Extremizers and Near-extremizers for the Radon Transform

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A Tale of Three Operators

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Radon Transform

The Radon transform takes a function

$$f : \mathbb{R}^d \to \mathbb{C}$$

and transforms it to a function

$$\mathcal{R}f : G_d \to \mathbb{C}$$

where $G_d =$ Grassmann manifold of all **affine** hyperplanes $\pi \subset \mathbb{R}^d$ by

$$\mathcal{R}(\pi) = \int_\pi f.$$

A Radon Transform Inequality

- Identify Grassmann manifold G_d with $\{(r,\theta) \in \mathbb{R} \times S^{d-1}\}$; $(r,\theta) \leftrightarrow$ hyperplane with normal vector θ, at signed distance r from 0. Essentially a two to one map.
- Equip G_d with measure $d\mu = dr\, d\theta$.
- $\mathcal{R}$ maps $L^1(\mathbb{R}^d)$ to $L^1(G_d)$. (Trivial; Fubini)
- $\mathcal{R}$ maps $L^p(\mathbb{R}^d)$ to $L^q(G_d)$ for

  $$\mathbf{p} = \frac{\mathbf{d+1}}{\mathbf{d}} \text{ and } \mathbf{q} = \mathbf{d+1}.$$

  ([Oberlin-Stein 1981], [Calderón 1983], ...)
- A one-parameter family of inequalities follows from these two. No others are true.
- Therefore: most interesting inequality is $L^{(d+1)/d} \rightarrow L^{d+1}$.

Inverse Problems

Let $\mathbb{A}$ be the optimal constant in the inequality

$$\|\mathcal{R}f\|_{d+1} \leq \mathbb{A}\|f\|_{(d+1)/d}.$$

What is the nature of those functions which **extremize**, or nearly extremize, or extremize it to within a small factor?

In this talk I will

- First review results on various versions of this question, obtained over several years.
- Then focus on a recent result.

Symmetry

- The $L^{(d+1)/d} \rightarrow L^{d+1}$ inequality for $\mathcal{R}$ has an extraordinarily

  $\boxed{\textbf{large group of symmetries}.}$

- Indented,

  $$\frac{\|\mathcal{R}(f \circ \phi)\|_{d+1}}{\|f \circ \phi\|_{(d+1)/d}} = \frac{\|\mathcal{R}(f)\|_{d+1}}{\|f\|_{(d+1)/d}}$$

  for any invertible **affine transformation** ϕ of $\mathbb{R}^d$.
- This high degree of symmetry gives the inequality a special interest.

Quasiextremals

A combinatorial analysis ([C 2006, 2011]) of the inequality yields:

Theorem. [C 2006/2011] If $\|\mathcal{R}\mathbf{1}_E\|_{d+1} \geq \delta\|\mathbf{1}_E\|_{(d+1)/d}$ then

there exists a **convex** set $\mathcal{C}$

which is not large:

$$|\mathcal{C}| \leq |\mathbf{E}|,$$

but which contains a significant fraction of *E*:

$$|\mathcal{C} \cap \mathbf{E}| \geq \mathbf{c}\delta^{\mathbf{C}}|\mathbf{E}|.$$

Here c, C are constants which depend only on the dimension d.

Extremizers Exist

Theorem. There exists a function satisfying

$$\|\mathcal{R}f\|_{d+1} = \mathbb{A}\|f\|_{(d+1)/d}.$$

Precompactness

Theorem. Suppose $\|f_n\|_{(d+1)/d} \to 1$ and $\|\mathcal{R}f_n\|_{d+1} \to \mathbb{A}$. Then

there exist invertible affine transformations $\phi_n \in G$

such that some subsequence of $\{c_n \, f_n \circ \phi_n\}$ **converges**, in $L^{(d+1)/d}$ norm.

Corollary. (**Near-extremizers**) If

$$\|\mathcal{R}f\|_{d+1} \geq (\mathbb{A} - \varepsilon)\|f\|_{(d+1)/d},$$

then f is $\delta(\varepsilon)$–close to some extremizer.

Critical Points and Euler-Lagrange Equation

- Extremizers are critical points of the functional

$$\boldsymbol{\Phi}_{\mathcal{R}}(\mathbf{f}) = \frac{\|\mathcal{R}f\|_{d+1}}{\|f\|_{(d+1)/d}}.$$

- Critical points are characterized by a generalized **Euler-Lagrange equation**:

$$\mathbf{f} = \lambda\Big(\mathcal{R}^*(\mathcal{R}\mathbf{f})^{\mathbf{d}}\Big)^{\mathbf{d}}.$$

- $\lambda = \lambda_f \in \mathbb{R}$ is a Lagrange multiplier (a ratio of powers of norms of $\mathcal{R}f$ and f).
- $\mathcal{R}^*$ is the transpose of $\mathcal{R}$ — a similar operator.

Properties of Extremizers

Analysis of the Euler-Lagrange equation leads to:

Theorem. [C-Xue 2010] All **critical points** f of $\Phi_{\mathcal{R}}$ satisfy

- $f \in C^\infty$
- $f(x) \to 0$ as $|x| \to \infty$.

Theorem. [C, May 2011]

Any critical point f of the functional $\Phi_{\mathcal{R}}$ admits an asymptotic expansion of the form

$$\mathbf{f}(\mathbf{x}) = \sum_{\mathbf{k}=0}^{\infty} \mathbf{g_k}\Big(\frac{\mathbf{x}}{|\mathbf{x}|}\Big)\ |\mathbf{x}|^{-\mathbf{d}-\mathbf{k}} \quad \text{as } |\mathbf{x}| \to \infty$$

with each $g_k \in C^\infty(S^{d-1})$.

Obstructions to Proving the Regularity Theorem

- The Euler-Lagrange equation is nonlinear, and nonlocal.
- Because the inequality has a scaling symmetry, the right-hand side of the equation, $\left(\mathcal{R}^*(\mathcal{R}f)^d\right)^d$, is in general no smoother than f itself.
- Therefore **one cannot prove the regularity theorem by a direct "bootstrapping" argument.**
- It is not true that arbitrary solutions of the Euler-Lagrange equation are C^∞; **there are nonsmooth solutions in weak $L^{(d+1)/d}$**.
- The proof will not be discussed today due to lack of time; [C-Xue] use weighted norm inequalities for a singular convolution operator.

Identification of Extremizers

I turn now to the main topic of today's lecture, the identification of extremizers.

Very few nontrivial inequalities have known extremizers

The most relevant:

- Young's convolution inequality (Beckner [1975]; Lieb [1976]):

  $$\iint_{\mathbb{R}^{d+d}} \mathbf{f(x)g(y)h(x-y)\,dx\,dy} \le \mathbf{A_{p,q,r}} \|\mathbf{f}\|_{\mathbf{p}} \|\mathbf{g}\|_{\mathbf{q}} \|\mathbf{h}\|_{\mathbf{r}}$$

  if $p^{-1} + q^{-1} + r^{-1} = 2$. Extremizers are certain triples (f, g, h) of (not necessarily radial) **Gaussians** $e^{-Q(x)}$.

- Hardy-Littlewood-Sobolev inequality (Lieb [1983]):

  $$\iint_{\mathbb{R}^{d+d}} \mathbf{f(x)g(y)|x-y|^{-d+\gamma}\,dx\,dy} \le \mathbf{A_{p,q,\gamma}} \|\mathbf{f}\|_{\mathbf{p}} \|\mathbf{g}\|_{\mathbf{q}}$$

  if $q = p$ and $2p^{-1} = 1 + \gamma d^{-1}$. Extremizers

  $\mathbf{F(x) = (1 + |x|^2)^{-d/p}}$ and $F \circ \phi$ for all $\phi(x) = a + rx$, $0 \neq r \in \mathbb{R}$.

Today's Main Result

Theorem.

- The function
$$\mathbf{F(x) = (1 + |x|^2)^{-d/2}}$$
is an extremizer for the Radon transform inequality.
- Each extremizer is of the form
$$\mathbf{c\,F} \circ \phi$$
for some invertible affine transformation ϕ of $\mathbb{R}^d$.

- That $(1 + |x|^2)^{-d/2}$ is an extremizer, had been conjectured by Baernstein and Loss [1997].
- They proved this for dimension $d = 3$ (and treated the 2-plane transform for all d).

Main Ideas

- **Symmetry.** Inequality has a very high degree of symmetry.
- **Symmetrization.**
  Replacing f by its radially symmetric decreasing rearrangement, never decreases the functional $\Phi_{\mathcal{R}}(f) = \|\mathcal{R}f\|_{d+1}/\|f\|_{(d+1)/d}$.
- **Inverse symmetrization.**
  If symmetrizing f does not strictly increase $\Phi_{\mathcal{R}}(f)$, then f must be appropriately symmetric.
- **Identifying symmetry.**
  One must be able to recognize functions with "ellipsoidal" symmetry; $f = F \circ \phi$, where F is radial and ϕ is affine.
- **Additional symmetry.**
  The inequality has additional symmetries, beyond the affine group.

The Magic Symmetry

- **Proposition.** **A non-affine symmetry.** If f is an extremizer, then so is
$$g(u,s) = |\mathbf{s}|^{-\mathbf{d}}\,\mathbf{f}(\frac{\mathbf{u}}{\mathbf{s}}, \frac{\mathbf{1}}{\mathbf{s}})$$
with coordinates $(u,s) \in \mathbb{R}^{d-1} \times \mathbb{R}^1$.
- In fact,
$$\|g\|_{(d+1)/d} = \|f\|_{(d+1)/d}$$
and
$$\|\mathcal{R}g\|_{d+1} = \|\mathcal{R}f\|_{d+1}.$$

Second and Third Main Steps

- **Proposition.** Any extremizer is “ellipsoidal”, that is, of the form $F \circ \phi$, for some **radial** function F and some **affine** ϕ.
- **Proposition.** If

  $$f \text{ is radial} \quad \text{and} \quad |s|^{-d} f(\frac{u}{s}, \frac{1}{s}) \text{ is “ellipsoidal”}$$

  then

  $$f(x) = c(1 + r|x|^2)^{-d/2} \ \text{ for some constants } c, r.$$
- The general scheme is along the lines introduced by **Lieb** in his work on the Hardy-Littlewood-Sobolev inequality.

A Convolution Operator

- $T : L^{(d+1)/d}(\mathbb{R}^d) \to L^{d+1}(\mathbb{R}^d)$:

  $$Tf(x) = f * \sigma(x) = \int f(x-y)\, d\sigma(y)$$

  where σ is the "affine surface measure" $\mathbf{dy'}$ on the parabola

  $$\mathbf{y_d} = \tfrac{1}{2}|\mathbf{y'}|^2 \quad \text{with } \mathbf{y} = (\mathbf{y'}, \mathbf{y_d}) \in \mathbb{R}^{\mathbf{d-1}} \times \mathbb{R}^{\mathbf{1}}.$$

- T is a canonical representative of the large class of operators defined by convolution with generalized surface measures on submanifolds.
- All of the theorems about quasiextremals, existence of extremizers, precompactness of extremizing sequences, and regularity of critical points of $\Phi_{\mathcal{R}}$, were **originally proved for T**.

A Third Operator

- $\mathcal{R}^\sharp : L^{(d+1)/d}(\mathbb{R}^d) \to L^{d+1}(\mathbb{R}^d)$:

$$\mathcal{R}^\sharp f(x) = \int_{\mathbb{R}^{d-1}} f(y', x_d + x' \cdot y')\, dy'.$$

- $\mathcal{R}^\sharp$ is essentially the Radon transform $\mathcal{R}$, but with different normalizations in two different respects.
- $\mathcal{R}^\sharp$ is connected with the Heisenberg group $\mathbb{H}^{2d-1}$.

A rose by any other name . . .

- Fact: For all functions f,

$$\|\mathcal{R}f\|_{d+1} = \|Tf\|_{d+1} = \|\mathcal{R}^\sharp f\|_{d+1}.$$

- The three $\boxed{\text{operators}}$ are not identical, but the $\boxed{\text{inequalities}}$ are!
- These identities rely on **(nonaffine)** changes of coordinates, Jacobian factors, and the particular exponents $p = \frac{d+1}{d}$ and $q = d + 1$.
- $\mathcal{R}^\sharp$ is the bridge between T and $\mathcal{R}$. The proof of today's theorem relies on a symmetry which was discovered for $\mathcal{R}^\sharp$, rather than for $\mathcal{R}$.

- All theorems in this talk have versions for all three operators.
- Among the extremizers for T are all functions

$$c\left(1 + a|\mathbf{x'}|^2 + b\left|\,\mathbf{x_d} - \tfrac{\mathbf{1}}{\mathbf{2}}|\mathbf{x'}|^{\mathbf{2}}\,\right|^2\right)^{-d/2}$$

with coordinates $(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}^1$.

An Identity (Drury [1983,1984]; C [2011])

$$\|\mathcal{R}f\|_{d+1}^{d+1} = \int_{(\mathbb{R}^{\mathbf{d-1}})^{\mathbf{d+1}}} \frac{\mathbf{I}(\mathbf{x}'_{\mathbf{0}},\cdots,\mathbf{x}'_{\mathbf{d}})}{\boldsymbol{\Delta}_{\mathbf{d-1}}(\mathbf{x}'_{\mathbf{1}},\cdots,\mathbf{x}'_{\mathbf{d}})} \prod_{j=0}^{d} dx'_j$$

where $x_j = (x'_j, t_j) \in \mathbb{R}^{d-1} \times \mathbb{R}^1$, where the inner integral is

$$I(x'_0,\cdots,x'_d) = \int_{\mathbb{R}^d} \mathbf{f}(\mathbf{x}'_{\mathbf{0}}, \mathbf{v}(\mathbf{x}') \cdot \mathbf{t}) \prod_{\mathbf{j=1}}^{\mathbf{d}} \mathbf{f}(\mathbf{x}'_{\mathbf{j}}, \mathbf{t}_{\mathbf{j}})\, \mathbf{dt}_{\mathbf{j}}$$

with $t = (t_1,\cdots,t_d)$,
$\Delta_{d-1} =$ **volume of** $d-1$**–simplex in** $\mathbb{R}^{d-1}$ with indicated vertices,
$v(x') = v(x'_1,\cdots,x'_d) =$ unique vector such that

$(x'_0, v(x') \cdot t)$ **is coplanar with** (x'_j, t_j) for all $j \in \{1,\cdots,d\}$.

Radial Nonincreasing Rearrangement

If $f : \mathbb{R}^d \to [0, \infty)$ is finite a.e. then there exists essentially unique $f^* : \mathbb{R}^d \to [0, \infty)$ satisfying:

- **Radial**
- **Nonincreasing**: $r > r' \Rightarrow f^*(r) \leq f^*(r')$
- **Equimeasurable** with f:

$$\left|\{x : f^*(x) > \lambda\}\right| = \left|\{x : f(x) > \lambda\}\right|$$

for all λ.

Rearrangement Inequalities

- Riesz [1930]; Sobolev [1936]

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} f(x)g(y)h(x+y)\,dx\,dy \leq \iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathbf{f}^*(x)\mathbf{g}^*(y)\mathbf{h}^*(x+y)\,dx\,dy.$$

- Brascamp-Lieb-Luttinger [1974]

$$\int_{\mathbb{R}^m}\prod_j \mathbf{f_j}(L_j(x))\,dx \leq \int_{\mathbb{R}^m}\prod_j \mathbf{f_j}^*(L_j(x))\,dx$$

for any linear mappings $L_j : \mathbb{R}^m \to \mathbb{R}^1$.

Rearrangement and the Radon Transform [C 1984]

- The inner integrals in the Drury/C identity are of the Brascamp-Lieb-Luttinger form.
- **Consequence:** Let $f^{\star\star}(x', x_d)$ be nondecreasing rearrangement of $f(x', x_d)$ **with respect to the x_d variable for each x'**. Then

$$\|\mathcal{R}\mathbf{f}^{\star\star}\|_{\mathbf{d+1}} \geq \|\mathcal{R}\mathbf{f}\|_{\mathbf{d+1}}.$$

- Doing this repeatedly with respect to a dense set of directions in $\mathbb{R}^d$ and extracting limit of a subsequence, one obtains for the **radial** decreasing rearrangement f^*:

$$\frac{\|\mathcal{R}f^*\|_{d+1}}{\|f^*\|_{(d+1)/d}} \geq \frac{\|\mathcal{R}f\|_{d+1}}{\|f\|_{(d+1)/d}}.$$

(Justification: Brascamp-Lieb-Luttinger [1974]; Carlen-Loss [1990])

Consequence

Corollary. $\boxed{\text{There exist radial extremizers}}$ for our Radon transform inequality.

Proof:

- Extremizers are already known to exist.
- The radial nonincreasing rearrangement of any extremizer is an extremizer.

Burchard's Inverse Theorem [1996]

$\boxed{\text{If}}$

$$\int_{\mathbb{R}^m} \prod_{j=0}^{m} f_j(L_j(x))\,dx \boxed{=} \int_{\mathbb{R}^m} \prod_{j=0}^{m} f_j^*(L_j(x))\,dx$$

$\boxed{\text{then}}$ there exist c_j such that

$$\mathbf{f_j(x) \equiv f_j^*(x - c_j)}$$

and moreover $\{c_j\}$ **are compatible** in the sense that there exists $v \in \mathbb{R}^m$ such that

$$c_j = L_j(v) \text{ for all } j \in \{0, \cdots, m\} \quad \text{—}$$

provided that all level sets of all f_j are null sets: For all λ,

$$\big|\{x : f_j(x) = \lambda\}\big| = 0.$$

Scholium

- Burchard's theorem, as stated, does not apply directly, because it requires that level sets be null sets, which is not obvious here.
- I believe that it is possible to refine the analysis of [C-Xue 2010] to show that all critical points of $\Phi_{\mathcal{R}}$ are real analytic. This implies the null level sets condition.
- Burchard proved a more fundamental inverse theorem, characterizing cases of equality in Young's inequality, when all functions are characteristic functions of sets. When the measures of these sets satisfy certain inequalities, the same characterization of equality holds.
- A tedious direct argument supplements this inverse theorem, allowing one to draw the desired conclusion in our application to the Radon transform. I will spare you all these details.

Upshot for Radon Transform

- **Proposition.** If f is an extremizer for the Radon transform inequality then for each unit vector $v \in \mathbb{R}^d$, there exists a **skew reflection** R_v in the direction v such that

$$f \equiv f \circ R_v.$$

- By a skew reflection R_v I mean an **affine involution** such that $R_v(x) - x$ **is parallel to** v for every x.
- The discussion so far provides a skew reflection R_v for every direction v, but no control over their relationship to one another.
- Burchard faced this same point in extending the inverse theorem to higher dimensions.

By their fruits ye shall know them

How to recognize functions with ellipsoidal symmetry?

- Consider $G =$ **group of all affine symmetries of** f. This includes the group generated by all the skew reflections of the preceding slide.
- G must be **compact**.
-

  Hail Group Theory!

  Fact. Any compact subgroup of the affine group, is conjugate (by an element of the affine group) to a subgroup of the orthogonal group $O(d)$.
- Therefore after an affine change of variables, any extremizer f is invariant with respect to (orthogonal) reflection about every codimension one subspace of $\mathbb{R}^d$. Thus f **has ellipsoidal symmetry.**

Step 3

Proposition. If f is radial and $|s|^{-d} f(s^{-1}u, s^{-1})$ is "ellipsoidal" then $f(x) = c(1 + r|x|^2)^{-d/2}$ for some constants c, r.
Proof for $\mathbb{R}^2$ case:

- $$s^{-2} f(s^{-1}t, s^{-1}) = h((s-a)^2 + \lambda t^2)$$

  for some unknown function h and numbers a, λ. Writing $f(x) = g(|x|^2)$, get

  $$g(\frac{t^2+1}{s^2}) = s^2 h((s-a)^2 + \lambda t^2).$$

  Here g, h, a, λ **are all unknown!**
- The vector field $V = (1+t^2)\partial_t + st\partial_s$ annihilates $\frac{t^2+1}{s^2}$, and therefore annihilates the left-hand side.
- Apply V to both sides.

- Algebra ⇒
$$-\frac{h(\psi)}{h'(\psi)} = (1+t^2) + s^2 - sa.$$
where $\psi = (\mathbf{s}-\mathbf{a})^2 + \lambda \mathbf{t}^2$.
- Left-hand side is a function of ψ alone, so
$$(1+t^2) + s^2 - sa \textbf{ is a function of } (s-a)^2 + \lambda t^2\textbf{.}$$
- Since these have same coefficient of s^2, **they are equal.**
- Therefore
$$\frac{h(\psi)}{h'(\psi)} \equiv -\psi.$$
- Solve ODE
$$\frac{d}{d\psi} \log h(\psi) = -\frac{1}{\psi}$$
to finish.