

Multilinear Kakeya, Factorisation and Algebraic Topology

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Loomis–Whitney inequality

$$\int_{\mathbb{R}^n} \prod_{j=1}^n f_j(\Pi_j x) dx \leq \prod_{j=1}^n \|f_j\|_{L^{n-1}(\mathbb{R}^{n-1})};$$

Let $F_j = f_j^{n-1}$; then this becomes

$$\int_{\mathbb{R}^n} \prod_{j=1}^n F_j(\Pi_j x)^{1/(n-1)} dx \leq \prod_{j=1}^n \left(\int_{\mathbb{R}^{n-1}} F_j \right)^{1/(n-1)}.$$

Take $F_j(x) = \sum_i \alpha_i^j \chi_{B_i}(x)$ where B_i are unit balls in \mathbb{R}^{n-1} ; then we have

$$\int_{\mathbb{R}^n} \prod_{j=1}^n \left(\sum_i \alpha_i^j \chi_{T_i^j}(x) \right)^{1/(n-1)} dx \leq \prod_{j=1}^n \left(\sum_i \alpha_i^j \right)^{1/(n-1)}$$

where T_i^j are doubly-infinite tubes in \mathbb{R}^n parallel to e_j with unit cross-section. Equiv. to L–W by density of step functions and scaling.

Directional perturbation

Loomis–Whitney:

$$\int_{\mathbb{R}^n} \prod_{j=1}^n \left(\sum_i \alpha_i^j \chi_{T_i^j}(x) \right)^{1/(n-1)} dx \leq \prod_{j=1}^n \left(\sum_i \alpha_i^j \right)^{1/(n-1)}$$

where T_i^j are doubly-infinite tubes in \mathbb{R}^n **parallel** to e_j with unit cross-section.

What if we relax the condition that the tubes be **exactly parallel** to e_j ?

If we take an invertible linear transformation $A \in GL(n, \mathbb{R})$ and replace the role of e_j by Ae_j then we are okay with a factor of $(\det A)^{-1/(n-1)}$ included on RHS – affine invariant Loomis–Whitney inequality.

But we are interested in the situation where **for each fixed j we allow the tubes $\{T_i^j\}_i$ to have different directions for different i** – within reason of course.

Multilinear Kakeya

So, for $1 \leq j \leq n$, let \mathcal{T}_j be a family of tubes whose directions are close (say within 10°) to e_j . Then do we have

$$\int_{\mathbb{R}^n} \prod_{j=1}^n \left(\sum_{T_j \in \mathcal{T}_j} \alpha_{T_j} \chi_{T_j}(x) \right)^{1/(n-1)} dx \leq C_n \prod_{j=1}^n \left(\sum_{T_j \in \mathcal{T}_j} \alpha_{T_j} \right)^{1/(n-1)} ?$$

This is the Multilinear Kakeya problem, so called because it also represents the contribution to the Kakeya maximal problem coming from transverse intersections of tubes.

Multilinear Keakeya theorem

\mathcal{T}_j is the set of doubly-infinite tubes in \mathbb{R}^n of cross-sectional diameter 1 whose direction is within 10° of the unit vector e_j .

Theorem (Multilinear Keakeya)

If $p \geq 1/(n-1)$ then

$$\int \prod_{j=1}^n \left(\sum_{T_j \in \mathcal{T}_j} c_{T_j} \chi_{T_j}(x) \right)^p dx \leq C_{p,n} \prod_{j=1}^n \left(\sum_{T_j \in \mathcal{T}_j} c_{T_j} \right)^p.$$

This is obvious when $n = 2$ and $p = 1$.

This theorem is due to Jon Bennett, Terry Tao and AC in the case $p > 1/(n-1)$, and to Guth in the endpoint case $p = 1/(n-1)$. The two methods of proof are very different.

Multilinear Restriction

Let $\Sigma_1, \dots, \Sigma_n$ be transverse hypersurfaces in \mathbb{R}^n with induced Lebesgue measures $d\sigma_1, \dots, d\sigma_n$ respectively. Let

$$g \mapsto \widehat{gd\sigma_j} = \int_{\Sigma_j} g(\xi) e^{-2\pi i x \cdot \xi} d\sigma_j(\xi)$$

be the extension operator for Σ_j .

Theorem (BCT)

For each $q \geq 2/(n-1)$ and $\epsilon > 0$ we have

$$\int_{B(0,R)} \prod_{j=1}^n |\widehat{g_j d\sigma_j}(x)|^q dx \leq C_{n,q,\epsilon} R^\epsilon \prod_{j=1}^n \|g_j\|_2^q.$$

Note: no curvature assumptions on the Σ_j .

Endpoint problem (i.e. $\epsilon = 0$) still open.

Linear Restriction

Theorem (J. Bourgain and L. Guth, 2010)

For the unit sphere S^{n-1} in \mathbb{R}^n we have

$$\|\widehat{gd\sigma}\|_{L^q(\mathbb{R}^n)} \leq C\|g\|_{L^q(S^{n-1})}$$

provided that

$$q > \frac{2n+1}{n-1}.$$

Guth's endpoint theorem

Guth's argument relies on heavy-duty algebraic geometry and algebraic topology.

Some of the tools needed:

- \mathbb{Z}_2 -cohomology
- Covering spaces
- Cup products
- Lusternik–Schnirelmann theory
- Commutative diagrams and long exact sequences

Main point: to try to understand Guth's argument in terms accessible to an analyst.

Abstract formulation

Consider, for $p \geq 1/(n-1)$,

$$\int_{\mathbb{R}^n} \prod_{j=1}^n \left(\sum_{T_j \in \mathcal{T}_j} c_{T_j} \chi_{T_j}(x) \right)^p dx \leq C_{p,n} \prod_{j=1}^n \left(\sum_{T_j \in \mathcal{T}_j} c_{T_j} \right)^p.$$

This has a more abstract formulation as:

$$\int_{\mathbb{R}^n} \prod_{j=1}^n U_j f_j(x)^p dx \leq C_{p,n} \prod_{j=1}^n \left(\int_{Y_j} f_j \right)^p$$

where Y_j is a discrete space and U_j is a linear map taking positive sequences defined on Y_j to positive functions defined on \mathbb{R}^n . Slightly more generally still, consider for linear U_j , $r_j > 0$, $r = \sum_j r_j \geq 1$ and $p_j \geq 1$ ($p_j = 1, r_j = p \geq 1/(n-1), r \geq n/(n-1) \geq 1$) the inequality

$$\int_X \prod_{j=1}^n |U_j f_j(x)|^{r_j} d\mu(x) \leq A^r \prod_{j=1}^n \|f_j\|_{L^{p_j}(Y_j)}^{r_j}.$$

Multilinear duality I

Proposition (\sim Guth)

Let $X, Y_j, 1 \leq j \leq n$ be measure spaces, let $U_j : \mathcal{M}(Y_j) \rightarrow \mathcal{M}(X)$ be linear. Let $r_j > 0, r = \sum_j r_j \geq 1$ and $p_j \geq 1$. Suppose that for every $M \in L^{r'}(X), M \geq 0$, there exist $S_1(x), \dots, S_n(x)$ such that

$$M(x) \leq \prod_{j=1}^n S_j(x)^{r_j/r} \text{ a.e.},$$

and, for all \tilde{S}_j with $|\tilde{S}_j(x)| = |S_j(x)|$ a.e.,

$$\|U_j^* \tilde{S}_j\|_{p_j'} \leq A \|M\|_{r'}.$$

Then

$$\int_X \prod_{j=1}^n |U_j f_j(x)|^{r_j} d\mu(x) \leq A^r \prod_{j=1}^n \|f_j\|_{L^{p_j}(Y_j)}^{r_j}.$$

Proof.

We want to examine $\int_X \prod_{j=1}^n |U_j f_j(x)|^{r_j} d\mu(x)$ which is the r 'th power of the L^r norm of $\prod_{j=1}^n |U_j f_j(x)|^{r_j/r}$.

Testing against a function M with unit norm in $L^{r'}$ we get

$$\begin{aligned} \int \prod_{j=1}^n |U_j f_j(x)|^{r_j/r} M(x) d\mu(x) &\leq \int \prod_{j=1}^n |U_j f_j(x)|^{r_j/r} S_j(x)^{r_j/r} d\mu(x) \\ &\leq \prod_{j=1}^n \left(\int S_j(x) |U_j f_j(x)| dx \right)^{r_j/r} = \prod_{j=1}^n \left(\int U_j^* \tilde{S}_j(x) f_j(x) dx \right)^{r_j/r} \\ &\leq \prod_{j=1}^n \left(\|U_j^* \tilde{S}_j\|_{p_j'} \|f_j\|_{p_j} \right)^{r_j/r} \leq \prod_{j=1}^n (A \|M\|_{r'} \|f_j\|_{p_j})^{r_j/r} \leq A \prod_{j=1}^n \|f_j\|_{p_j}^{r_j/r}. \end{aligned}$$



Multilinear duality II

Theorem (AC and S. Valdimarsson)

Suppose that X and Y_j are measure spaces satisfying reasonable conditions, U_j is linear taking positive functions on Y_j to positive functions on X , and that $r_j > 0$, $r = \sum_j r_j \geq 1$ and $p_j \geq 1$. If

$$\int_X \prod_{j=1}^n U_j f_j(x)^{r_j} d\mu(x) \leq A^r \prod_{j=1}^n \|f_j\|_{L^{p_j}(Y_j)}^{r_j}$$

then, $\forall 0 \leq M \in L^{r'}(X)$, $\forall \epsilon > 0$, $\exists S_1(x), \dots, S_n(x)$ such that

$$M(x) \leq \prod_{j=1}^n S_j(x)^{r_j/r} \quad \text{a.e.}$$

and

$$\|U_j^* S_j\|_{p_j'} \leq (A + \epsilon) \|M\|_{r'}.$$

This does not seem to be a standard result of functional analysis, but instead comes about as a result of **duality methods in the theory of convex optimisation**. These ultimately rely upon some form of the Ky Fan minimax Theorem, which itself is closely related to the **Brouwer fixed point theorem**. The proof is highly non-constructive.

Example. Let $(T_1 f)(x_1, x_2) = f(x_1)$ and $(T_2 g)(x_1, x_2) = g(x_2)$; then

$$\int_{\mathbb{R}^2} T_1 f(x) T_2 g(x) dx = \int_{\mathbb{R}^2} f(x_1) g(x_2) dx = \int_{\mathbb{R}} f \int_{\mathbb{R}} g.$$

So with $n = 2$, $r_1 = r_2 = p_1 = p_2 = 1$ the duality principle gives that for every $M \geq 0$ in $L^2(\mathbb{R}^2)$ we can factorise it as $M = (GH)^{1/2}$ such that

$$\sup_x \int G(x, y) dy \leq \|M\|_2$$

and

$$\sup_y \int H(x, y) dx \leq \|M\|_2.$$

Loomis–Whitney factorisation

Any multilinear inequality of the appropriate form gives a corresponding factorisation result. Corresponding to the Loomis–Whitney inequality

$$\int_{\mathbb{R}^n} \prod_{j=1}^n f_j(\Pi_j x) dx \leq \prod_{j=1}^n \|f_j\|_{L^{n-1}(\mathbb{R}^{n-1})} :$$

Proposition

Let $1 < p \leq n$ and let $\frac{1}{q} = \frac{1}{n-1} \left(\frac{n}{p} - 1 \right)$. Then for all $M \in L^p(\mathbb{R}^n)$ there exist S_1, \dots, S_n such that $M(x) \leq \prod_{j=1}^n S_j(x)^{1/n}$ a.e. and such that if $V_j = \{e_j\}^\perp$ we have

$$\left\| \int_{V_j} S_j(x) dx' \right\|_{L^q(dx_j)} \leq \|M\|_{L^p(\mathbb{R}^n)}.$$

(Factorisation result for any one p is equivalent to that for any other.)

Brascamp–Lieb factorisation

More generally, if $B_j : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are self-adjoint projections onto subspaces V_j of \mathbb{R}^n and $p_j > 0$ are such that

$$\sum_j p_j B_j = I_n$$

then with $p = \sum_j p_j \geq 1$, for all $M \in L^{p'}$ there exist S_j with

$$M(x) \leq \prod_{j=1}^n S_j(x)^{p_j/p} \quad \text{a.e.}$$

and

$$\sup_{x'' \in V_j^\perp} \int_{V_j} S_j(x) dx' \leq \|M\|_{L^{p'}(\mathbb{R}^n)}.$$

This follows from the geometric Brascamp–Lieb inequality of Ball and Barthe and in particular applies to Beckner's sharp Young convolution inequality.

Guth's reduction

Using the (easy half of) the duality principle, plus the fact that 1 is the smallest scale occurring in Multilinear Keakeya, Guth reduced the endpoint Multilinear Keakeya result to:

Given a nonnegative function $M(Q)$ defined on the lattice of unit cubes in \mathbb{R}^n , there exist functions $S_j(Q)$, $1 \leq j \leq n$, such that

$$M(Q) \leq C_n \prod_{j=1}^n S_j(Q)^{1/n} \text{ for all } Q \in \text{supp } M.$$

and, for each fixed j and $T \in \mathcal{T}_j$

$$\sum_{Q: Q \cap T \neq \emptyset} S_j(Q) \leq C_n \left(\sum_Q M(Q)^n \right)^{1/n}.$$

Guth's construction

We need S_j such that :

$$M(Q) \leq C_n \prod_{j=1}^n S_j(Q)^{1/n} \text{ for all } Q \in \text{supp } M. \quad (1)$$

and, for each fixed j and $T \in \mathcal{T}_j$

$$\sum_{Q: Q \cap T \neq \emptyset} S_j(Q) \leq C_n \left(\sum_Q M(Q)^n \right)^{1/n}. \quad (2)$$

Guth does this (essentially) by constructing an algebraic hypersurface $Z = \{p(x) = 0\}$ with p of degree at most $C_n (\sum_Q M(Q)^n)^{1/n}$ à la Dvir, and takes $S_j(Q)$ to be the smallest component of surface area of $Z \cap Q$ in directions occurring amongst \mathcal{T}_j . Point (2) will then automatically be satisfied.

Directional surface area

For a hypersurface Z and a unit vector $e \in \mathbb{R}^n$ we define the **component of surface area of Z in the direction e** or the **directional surface area of Z in the direction e** as:

$$\text{surf}_e(Z) = \int_Z |e \cdot n(x)| dS(x)$$

where $n(x)$ is the unit normal at x and dS is the element of $(n - 1)$ -dimensional Hausdorff measure on Z .

If Z is given by the graph of a function $\Gamma : D \subseteq \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ above the hyperplane $x_n = 0$, then its the directional surface area in the direction e_n is simply the $(n - 1)$ -dimensional area of D .

If Z is given by disjoint graphs of functions above the hyperplane $x_n = 0$ then its directional surface area in the direction e_n is just $\int_{\mathbb{R}^{n-1}} J(y) dy$ where $J(y)$ is the number of times the line through y parallel to e_n passes through Z .

Point (2)

Indeed, for j fixed and $T \in \mathcal{T}_j$ we have

$$\sum_{Q: Q \cap T \neq \emptyset} S_j(Q) \leq \sum_{Q: Q \cap T \neq \emptyset} \text{surf}_{e(T)}(Z \cap Q)$$

(where $e(T)$ is the direction of T), and this essentially

$$\text{surf}_{e(T)}(Z \cap T) \sim \int_{D(T)} \#(Z \cap l(\zeta)) d\zeta$$

where $D(T)$ is a unit disc perpendicular to $e(T)$, and the $l(\zeta)$ are lines parallel to $e(T)$ indexed by ζ .

But for almost every ζ ,

$$\#(Z \cap l(\zeta)) \leq \deg p \leq C_n \left(\sum_Q M(Q)^n \right)^{1/n}$$

and so we're done.

Guth's construction, cont'd

So, given $M(Q)$, we are left with finding a polynomial p of degree at most $C_n (\sum_Q M(Q)^n)^{1/n}$, with zero set Z , such that

$$M(Q) \leq C_n \prod_{j=1}^n S_j(Q)^{1/n} \text{ for all } Q \in \text{supp } M$$

where $S_j(Q)$ is the component of surface area of $Z \cap Q$ in some direction close to e_j .

In fact a moment's consideration shows that there may be some technical problems in achieving this: take $M = \chi_V$ with V a finite subset of \mathcal{Q} ; if Z has normal direction e_j at Q then $S_k(Q)$ could be essentially zero for $k \neq j$. So we should perhaps only be asking for p to have degree $O(\lambda (\sum_Q M(Q)^n)^{1/n})$ and then define $S_j(Q)$ to be λ^{-1} times its previous definition for a suitable sufficiently large scaling constant λ . This is what Guth actually does.

An easier version

By the G.M./A.M. inequality we have

$$\prod_{j=1}^n S_j(Q)^{1/n} \leq C_n \sum_{j=1}^n S_j(Q) \sim \mathcal{H}_{n-1}(Z \cap Q)$$

because the directions involved in the S_j are approximately orthonormal.

So an *easier* task is, given M , to find a polynomial p of degree at most $C_n (\sum_Q M(Q)^n)^{1/n}$, with zero set Z , such that

$$M(Q) \leq C_n \mathcal{H}_{n-1}(Z \cap Q) \text{ for all } Q \in \text{supp } M.$$

This is not too difficult...

The easier task...

Given $M(Q)$. Then $\exists p$ with $\deg p \leq C_n (\sum_Q M(Q)^n)^{1/n}$ and zero set Z such that $\mathcal{H}^{n-1}(Z \cap Q) \geq C_n M(Q) \forall Q$.

First chop up each Q into $M(Q)^n$ equal subcubes. So altogether we have $\sum_Q M(Q)^n$ cubes S of various sizes. Consider a map

$$F : p \mapsto \left\{ \int_{\{p>0\} \cap S} 1 - \int_{\{p<0\} \cap S} 1 \right\}_S$$

defined on the vector space of polynomials of degree $\leq d$ in n real variables, with dimension $\sim d^n$.

Clearly F is continuous, homogeneous of degree 0 and odd.

So we can think of F as

$$F : \mathbb{S}^k \rightarrow \mathbb{R}^m$$

where $k \sim d^n$ and $m = \sum_Q M(Q)^n$.

The easier task, cont'd

Recall that $F : p \mapsto \left\{ \int_{\{p>0\} \cap S} 1 - \int_{\{p<0\} \cap S} 1 \right\}_S$ is continuous and odd and that

$F : \mathbb{S}^k \rightarrow \mathbb{R}^m$ where $k \sim d^n$ and $m = \sum_Q M(Q)^n$.

So provided $k \geq m$ – which we can arrange if $d \sim (\sum_Q M(Q)^n)^{1/n}$ – the **Borsuk–Ulam theorem** tells us that F vanishes at some p .

This means that the zero set Z of p exactly bisects each S .

Now if S is a subcube of Q , S will have volume $\sim M(Q)^{-n}$ and diameter $\sim M(Q)^{-1}$ and hence any bisecting surface will meet it in a set of $(n-1)$ -dimensional measure $\sim M(Q)^{-(n-1)}$.

This will be true for each of the $M(Q)^n$ disjoint S 's whose union is Q , so Z will meet Q in a set of $(n-1)$ -dimensional measure $\sim M(Q)^n \times M(Q)^{-(n-1)} = M(Q)$, as was needed.

Borsuk–Ulam

Theorem (Borsuk–Ulam)

If $F : \mathbb{S}^k \rightarrow \mathbb{R}^k$ is continuous, then there is some x with $F(x) = F(-x)$.

So if F is also odd, i.e. $F(-x) = -F(x)$ for all x , then there is some x with $F(x) = 0$.

Trivially the same applies to functions $F : \mathbb{S}^k \rightarrow \mathbb{R}^m$ with $k \geq m$ – just add extra zero components of F until there are k of them.

This is just the topological version of the fact that a *linear* map $F : \mathbb{F}^{k+1} \rightarrow \mathbb{F}^m$ with $k \geq m$ has a nontrivial nullspace – a key point in the arguments of Dvir for the Kakeya set problem in vector spaces over finite fields and those of Ellenberg, R. Oberlin and Tao for the Kakeya maximal theorem for vector spaces over finite fields.

Algebraic geometry and algebraic topology

Until now the full Guth argument for the *geometric* mean relies on heavy-duty algebraic geometry and algebraic topology in place of the Borsuk–Ulam theorem.

Some of the tools needed:

- \mathbb{Z}_2 -cohomology
- Covering spaces
- Cup products
- Lusternik–Schnirelmann theory
- Commutative diagrams and long exact sequences

Now there is an argument (joint with S. Valdimarsson) for the full result using nothing more than Borsuk–Ulam.

Main differences in the argument...

Instead of chopping up each cube Q into $M(Q)^n$ equal **subcubes** of volume $M(Q)^{-n}$ we chop it up into roughly $M(Q)^n$ congruent **rectangles** or **ellipsoids** each of volume roughly $M(Q)^{-n}$.

We replace the use of the Borsuk–Ulam theorem by an equivalent covering statement:

Suppose $U_i \subseteq \mathbb{S}^N$, $1 \leq i \leq k$. Suppose that for each i , $U_i \cap (\overline{-U_i}) = \emptyset$, $\overline{U_i} \cap (-U_i) = \emptyset$. Then if $k \leq N$, the $2k$ sets U_i and $-U_i$ do not cover \mathbb{S}^N .

The **indeterminate shape** of the ellipsoids is chosen by the topology.

A covering lemma

Lemma

Suppose $U_i \subseteq \mathbb{S}^N$, $1 \leq i \leq k$. Suppose that for each i , $U_i \cap (\overline{-U_i}) = \emptyset$, $\overline{U_i} \cap (-U_i) = \emptyset$. Then if $k \leq N$, the $2k$ sets U_i and $-U_i$ do not cover \mathbb{S}^N .

Proof.

Let $f_i(x) = d(x, -U_i) - d(x, U_i)$. Then $f(-x) = -f(x)$, so by Borsuk–Ulam there is an x with $f(x) = 0$. Then we claim that this x does not belong to any U_i or $-U_i$. For if $x \in U_i$ we have $d(x, U_i) = 0$, hence $d(x, -U_i) = 0$, hence $x \in \overline{-U_i}$, a contradiction. Likewise $x \in -U_i$ gives $x \in \overline{U_i}$, another contradiction. □

Visibility

Following Guth we associate a **fundamental convex body** $K(Z)$ to a hypersurface Z . With \hat{u} the unit vector in the direction of u , define

$$K(Z) := \{u \in \mathbb{B} : \text{surf}_{\hat{u}}(Z) \leq 1/|u|\}.$$

$K(Z)$ convex: u satisfies $\text{surf}_{\hat{u}}(Z) \leq 1/|u|$ if and only if $\int_Z |u \cdot n| dS \leq 1$; this condition is clearly retained under convex combinations of u 's.

Under the assumption that $\text{surf}_e(Z) \geq 1$ for all e , if $\{f_j\}$ is any approximately ONB, $\pm f_j / \text{surf}_{f_j}(Z) \in K(Z)$ for all j , so that by convexity of $K(Z)$

$$\text{vol } K(Z) \geq C_n \prod_{j=1}^n \text{surf}_{f_j}(Z)^{-1},$$

that is

$$\text{vol } K(Z)^{-1/n} \leq C_n \prod_{j=1}^n \text{surf}_{f_j}(Z)^{1/n}.$$

Visibility, cont'd

Define

$$\text{vis}(Z) := \text{vol } K(Z)^{-1/n}$$

so that

$$\text{vis}(Z) \leq C_n \prod_{j=1}^n \text{surf}_{f_j}(Z)^{1/n}$$

whenever $\{f_j\}$ is any approximately ONB and $\text{surf}_e(Z) \geq 1$ for all e .

John ellipsoid theorem $\implies \exists$ centred ellipsoid E with

$$E \subseteq K(Z) \subseteq n^{1/2}E.$$

If the principal directions of E are e_1, \dots, e_n it's not hard to see that

$$\text{vol } K(Z) \sim \prod_{j=1}^n \text{surf}_{e_j}(Z)^{-1}; \text{ and so}$$

$$\text{vis}(Z) \sim \inf_{\{f_j\} \text{ approx orthonormal}} \prod_{j=1}^n \text{surf}_{f_j}(Z)^{1/n}.$$

Main result

So, we will be done if we can prove:

Theorem (Guth)

Given a (finitely supported) function M defined on the unit cubes of \mathbb{R}^n there exists a polynomial p such that with $Z = Z_p = \{x : p(x) = 0\}$,

$$\deg p \leq C_n \left(\sum_Q M(Q)^n \right)^{1/n}$$

and such that for all Q

$$\text{vis}(Z \cap Q) \geq C_n M(Q).$$

Scheme of new argument

For each polynomial p its zero set is the hypersurface Z_p , and we let

$$S(Q) = \{p : \text{vis}(Z_p \cap Q) \leq M(Q)\}.$$

AIM: show that we can find a polynomial of degree at most $C_n(\sum_Q M(Q)^n)^{1/n}$ **which is not in any of the $S(Q)$.** Write

$$\bigcup_Q S(Q) = \bigcup_Q \bigcup_{r \geq 0} \bigcup_{\Theta \in \mathcal{C}_{Q,r}} \bigcup_{\alpha \in \mathcal{A}_{Q,r,\Theta}} \left(S_\alpha^{(r),\Theta^+}(Q) \cup S_\alpha^{(r),\Theta^-}(Q) \right),$$

where

$$S_\alpha^{(r),\Theta^-}(Q) = -S_\alpha^{(r),\Theta^+}(Q),$$

$$S_\alpha^{(r),\Theta^+}(Q) \cap \overline{S_\alpha^{(r),\Theta^-}(Q)} = \emptyset$$

and

$$S_\alpha^{(r),\Theta^-}(Q) \cap \overline{S_\alpha^{(r),\Theta^+}(Q)} = \emptyset.$$

Scheme of new proof, cont'd

Work within the class \mathcal{P}_k of polynomials of degree bounded by some $k \in \mathbb{N}$; if we can show that the *total* cardinality of the indexing set in

$$\bigcup_Q S(Q) = \bigcup_Q \bigcup_{r \geq 0} \bigcup_{\Theta \in \mathcal{C}_{Q,r}} \bigcup_{\alpha \in \mathcal{A}_{Q,r,\Theta}} \left(S_\alpha^{(r),\Theta^+}(Q) \cup S_\alpha^{(r),\Theta^-}(Q) \right)$$

is at most the dimension of the unit sphere of the class of polynomials \mathcal{P}_k , then the sets in the union cannot cover this unit sphere, by the covering lemma equivalent to Borsuk–Ulam.

Claim. The number of terms in the union is at most $C_n \sum_Q M(Q)^n$.

So amongst the polynomials in the unit sphere \mathbb{S}^N of \mathcal{P}_k , there will exist one which is not in any of the $S(Q)$ – provided that $N \geq C_n \sum_Q M(Q)^n$. Since $N \sim k^n$ we shall have that there exists a polynomial of degree $\sim C_n (\sum_Q M(Q)^n)^{1/n}$ satisfying $\text{vis}(Z_p \cap Q) > M(Q)$ for all Q , as desired.

New proof – level II

Let, for $r \geq 0$,

$$S^{(r)}(Q) = \{p : \text{vis}(Z_p \cap Q) \sim 2^{-r}M(Q)\}.$$

Then

$$S(Q) = \bigcup_{r \geq 0} S^{(r)}(Q).$$

For each fixed Q and $r \geq 0$ we introduce a collection $\mathcal{C}_{Q,r}$ of *colours* whose cardinality is bounded by A_n independently of Q and r . For each colour Θ we shall define subsets $S^{(r),\Theta}(Q)$ of $S^{(r)}(Q)$ which have the property that

$$S^{(r)}(Q) = \bigcup_{\Theta \in \mathcal{C}_{Q,r}} S^{(r),\Theta}(Q).$$

For each fixed Q , $r \geq 0$ and colour $\Theta \in \mathcal{C}_{Q,r}$ we will be able to define an indexing set $\mathcal{A}_{Q,r,\Theta}$ of cardinality at most¹ $C_n 2^{-rn} M(Q)^n$, and for each $\alpha \in \mathcal{A}_{Q,r,\Theta}$, a subset $S_\alpha^{(r),\Theta}(Q)$ of $S^{(r),\Theta}(Q)$ such that

$$S^{(r),\Theta}(Q) = \bigcup_{\alpha \in \mathcal{A}_{Q,r,\Theta}} S_\alpha^{(r),\Theta}(Q).$$

Finally we shall decompose each $S_\alpha^{(r),\Theta}(Q)$ as

$$S_\alpha^{(r),\Theta}(Q) = S_\alpha^{(r),\Theta+}(Q) \cup S_\alpha^{(r),\Theta-}(Q)$$

in such a way that for all Q, r, Θ and α ,

$$S_\alpha^{(r),\Theta+}(Q) \cap \overline{S_\alpha^{(r),\Theta-}(Q)} = \emptyset$$

and

$$S_\alpha^{(r),\Theta-}(Q) \cap \overline{S_\alpha^{(r),\Theta+}(Q)} = \emptyset.$$

¹If $2^{-rn} M(Q)^n \ll 1$ then $\mathcal{A}_{Q,r,\Theta}$ will be empty.

Now the number of terms in the indexing set for the union

$$\bigcup_Q S(Q) = \bigcup_Q \bigcup_{r \geq 0} \bigcup_{\Theta \in \mathcal{C}_{Q,r}} \bigcup_{\alpha \in \mathcal{A}_{Q,r,\Theta}} \left(S_{\alpha}^{(r),\Theta+}(Q) \cup S_{\alpha}^{(r),\Theta-}(Q) \right),$$

(noting that the r sum is empty for $2^r \gg M(Q)$), is at most

$$\sum_Q \sum_{1 \leq 2^r \leq M(Q)} \sum_{\Theta \in \mathcal{C}_{Q,r}} \sum_{\alpha \in \mathcal{A}_{Q,r,\Theta}} 2 \leq C_n \sum_Q \sum_{r \geq 0} 2^{-rn} M(Q)^n \leq C_n \sum_Q M(Q)^n.$$

New proof – level III

Let us ignore the somewhat technical issue of colours and see how, for each Q and r fixed, we can naturally decompose each $S^{(r)}(Q)$ into at most $C_n 2^{-rn} M(Q)^n$ subsets $S_\alpha^{(r)}(Q)$ each of which can be further decomposed into two disjoint subsets (halves) $S_\alpha^{(r)\pm}(Q)$ with $S_\alpha^{(r)-}(Q) = -S_\alpha^{(r)+}(Q)$.

One has to work a little harder to ensure that the closure of one of these halves is disjoint from the other half, and that there is a single indexing set of size at most $C_n 2^{-rn} M(Q)^n$ which works for all $p \in S^{(r)}(Q)$.

$$p \in S^{(r)}(Q) \implies \text{vol } K(Z_p \cap Q) \sim 2^m / M(Q)^n.$$

So we can fit $\sim 2^{-rn} M(Q)^n$ disjoint parallel translates of $K(Z_p \cap Q)$ in Q , with the translations along the principal directions of the ellipsoid E associated to $K(Z_p \cap Q)$.

Lemma

There is a C_n such that if $p \in S^{(r)}(Q)$ and $\eta < 1$, then Z_p bisects at most $C_n \eta^{-(n-1)} 2^{-rn} M(Q)^n$ disjoint copies of $\eta K(Z_p \cap Q)$ in Q .

– if it bisected too many, then $\text{vol } K(Z_p \cap Q)$ would be too big to be compatible with what's assumed about $\text{vis}(Z_p \cap Q)$.

So if η is chosen sufficiently small depending on n , and if $p \in S^{(r)}(Q)$, then Z_p can bisect only a proportion $C_n \eta < 1$ of the approximately $\eta^{-n} 2^{-rn} M(Q)^n$ disjoint copies of $\eta K(Z_p \cap Q)$ in Q .

In particular Z_p **will not bisect all of the available disjoint copies of $\eta K(Z_p \cap Q)$ in Q** , which number approximately $2^{-rn} M(Q)^n$ in total.

Let $S_\alpha^{(r)}(Q)$ consist of those $p \in S^{(r)}(Q)$ such that $Z_p \cap Q$ does not bisect the α 'th translate ηK_α of $\eta K(Z_p \cap Q)$ in Q .

Then, since Z_p cannot bisect all of the copies of $\eta K(Z_p \cap Q)$ in Q ,

$$S^{(r)}(Q) \subseteq \bigcup_{\alpha} S_\alpha^{(r)}(Q)$$

as required, where there are approximately $2^{-rn} M(Q)^n$ terms in the union.

Moreover there is the natural decomposition of $S_\alpha^{(r)}(Q)$ depending on whether

$$\text{vol}(\{p > 0\} \cap \eta K_\alpha) > \text{vol}(\{p < 0\} \cap \eta K_\alpha)$$

– in which case we say that $p \in S_\alpha^{(r)+}(Q)$ – or

$$\text{vol}(\{p > 0\} \cap \eta K_\alpha) < \text{vol}(\{p < 0\} \cap \eta K_\alpha),$$

– in which case we say that $p \in S_\alpha^{(r)-}(Q)$.

Conclusion

By attempting to understand Guth's algebraic-topological proof of the Multilinear Kakeya theorem we've discovered:

- A) A new multilinear duality principle and some new and apparently non-trivial factorisation properties of functions in $L^p(\mathbb{R}^n)$

- B) A way to reduce the Algebraic Topology in the argument to just the Borsuk–Ulam theorem – for which proofs readily accessible to the analyst exist.