infty)) \prod_{1\le i<j\le d} (a_j - a_i) \\
&\sim \prod_{j=1}^d |L_\Gamma(t+h_j)|^{\frac{1}{d}}.
\end{aligned}$$

Here, we have used (4) and (5), together with the inequality relating the arithmetic and geometric means of the quantities $e^{a_i h_j}$.

Formula expressing J_{Φ_Γ} in terms of the minors of L_Γ: Define inductively multivariate functions I_r, $1 \leq r \leq d$. Each I_r will be defined on $(T, \infty)^r$, with T large enough. For $r = 1$ we set

$$I_1(x) = \frac{L_{\Gamma_1 \ldots \Gamma_{d-2}}(x) L_\Gamma(x)}{L_{\Gamma_1 \ldots \Gamma_{d-1}}(x)^2}$$

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and then inductively

$$I_r(x_1,\ldots,x_r) = \prod_{s=1}^{r} \frac{L_{\Gamma_1\ldots\Gamma_{d-r-1}}(x_s) L_{\Gamma_1\ldots\Gamma_{d-r+1}}(x_s)}{L_{\Gamma_1\ldots\Gamma_{d-r}}(x_s)^2} \int_{x_1}^{x_2} dy_1 \ldots \int_{x_{r-1}}^{x_r} dy_{r-1} I_{r-1}(y_1,\ldots,y_{r-1}),$$

where $L_{\Gamma_1\ldots\Gamma_r} \equiv 1$ for $r=-1,0$. We have

$$J_{\Phi_\Gamma}(t_1,\ldots,t_d) = I_d(t_1,\ldots,t_d)$$

Let $x_1 < \ldots < x_{k-1} < w < z < x_{k+1} < \ldots < x_l$ and $0 < \eta < 1$. Then

$$\int_w^z e^{x_k} \prod_{1 \le i < j \le l} (x_j - x_i) dx_k \ge C_\eta e^{w+\eta(z-w)} \int_w^z \prod_{1 \le i < j \le l} (x_j - x_i) dx_k.$$

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This implies for $t_1 < \ldots < t_l$ and $a > 0$:

$$\int_{t_1}^{t_2} dx_1 \ldots \int_{t_{l-1}}^{t_l} dx_{l-1} e^{a \sum_{i=1}^{l-1} x_i} \prod_{1 \le i < j \le l-1} (x_j - x_i)$$
$$\gtrsim e^{\frac{a(l-1)}{l} \sum_{i=1}^{l} t_i} \prod_{1 \le i < j \le l} (t_j - t_i).$$

2nd Geometric Inequality:

Suppose $T \in \mathbb{R}$ is large enough, so that (4) holds. Then

$$J_{\Phi_\Gamma}(t_1, \dots, t_d) \gtrsim \prod_{i=1}^{d} |L_\Gamma(t_i)|^{\frac{1}{d}} \prod_{1 \le i < j \le d} (t_j - t_i).$$

Higher order offspring curves

The n'th order offspring curves of Γ have the form

$$\Gamma_\eta(t) = (\Gamma_{\eta,1}, \ldots, \Gamma_{\eta,d}) = \sum_{j=1}^{J} \Gamma(t + \eta_j),$$

where $J = d^n$, $\eta = (\eta_1, \ldots, \eta_J) \in \mathbb{R}^J$ and $0 = \eta_1 \leq \eta_2 \leq \ldots \leq \eta_J$.

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where $J = d^n$, $\eta = (\eta_1, \ldots, \eta_J) \in \mathbb{R}^J$ and $0 = \eta_1 \leq \eta_2 \leq \ldots \leq \eta_J$. Due to the exponential parametrisation of the original curve Γ, any offspring curve of Γ, of any order, resembles Γ. In particular,

$$\gamma(t) = (\gamma_1(t), \ldots, \gamma_d(t)) := \Gamma_\eta(t) = \sum_{j=1}^{J} \Gamma(t+\eta_j)$$

satisfies

$$\gamma_i(t) = e^{a_i t} \sum_{j=1}^{J} e^{a_i \eta_j} \psi_i(t+\eta_j), \quad 1 \leq i \leq d.$$

Therefore, if we let $\varphi_i(t) = \sum_{j=1}^{J} e^{a_i\eta_j}\psi_i(t+\eta_j)$, we have

$$\lim_{t\to\infty} \varphi_i(t) = \psi_i(\infty)\sum_{j=1}^{J} e^{a_i\eta_j} =: \varphi_i(\infty)$$

and, for all $1 \leq m \leq d$,

$$\lim_{t\to\infty} \varphi_i^{(m)}(t) = 0.$$

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Define, for $1 \leq m \leq d$ and η as above,

$$L_{\eta,\Gamma_1\dots\Gamma_m}(t) = \begin{vmatrix} \Gamma'_{\eta,1}(t) & \cdots & \Gamma^{(m)}_{\eta,1}(t) \\ \vdots & & \vdots \\ \Gamma'_{\eta,m}(t) & \cdots & \Gamma^{(m)}_{\eta,m}(t) \end{vmatrix}.$$

Let Γ_η be an n'th generation offspring curve of Γ, $J = d^n$, and $1 \leq m \leq d$. Then, there exists T, depending on d, the a_i, and the ψ_i, such that

$$L_{\eta,\Gamma_1\ldots\Gamma_m}(t) \sim e^{t\sum_{i=1}^m a_i} \prod_{i=1}^m \left(a_i\psi_i(\infty) \sum_{j=1}^J e^{a_i\eta_j} \right) \prod_{1\leq i<j\leq m} (a_j - a_i) \tag{6}$$

for all $t > T$.

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for all $t > T$.

Note that we may pick the same T for all offspring curves of Γ of all orders.

Proof of main theorem

Define

$$d\sigma_{T,R}(\phi) = \int_T^R \phi(\Gamma(t))|L_\Gamma(t)|^{\frac{2}{d(d+1)}}\,dt$$

and

$$d\omega_{T,R}(\phi) = \int_T^R \phi(t)|L_\Gamma(t)|^{\frac{2}{d(d+1)}}\,dt.$$

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We prove the dual inequality

$$\|\widehat{gd\sigma_{T,R}}\|_{L^q(B_r)} \leq C_p\|g\|_{L^p(d\omega_{T,R})},$$

for

$$\frac{1}{p} + \frac{d(d+1)}{2q} = 1, \quad 1 \leq p < \frac{d^2+d+2}{2} =: D,$$

uniformly in R and all balls B_r of radius r and centre at the origin, and a fixed large enough T, to be chosen later.

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Let

$$\mathcal{A}(T,R,r) = \sup_{\Gamma \in \mathcal{K}_T} \left(\sup_{\|g\|_{L^p(d\omega_{T,R})} \le 1} \|\widehat{g d\sigma_{T,R}}\|_{L^q(B_r)} \right).$$

It is easy to see that $\mathcal{A}(T,R,r) < \infty$ for every T sufficiently large, R and r, but it is our goal to prove a bound for $\mathcal{A}(T,R,r)$ which is uniform in T, R, and r.

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- Consider d-fold product $\prod_{i=1}^{d} \widehat{g_i d\sigma_{T,R}}$.

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- Perform complex interpolation.
- Consider D-fold product $\prod_{i=1}^{D} (\psi_i |L_\Gamma|^{-\frac{2}{d(d+1)p}} \widehat{d\sigma_{T,R}})$ and M. Christ's multilinear interpolation to bound

$$\mathcal{A}(T,R,r) \leq C_p.$$