Multilinear Radon-like transforms

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Mapping R of the form

$$Rf(x) = \int_{\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_m}} f_1(y_1) \cdots f_m(y_m) \delta(F(y,x)) \psi(y,x) \, \mathrm{d}y$$

is a natural description of a multilinear Radon-like transform.

Here, $x \in \mathbb{R}^n$, $f = (f_1, \ldots, f_m)$, $f_j : \mathbb{R}^{d_j} \to \mathbb{C}$ test function, $F : \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_m} \times \mathbb{R}^n \to \mathbb{R}^k$ suitably smooth function typically nondegenerate on support of ψ.

If $m = 1$, $Rf(x)$ is perhaps more familiar looking "average" of f over submanifold $M_x = \{y \in \mathbb{R}^{d_1} : F(y,x) = 0\}$.

We seek "L^p improving" properties of R, i.e. $L^{p_1}(\mathbb{R}^{d_1}) \times \cdots \times L^{p_m}(\mathbb{R}^{d_m}) \to L^q(\mathbb{R}^n)$ estimates.

Dualising:

$$\int_{\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_{m+1}}} \prod_{j=1}^{m+1} f_j(y_j)\delta(F(y))\psi(y)\,\mathrm{d}y \leq C \prod_{j=1}^{m+1} \|f_j\|_{L^{p_j}(\mathbb{R}^{d_j})}. \tag{1}$$

Parametrising support of distribution $\delta \circ F$, we are led to

$$\int_{\mathbb{R}^d} \prod_{j=1}^{m+1} f_j(B_j(x))\psi(x)\,\mathrm{d}x \leq C \prod_{j=1}^{m+1} \|f_j\|_{L^{p_j}(\mathbb{R}^{d_j})}, \tag{2}$$

where $B_j : \mathbb{R}^d \to \mathbb{R}^{d_j}$ and $d = (\sum_{j=1}^{m+1} d_j) - k$.

Examples: multilinear singular convolution

S_1, S_2 transversal compact co-dimension one submanifolds of $\mathbb{R}^d$ ($d \geq 2$) which are smooth with nonvanishing gaussian curvature. Then

$$\|f_1 \mathrm{d}\sigma_1 * f_2 \mathrm{d}\sigma_2\|_{L^2(\mathbb{R}^d)} \leq C \|f_1\|_{L^{\frac{4d}{3d-2}}(S_1, \mathrm{d}\sigma_1)} \|f_2\|_{L^{\frac{4d}{3d-2}}(S_2, \mathrm{d}\sigma_2)}. \quad (3)$$

Here, $\int_{\mathbb{R}^d} F(x)\, \mathrm{d}\sigma_j(x) = \int_{U_j} F(\Sigma_j(x'))\, \mathrm{d}x'$, where $\Sigma_j : U_j \to \mathbb{R}^d$ parametrises S_j.

Estimate (3) follows from $L^{d'}(\mathbb{R}^d) \to L^d(\mathbb{R}^d)$ estimate on R, with $m = 1$, and F satisfying rotational curvature condition.

Due to Moyua-Vargas-Vega $d = 3$ (1999); Tao-Vargas-Vega $d \geq 4$ (1998).

$S_1, \ldots, S_d$ transversal $C^{1,\beta}$ submanifolds of $\mathbb{R}^d$ in (say) a neighbourhood of zero. Then

$$\|f_1 \mathrm{d}\sigma_1 * \cdots * f_d \mathrm{d}\sigma_d\|_{L^2(\mathbb{R}^d)} \leq C \prod_{j=1}^{d} \|f_j\|_{L^{\frac{2d-2}{2d-3}}(S_j, \mathrm{d}\sigma_j)} \tag{4}$$

for $f_j \in L^{2d-2/2d-3}(S_j, \mathrm{d}\sigma_j)$ supported in sufficiently small neighbourhood of zero.

Due to Bejenaru-Herr-Tataru $d = 3$ (2010);
Bennett-B $d \geq 4$ (2010).

In fact

$$\|f_1 \mathrm{d}\sigma_1 * \cdots * f_d \mathrm{d}\sigma_d\|_{L^\infty(\mathbb{R}^d)} \leq C \prod_{j=1}^{d} \|f_j\|_{L^{(d-1)'}(S_j, \mathrm{d}\sigma_j)}. \tag{5}$$

Undualising $\Rightarrow$ if $f_1, \ldots, f_{d-1}$ supported sufficiently small neighbourhood of zero, the restriction operator

$$(f_1, \ldots, f_{d-1}) \mapsto (f_1 \mathrm{d}\sigma_1 * \cdots * f_{d-1} \mathrm{d}\sigma_{d-1})\Big|_{S_d}$$

is $L^{(d-1)'}(S_1, \mathrm{d}\sigma_1) \times \cdots \times L^{(d-1)'}(S_{d-1}, \mathrm{d}\sigma_{d-1}) \to L^{d-1}(S_d, \mathrm{d}\sigma_d)$ bounded.

For $d = 3$ this is L^2-phenomenon observed by Bejenaru-Herr-Tataru.

Again, (4) has consequences for multilinear restriction problem.

Transversal singular convolution estimates (4) and (5) follow from

Theorem 1

Suppose that $G:(\mathbb{R}^{d-1})^{d-1}\to\mathbb{R}$ is smooth in a neighbourhood of $u_*\in(\mathbb{R}^{d-1})^{d-1}$ and that

$$\det(\nabla_{u_1}G(u_*)\cdots\nabla_{u_{d-1}}G(u_*))\neq 0.$$

Then there exists a neighbourhood V of u_* and a constant C such that

$$\int_V f_1(u_1)\cdots f_{d-1}(u_{d-1})f_d(u_1+\cdots+u_{d-1})\delta(G(u))\,\mathrm{d}u\leq C\prod_{j=1}^{d}\|f_j\|_{(d-1)'}$$

for all nonnegative $f_j\in L^{(d-1)'}(\mathbb{R}^{d-1})$.

Due to Bennett-Carbery-Wright $d=3$ (2005); Bennett-B $d\geq 4$ (2010).

Theorem 2

Suppose that for each $1 \leq j \leq m$ the mappings $B_j : \mathbb{R}^d \to \mathbb{R}^{d_j}$ are $C^{1,\beta}$ submersions in a neighbourhood of $x_0 \in \mathbb{R}^d$. Suppose further that

$$\bigoplus_{j=1}^{m} \ker \mathrm{d}B_j(x_0) = \mathbb{R}^d. \tag{6}$$

Then there exists a neighbourhood U of x_0 and a constant C such that

$$\int_U \prod_{j=1}^{m} (f_j \circ B_j)^{\frac{1}{m-1}} \leq C \prod_{j=1}^{m} \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{\frac{1}{m-1}} \tag{7}$$

for all nonnegative $f_j \in L^1(\mathbb{R}^{d_j})$, $1 \leq j \leq m$.

Remarks

(1) $d_j = d - 1$ is nonlinear Loomis-Whitney inequality of Bennett-Carbery-Wright.

(2) B-C-W used the Christ "method of refinements" (characteristic functions of sets) + "tensor power trick" (general functions).

(3) Bennett-B used "induction-on-scales", building on work of Bejenaru-Herr-Tataru.

Remarks

(4) The case $d_j = d - 1$ is now well understood. Here fibres of B_j are one-dimensional and if transversality fails, curvature is key.

Essentially necessary and sufficient conditions on $((B_j)_{j=1}^m, (p_j)_{j=1}^m)$ are known for which one has a local inequality

$$\int_{\mathbb{R}^d} \prod_{j=1}^m (f_j \circ B_j)^{p_j} \psi \leq C \prod_{j=1}^m \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{p_j}.$$

Due to Tao-Wright $m = 2$ (2003); Stovall $m \geq 3$ (2010).

Remarks

(5) Bejenaru–Herr (2011) recently used Theorem 2 to obtain some L^∞ singular convolution estimates for three transversal submanifolds whose co-dimensions sum to d (leading to L^2 norms on the right).

Quantitative version

Let

$$X_j(\mathrm{d}B_j(x_0)) = \bigwedge_{k=1}^{d_j} \mathrm{row}_k(\mathrm{d}B_j(x_0)) \in \Lambda^{d_j}(\mathbb{R}^d).$$

Then

$$\bigoplus_{j=1}^{m} \ker \mathrm{d}B_j(x_0) = \mathbb{R}^d \quad \Rightarrow \quad \star \bigwedge_{j=1}^{m} \star X_j(\mathrm{d}B_j(x_0)) \neq 0$$

where $\star$ denotes the Hodge star operator (natural isomorphism $\Lambda^n(\mathbb{R}^d) \to \Lambda^{d-n}(\mathbb{R}^d)$).

When $d_j = d - 1$, $\star X_j(\mathrm{d}B_j(x_0))$ is usual cross product of rows of $\mathrm{d}B_j(x_0)$ and

$$\star \bigwedge_{j=1}^{d} \star X_j(\mathrm{d}B_j(x_0)) = \det(X_1(\mathrm{d}B_1(x_0)), \ldots, X_d(\mathrm{d}B_d(x_0))).$$

Theorem 3

Let $\beta,\varepsilon,\kappa>0$ be given. Suppose that $B_j:\mathbb{R}^d\to\mathbb{R}^{d_j}$ is a $C^{1,\beta}$ submersion satisfying $\|B_j\|_{C^{1,\beta}}\leq\kappa$ in a neighbourhood of $x_0\in\mathbb{R}^d$ for each $1\leq j\leq m$. Suppose further that

$$\left|\star\bigwedge_{j=1}^{m}\star X_j(\mathrm{d}B_j(x_0))\right|\geq\varepsilon.$$

Then there exists a neighbourhood U of x_0 depending on at most β,ε,κ and d, such that for all cutoff functions ψ supported in U, there is a constant C depending only on d and ψ such that

$$\int_{\mathbb{R}^d}\prod_{j=1}^{m}(f_j\circ B_j)^{\frac{1}{m-1}}\psi\leq C\varepsilon^{-\frac{1}{m-1}}\prod_{j=1}^{m}\left(\int_{\mathbb{R}^{d_j}}f_j\right)^{\frac{1}{m-1}}\qquad(8)$$

for all nonnegative $f_j\in L^1(\mathbb{R}^{d_j})$, $1\leq j\leq m$.

Global estimates

Theorem 4

Suppose that for each $1 \leq j \leq m$ the mappings $B_j : \mathbb{R}^d \to \mathbb{R}^{d_j}$ are homogeneous of degree one and $C^{1,\beta}$ submersions on $\mathbb{S}^{d-1}$ for some $\beta > 0$. Suppose further that, for each $\omega \in \mathbb{S}^{d-1}$,

$$\bigoplus_{j=1}^{m} \ker \mathrm{d}B_j(\omega) = \mathbb{R}^d. \tag{9}$$

Then there exists a constant C such that

$$\int_{\mathbb{R}^d} \prod_{j=1}^{m} (f_j \circ B_j)^{\frac{1}{m-1}} \leq C \prod_{j=1}^{m} \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{\frac{1}{m-1}}$$

for all nonnegative $f_j \in L^1(\mathbb{R}^{d_j})$, $1 \leq j \leq m$.

Joint work with Barceló, Bennett and Gutiérrez.

Proof

There is $\delta > 0$ and finite constant C such that

$$\int_{B(\omega,\delta)} \prod_{j=1}^{m} (f_j \circ B_j)^{\frac{1}{m-1}} \leq C \prod_{j=1}^{m} \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{\frac{1}{m-1}}$$

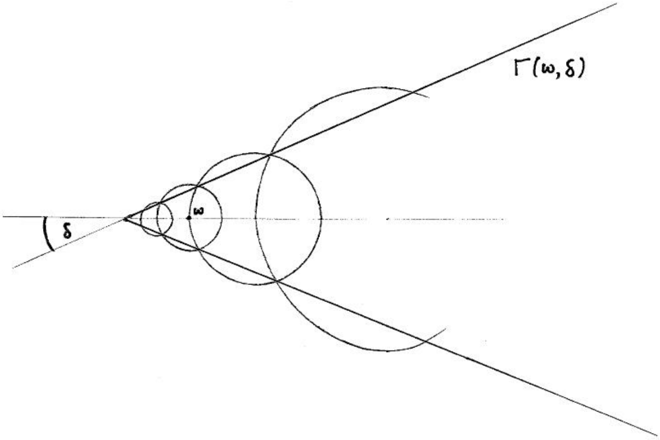
for all $f_j \in L^1(\mathbb{R}^{d_j})$ and all $\omega \in \mathbb{S}^{d-1}$.

Scaling isotropically + homogeneity of $B_j \Rightarrow$

$$\int_{\lambda B(\omega,\delta)} \prod_{j=1}^{m} (f_j \circ B_j)^{\frac{1}{m-1}} \leq C \prod_{j=1}^{m} \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{\frac{1}{m-1}}$$

uniformly in $\lambda > 0$.

The cone $\Gamma(\omega, \delta)$



With $\lambda = 1 + c\delta$ and $U = B(\omega, \delta)$,

$$\Gamma(\omega, \delta) \subseteq \bigcup_{k \in \mathbb{Z}} \lambda^k U$$

and therefore

$$\begin{aligned}
\int_{\Gamma(\omega,\delta)} \prod_{j=1}^{m} (f_j \circ B_j)^{\frac{1}{m-1}} &\leq \sum_{k\in\mathbb{Z}} \int_{\lambda^k U} \prod_{j=1}^{m} (f_j \circ B_j)^{\frac{1}{m-1}} \\
&\leq C \sum_{k\in\mathbb{Z}} \prod_{j=1}^{m} \left(\int_{B_j(\lambda^k U)} f_j \right)^{\frac{1}{m-1}} \\
&= C \sum_{k\in\mathbb{Z}} \prod_{j=1}^{m} \left(\int_{\lambda^k B_j(U)} f_j \right)^{\frac{1}{m-1}} .
\end{aligned}$$

Enemy here are $\omega \in \mathbb{S}^{d-1}$ and $1 \leq j \leq m$ for which

$$B_j(\omega) = 0.$$

The sets $\{\lambda^k B_j(U) : k \in \mathbb{Z}\}$ have bounded (independent of $\omega \in \mathbb{S}^{d-1}$) overlap for all but at most one $1 \leq j \leq m$. Call $j(\omega)$ the "worst" j.

By $(m-1)$-linear Hölder, uniformly in $\omega \in \mathbb{S}^{d-1}$,

$$\begin{aligned}
\int_{\Gamma(\omega,\delta)} \prod_{j=1}^{m} (f_j \circ B_j)^{\frac{1}{m-1}} &\leq C \sum_{k\in\mathbb{Z}} \prod_{j=1}^{m} \left(\int_{\lambda^k B_j(U)} f_j \right)^{\frac{1}{m-1}} \\
&\leq C \left(\int_{\mathbb{R}^{d_{j(\omega)}}} f_{j(\omega)} \right)^{\frac{1}{m-1}} \sum_{k\in\mathbb{Z}} \prod_{j\neq j(\omega)} \left(\int_{\lambda^k B_j(U)} f_j \right)^{\frac{1}{m-1}} \\
&\leq C \left(\int_{\mathbb{R}^{d_{j(\omega)}}} f_{j(\omega)} \right)^{\frac{1}{m-1}} \prod_{j\neq j(\omega)} \left(\sum_{k\in\mathbb{Z}} \int_{\lambda^k B_j(U)} f_j \right)^{\frac{1}{m-1}} \\
&\leq C' \prod_{j=1}^{m} \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{\frac{1}{m-1}} .
\end{aligned}$$

Bounded overlap claim

Using homogeneity of B_j,

$$|B_j(\omega)| \text{ "small" } \Rightarrow \omega \text{ "close" to } \ker \mathrm{d}B_j(\omega)$$

So, if $|B_j(\omega)|$ and $|B_{j'}(\omega)|$ are both small, $j \neq j'$, then ω is close to an element of both $\ker \mathrm{d}B_j(\omega)$ and $\ker \mathrm{d}B_{j'}(\omega)$.

However, we know there is some $\varepsilon > 0$ such that

$$\left| \star \bigwedge_{j=1}^{m} \star X_j(\mathrm{d}B_j(\omega)) \right| \geq \varepsilon$$

uniformly for $\omega \in \mathbb{S}^{d-1}$. This prevents $|B_j(\omega)|$ and $|B_{j'}(\omega)|$ being too small (depending on ε).

Abstract globalisation

Theorem 5
Suppose $\sum_{j=1}^{m} p_j d_j = d$ and that for each $1 \leq j \leq m$ the mappings $B_j : \mathbb{R}^d \to \mathbb{R}^{d_j}$ are homogeneous of degree one and $C^{1,\beta}$ submersions on a neighbourhood of $\omega \in \mathbb{R}^d$ for some $\beta > 0$. Suppose further that for some $\delta > 0$ and constant C,

$$\int_{B(\omega,\delta)} \prod_{j=1}^{m} (f_j \circ B_j)^{p_j} \leq C \prod_{j=1}^{m} \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{p_j}$$

holds for all nonnegative $f_j \in L^1(\mathbb{R}^{d_j})$, $1 \leq j \leq m$. Then there is a constant C' such that

$$\int_{\Gamma(\omega,\delta)} \prod_{j=1}^{m} (f_j \circ B_j)^{p_j} \leq C' \prod_{j=1}^{m} \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{p_j}$$

holds for all nonnegative $f_j \in L^1(\mathbb{R}^{d_j})$, $1 \leq j \leq m$.

Lemma 1

Suppose $\sum_{j=1}^m p_j d_j = d$ and that for each $1 \leq j \leq m$ the mappings $B_j : \mathbb{R}^d \to \mathbb{R}^{d_j}$ are $C^{1,\beta}$ submersions on a neighbourhood of some point $\omega \in \mathbb{R}^d$ for some $\beta > 0$. Suppose further that for some $\delta > 0$ and constant C, the inequality

$$\int_{B(\omega,\delta)} \prod_{j=1}^m (f_j \circ B_j)^{p_j} \leq C \prod_{j=1}^m \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{p_j}$$

holds for all nonnegative $f_j \in L^1(\mathbb{R}^{d_j})$, $1 \leq j \leq m$. Then

$$\int_{\mathbb{R}^d} \prod_{j=1}^m (f_j \circ \mathrm{d}B_j(\omega))^{p_j} \leq C \prod_{j=1}^m \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{p_j}$$

holds for all nonnegative $f_j \in L^1(\mathbb{R}^{d_j})$, $1 \leq j \leq m$.

To prove Lemma 1, consider $\beta = 1$, $\omega = 0$ and $B_j(\omega) = 0$ for each j.

Fix f_j constant at scale $1/M$. Set $g_j(x) = f_j(Mx)$.

So g_j constant at scale $1/M^2$ and cannot distinguish $B_j(x)$ from $\mathrm{d}B_j(0)x$ at scale $1/M$.

Given local inequality $\Rightarrow$

$$\int_{B(0,1/M)} \prod_{j=1}^{m} (g_j \circ \mathrm{d}B_j(0))^{p_j} \leq C \prod_{j=1}^{m} \left(\int_{\mathbb{R}^{d_j}} g_j \right)^{p_j}.$$

Scaling (using $\sum_{j=1}^{m} p_j d_j = d$) $\Rightarrow$

$$\int_{B(0,1)} \prod_{j=1}^{m} (f_j \circ \mathrm{d}B_j(0))^{p_j} \leq C \prod_{j=1}^{m} \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{p_j}.$$

Proof of Theorem 5

As before, by scaling and homogeneity,

$$\int_{\Gamma(\omega,\delta)} \prod_{j=1}^{m} (f_j \circ B_j)^{p_j} \leq C \sum_{k\in\mathbb{Z}} \prod_{j=1}^{m} \left(\int_{\lambda^k B_j(U)} f_j \right)^{p_j},$$

where $U = B(\omega,\delta)$ and $\lambda = 1 + c\delta$.

Lemma 1 $\Rightarrow$ Brascamp–Lieb constant for data $((\mathrm{d}B_j(\omega))_{j=1}^m, (p_j)_{j=1}^m)$ is finite.

Hence, for all subspaces V of $\mathbb{R}^d$,

$$\dim(V) \leq \sum_{j=1}^{m} p_j \dim(\mathrm{d}B_j(\omega)V).$$

Testing on $V = \mathrm{span}(\omega)$,

$$\sum_{j:B_j(\omega)\neq 0} p_j \geq 1.$$

Hölder $\Rightarrow$

$$\int_{\Gamma(\omega,\delta)} \prod_{j=1}^{m} (f_j \circ B_j)^{p_j} \leq C \prod_{j:B_j(\omega)=0} \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{p_j} \sum_{k\in\mathbb{Z}} \prod_{j:B_j(\omega)\neq 0} \left(\int_{\lambda^k B_j(U)} f_j \right)^{p_j}$$

$$\leq C \prod_{j:B_j(\omega)=0} \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{p_j} \prod_{j:B_j(\omega)\neq 0} \left(\sum_{k\in\mathbb{Z}} \int_{\lambda^k B_j(U)} f_j \right)^{p_j}.$$

For j satisfying $B_j(\omega) \neq 0$, we have bounded overlap of $\{\lambda^k B_j(U) : k \in \mathbb{Z}\}$.

Hence

$$\int_{\Gamma(\omega,\delta)} \prod_{j=1}^{m} (f_j \circ B_j)^{p_j} \leq C' \prod_{j=1}^{m} \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{p_j}.$$

Multilinear Radon-like transforms

"Loomis–Whitney case in three dimensions":

Suppose $F : (\mathbb{R}^2)^3 \to \mathbb{R}^3$ is $C^{1,\beta}$ in a neighbourhood of some point y_* in $(\mathbb{R}^2)^3$ and

$$\det(Y_1(y_*)\ Y_2(y_*)\ Y_3(y_*)) \neq 0, \tag{10}$$

where

$$Y_j(y_*) := \nabla_{(y_j)_1} F(y_*) \times \nabla_{(y_j)_2} F(y_*).$$

Then there exists a neighbourhood V of y_* in $(\mathbb{R}^2)^3$ and a constant C such that

$$\int_V f_1(y_1) f_2(y_2) f_3(y_3) \delta(F(y))\, \mathrm{d}y \leq C \|f_1\|_2 \|f_2\|_2 \|f_3\|_2 \tag{11}$$

for all nonnegative $f_1, f_2, f_3 \in L^2(\mathbb{R}^2)$.

If (10) holds on the unit sphere and F is homogeneous of degree one which is $C^{1,\beta}$ on the sphere (intersected with a neighbourhood of the zero set of F) then (11) holds with $V = (\mathbb{R}^2)^3$.

In general, for a differentiable mapping $F : \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_m} \to \mathbb{R}^d$ and $y \in \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_m}$, where $\sum_{j=1}^{m} d_j = (m-1)d$, let $Y_j(\mathrm{d}F(y)) \in \Lambda^{d_j}(\mathbb{R}^d)$ be given by

$$Y_j(\mathrm{d}F(y)) = \bigwedge_{k \in \mathcal{K}_j} \mathrm{d}F(y)(e_k),$$

where e_k denotes the kth standard basis vector in $\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_m}$, and $\mathcal{K}_j$ is the obvious partition of $\{1, \ldots, (m-1)d\}$ with $|\mathcal{K}_j| = d_j$.

Theorem 6
Suppose $\sum_{j=1}^m d_j = (m-1)d$. If $F : \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_m} \to \mathbb{R}^d$ is $C^{1,\beta}$ in a neighbourhood of some point y_* and

$$\left| \star \bigwedge_{j=1}^{m} \star Y_j(\mathrm{d}F(y_*)) \right| \neq 0 \tag{12}$$

then there exists a neighbourhood V of y_* and a constant C such that

$$\int_V \prod_{j=1}^m f_j(y_j)\delta(F(y))\,\mathrm{d}y \leq C \prod_{j=1}^m \|f_j\|_{(m-1)'}$$

for all nonnegative $f_j \in L^{(m-1)'}(\mathbb{R}^{d_j}), 1 \leq j \leq m$.

Furthermore, if F is homogeneous of degree one, $C^{1,\beta}$ on the sphere (intersected with a neighbourhood of the zero set of F) and such that (12) holds on the unit sphere, then

$$\int_{\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_m}} \prod_{j=1}^{m} f_j(y_j)\delta(F(y))\,\mathrm{d}y \leq C\prod_{j=1}^{m} \|f_j\|_{(m-1)'}$$

for all nonnegative $f_j \in L^{(m-1)'}(\mathbb{R}^{d_j}), 1 \leq j \leq m$.

Remarks on the proof

It is instructive to consider the "canonical" case where $F : \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_m} \cong \mathbb{R}^{(m-1)d} \to \mathbb{R}^d$ is the **linear** map

$$F = (I_d \; I_d \; \cdots \; I_d).$$

Changes of variables allow one to reduce to the case $\mathrm{d}F(y_*)$ is the above map.

Parametrising the kernel of F leads to surjective linear mappings $B_j : \mathbb{R}^{(m-2)d} \to \mathbb{R}^{d_j}$.

Unfortunately

$$(m-2)d - \sum_{j=1}^{m} \dim\ker B_j = (m-3)(m-1)d.$$

Our "get out" is to block together appropriate B_j to form linear maps $B_j^{\oplus}: \mathbb{R}^{(m-2)d} \to \mathbb{R}^{d_j^{\oplus}}$ given by

$$B_j^{\oplus}(x) = (B_{S_1^{(j)}}(x), \ldots, B_{S_{m-2}^{(j)}}(x))$$

where $S^{(j)}$ is the $(m-2)$-tuple obtained by deleting $j-2$ and $j-1$ from $(1, \ldots, m)$, and

$$d_j^{\oplus} = \sum_{\ell=1}^{m-2} d_{S_l^{(j)}}.$$

The kernels of such mappings can be shown to form a direct sum decomposition of $\mathbb{R}^{(m-2)d}$.