Cosmological Singularities and AdS/CFT Correspondence

Sumit R. Das

S.R.D, J. Michelson, K. Narayan and S. Trivedi, PRD 74 (2006) 026002
S.R.D, J. Michelson, K. Narayan and S. Trivedi, PRD 74 (2007) 026002
A.Awad, S.R.D, K. Narayan and S. Trivedi, PRD 77 (2008) 046008.
A. Awad, S.R.D, S.Nampuri, K. Narayan and S. Trivedi, hep-th/0806.XXXX

Usual AdS/CFT

- IIB string theory in asymptotically $AdS_5 \times S^5$ space-times is dual to large-N expansion of N=4 SYM theory on the boundary with appropriate sources or excitations.
- The usual relationship between the dimensionless parameters on the two sides are

$$g_s = g_{YM}^2 \qquad\qquad (R/l_s)^4 = 4\pi \, g_{YM}^2 \, N$$

- Where g_s is the string coupling, g_{YM}^2 is the square of the Yang-Mills coupling, l_s is the string length and R is the AdS length scale

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- For generic values of the parameters , the gauge theory hopefully continues to make sense, though there is no interpretation in terms of General Relativity.
- Could we make use of this fact to understand singularities ?

- In this talk : I will describe one particular approach.
- Some parts of our work overlap with

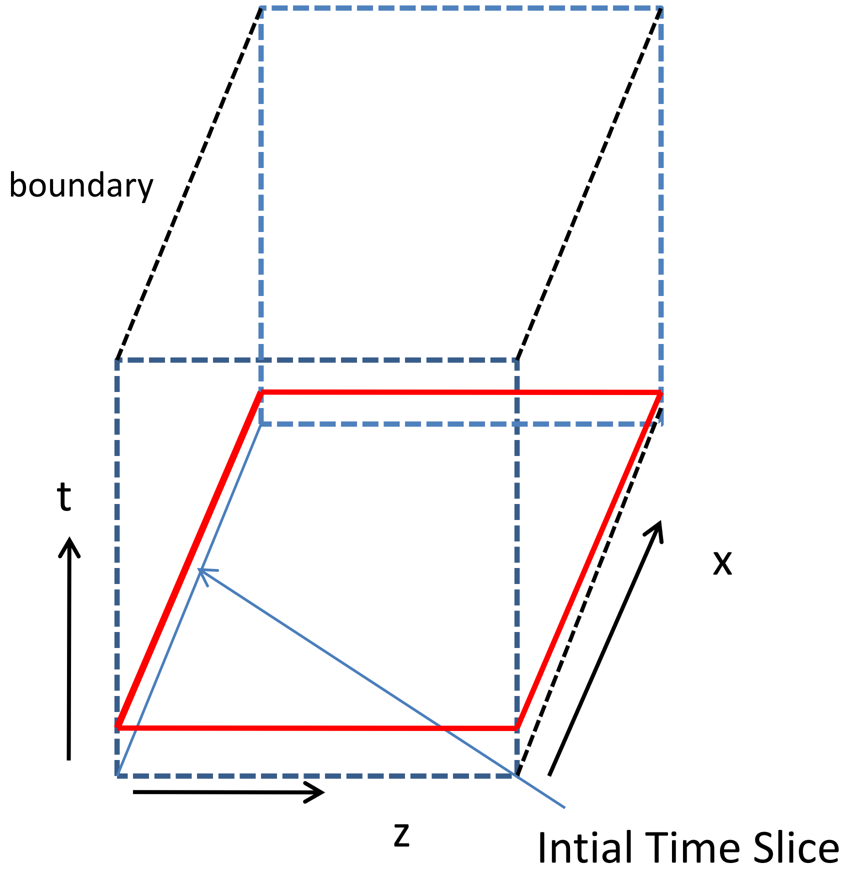
  Chu and Ho (2006,2007)
- Other approaches of interest –

  Hertog and Horowitz (2005);

  Craps, Hertog and Turok (2007,2008)

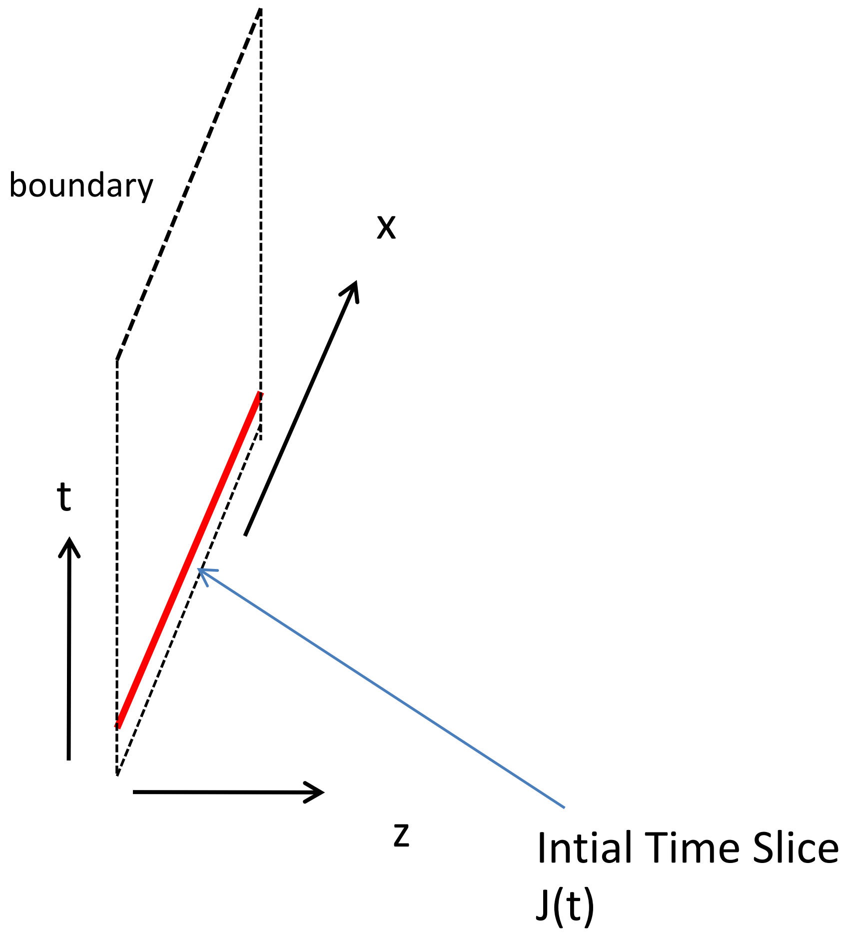
A Scenario

- At early times, start with the ground state of the gauge theory with large ‘t Hooft coupling.
- The physics is now well described by supergravity in usual $AdS_5 \times S^5$



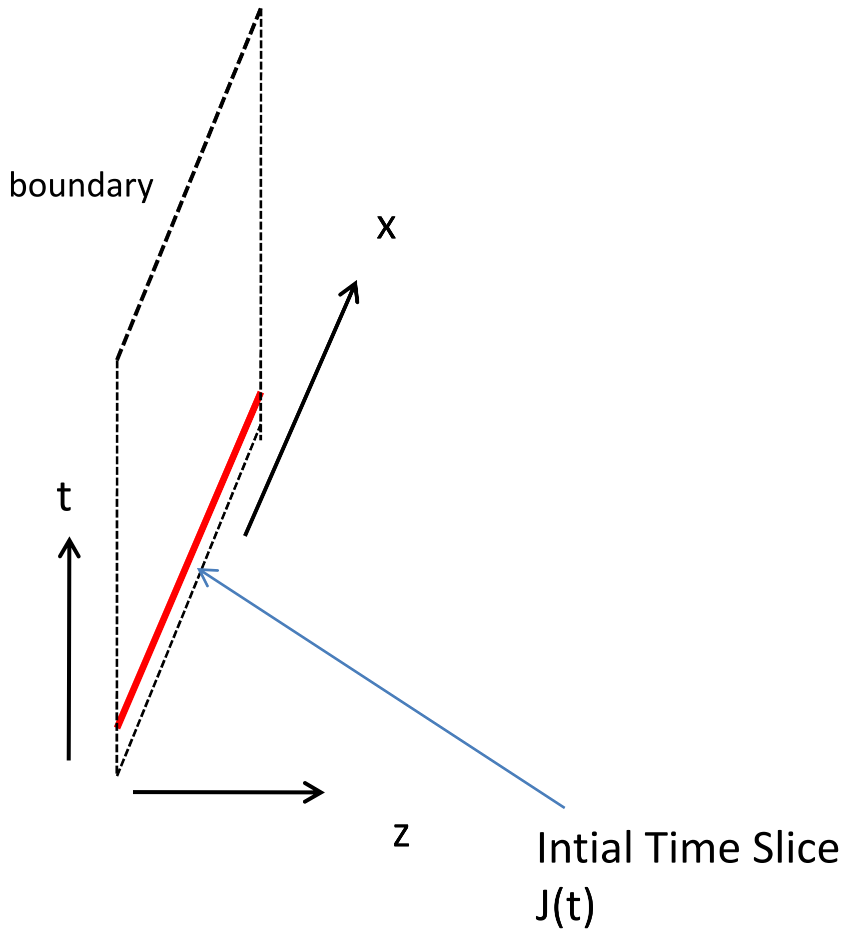
A Scenario

- Now turn on a **time dependent source** in the Yang-Mills theory which deforms the lagrangian.



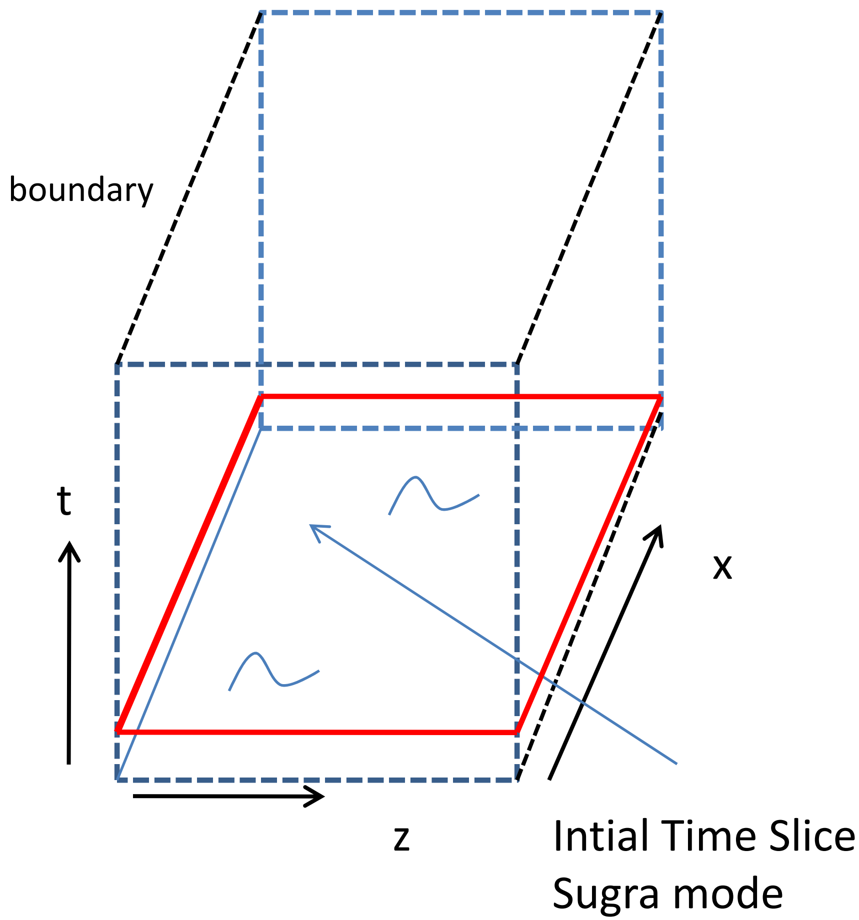
A Scenario

- Now turn on a **time dependent source** in the Yang-Mills theory which deforms the lagrangian.
- This source will be specified for all times.



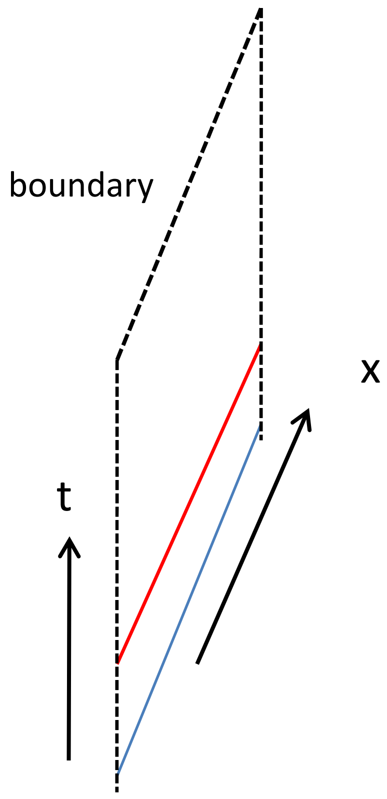
A Scenario

- Now turn on a time-dependent source in the Yang-Mills theory which deforms the lagrangian.
- This corresponds to turning on a non-normalizable mode of the supergravity in the bulk, thus deforming the original $AdS_5 \times S^5$



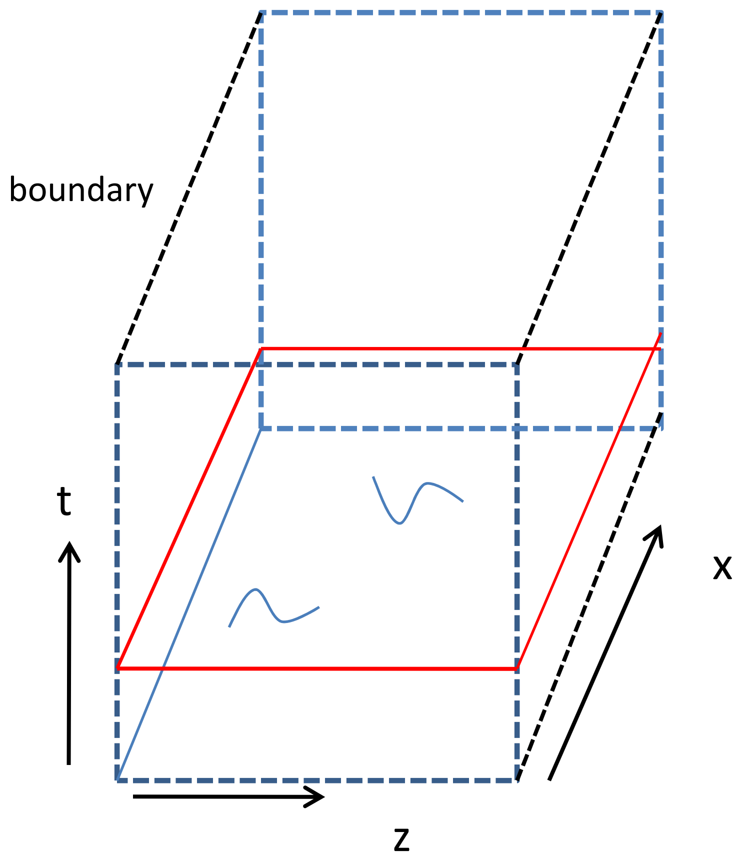
A Scenario

- The gauge theory evolves according to the deformed (time dep.) hamiltonian



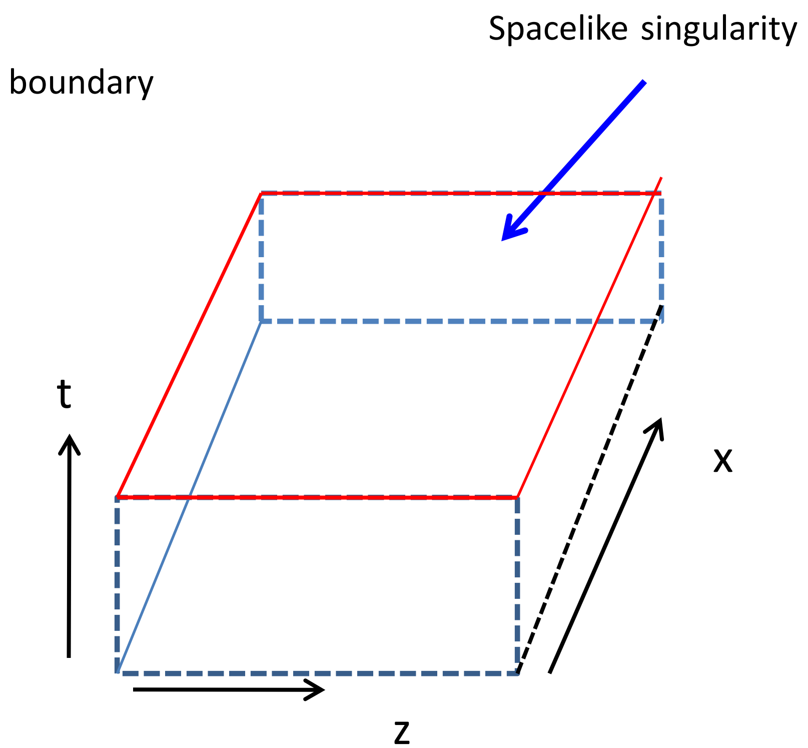
A Scenario

- The gauge theory evolves according to the deformed (time dep.) hamiltonian
- At sufficiently early times the supergravity background evolves according to the classical equations of motion



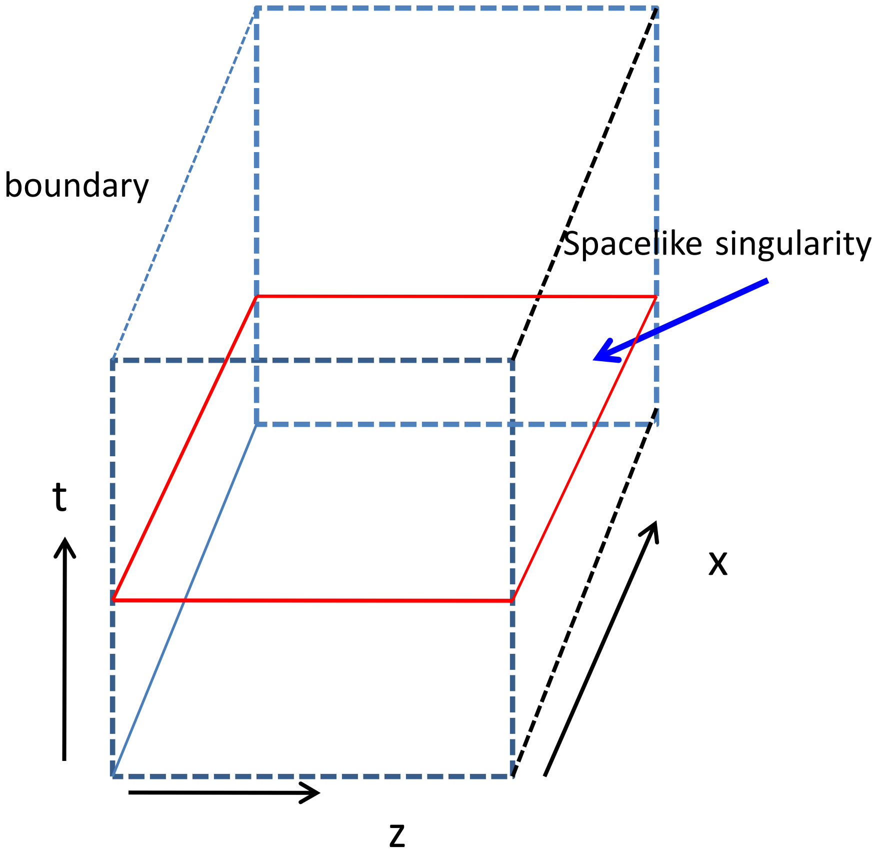
A Scenario

- At later times, the curvatures or other invariants of supergravity start becoming large
- If we nevertheless insist on the supergravity solution we encounter a singularity at some finite time
- Beyond this time, it is meaningless to evolve any further.



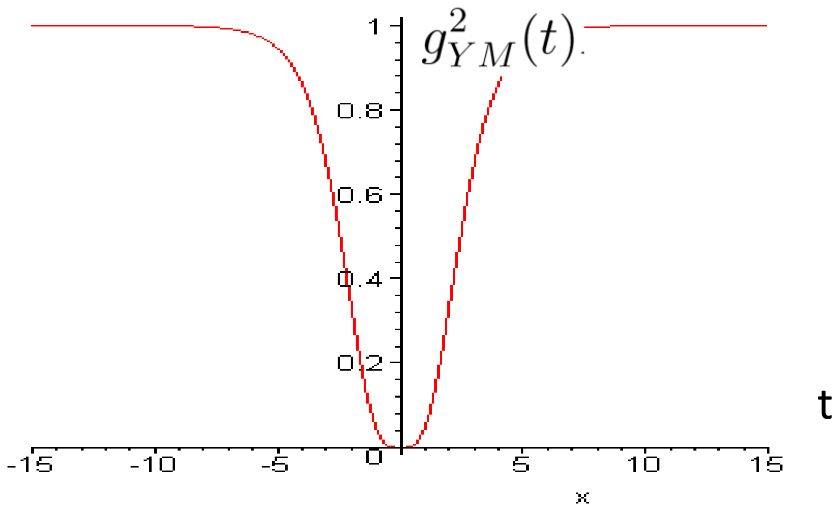
A Scenario

- However, the gauge theory could be still well defined at this time.
- And if we are lucky enough the gauge theory may be evolved beyond this point
- If not, we want to understand what is precisely the obstruction in the gauge theory.



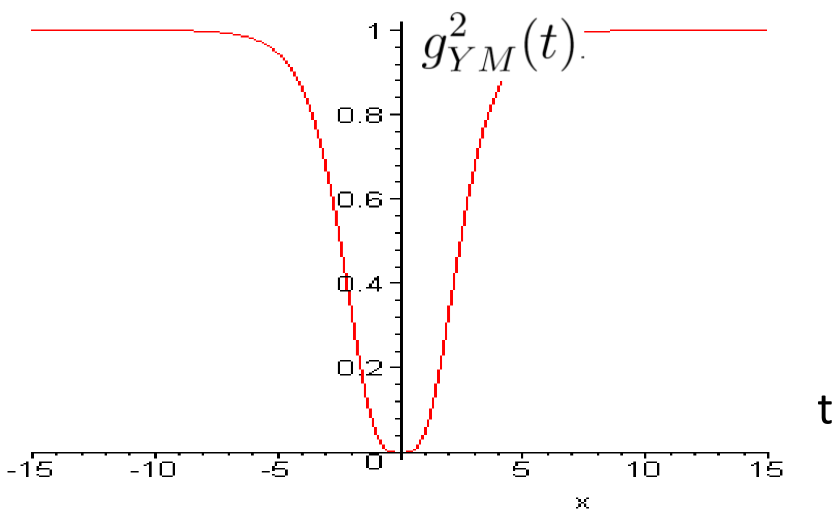
Models implementing this Scenario

- We will try to implement this scenario by turning on sources in the gauge theory which correspond to time dependent couplings $g_{YM}^2(t)$.
- The gauge theory would still live on flat space-time and there would be no other source.
- We will choose the gauge theory coupling to be bounded everywhere and becoming vanishingly small at some time.



Models implementing this Scenario

- We will try to implement this scenario by turning on sources in the gauge theory which correspond to time dependent couplings $g^2_{YM}(t)$.
- The gauge theory would still live on flat space-time and there would be no other source.
- This raises the hope that we could be able to analyze the gauge theory in some controlled fashion.



- In supergravity, this would correspond to a metric which is constrained to be **FLAT** on the boundary and a dilaton whose boundary value matches the gauge theory coupling.

$$ds^2 = \frac{1}{z^2}\left[dz^2 + g_{\mu\nu}(x,z)dx^\mu dx^\nu\right] + d\Omega_5^2$$

$$F_5 = \omega_5 + \star\omega_5$$

$$\mathrm{Lim}_{z\to 0} \quad e^{\Phi(x,z)} = g_{YM}^2(t)$$

$$\mathrm{Lim}_{z\to 0} \quad g_{\mu\nu}(x,z) = \eta_{\mu\nu}$$

At early times this should be $AdS_5 \times S^5$

Null Solutions

- The best controlled solutions of this type are those with null rather than spacelike singularities

$$ds^2 = \frac{1}{w^2}\left[dw^2 - 2dy^+dy^- + d\vec{y}^2 + \frac{1}{4}w^2(\Phi')^2(dy^+)^2\right]$$

- Where $\Phi(y^+)$ is the dilaton which is a function of y^+ alone.
- These null solutions have been independently obtained and studied by

Chu and Ho, JHEP 0604 (2006) 013

Chu and Ho, hep-th/0710.2640

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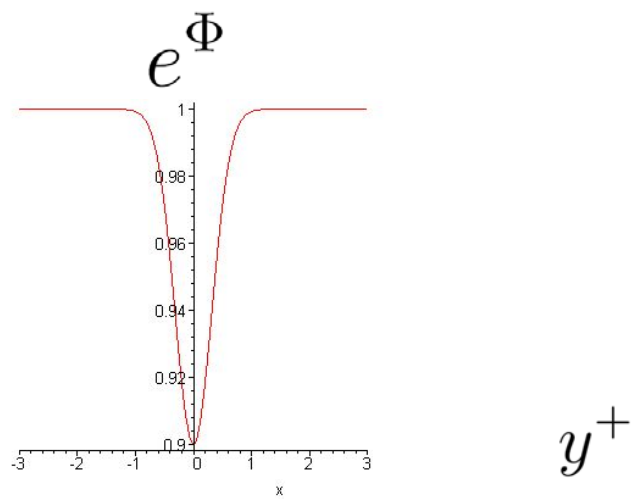
- Where $\Phi(y^+)$ is the dilaton which is a function of y^+ alone.
- This function may be chosen freely..in particular we can choose this function of the desired form.
- For example,

$$e^\Phi = g_s\left[1 - \alpha e^{-(y^+)^2/a^2}\right]$$



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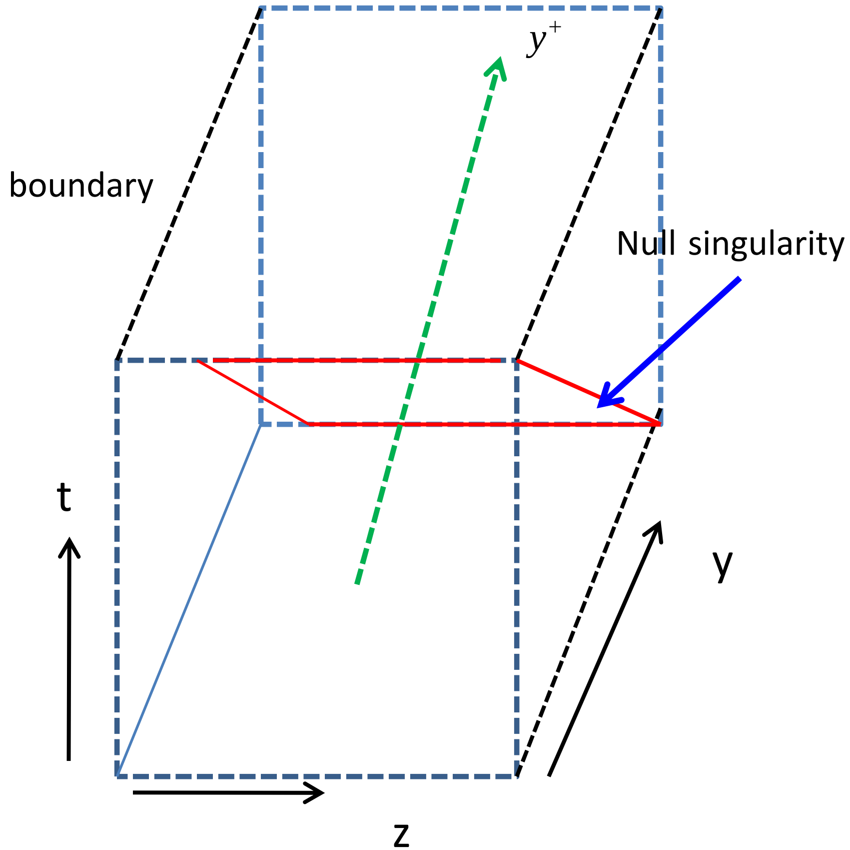
- This function may be chosen freely..in particular we can choose this function of the desired form.
- These solutions maintain ½ of the supersymmetries.

- There is a singularity at $y^{+}=0$ with a suitable choice of the function $\Phi(y^{+})$, e.g. with

$$e^{\Phi} = g_s \left[1 - e^{-(y^{+})^2/a^2}\right]$$

- Null geodesics can reach this point in a finite affine parameter.



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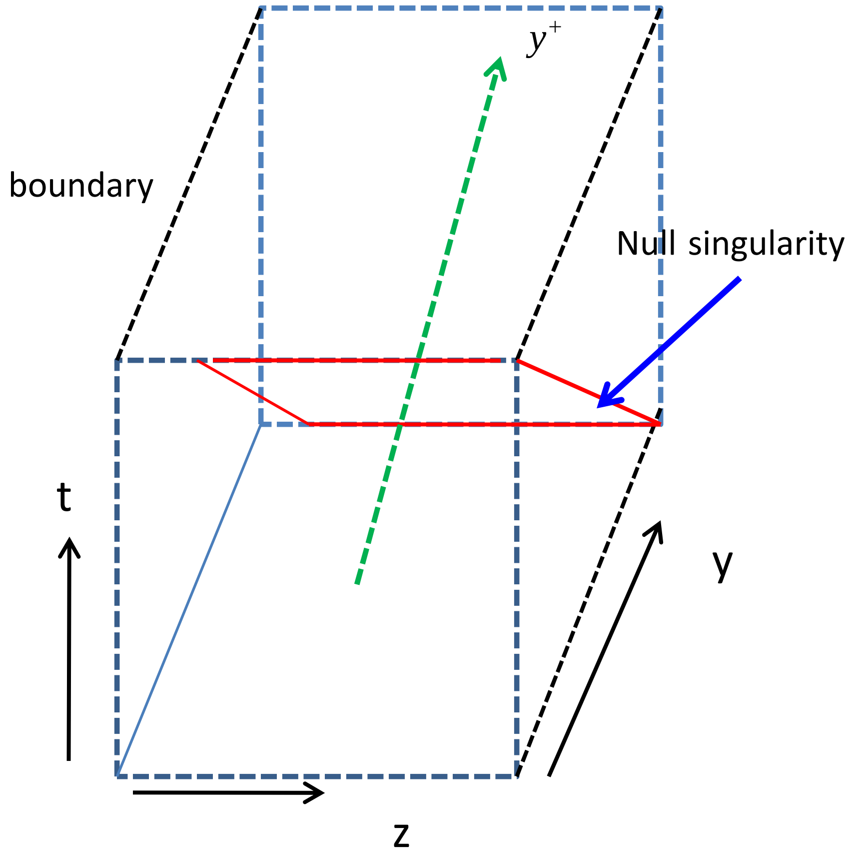

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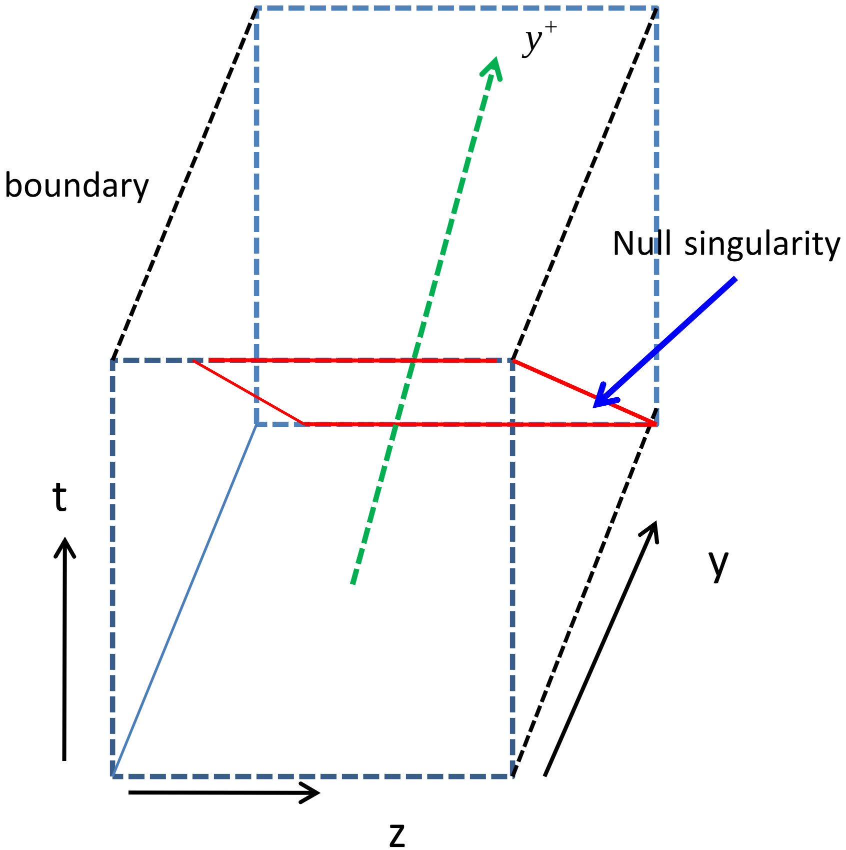

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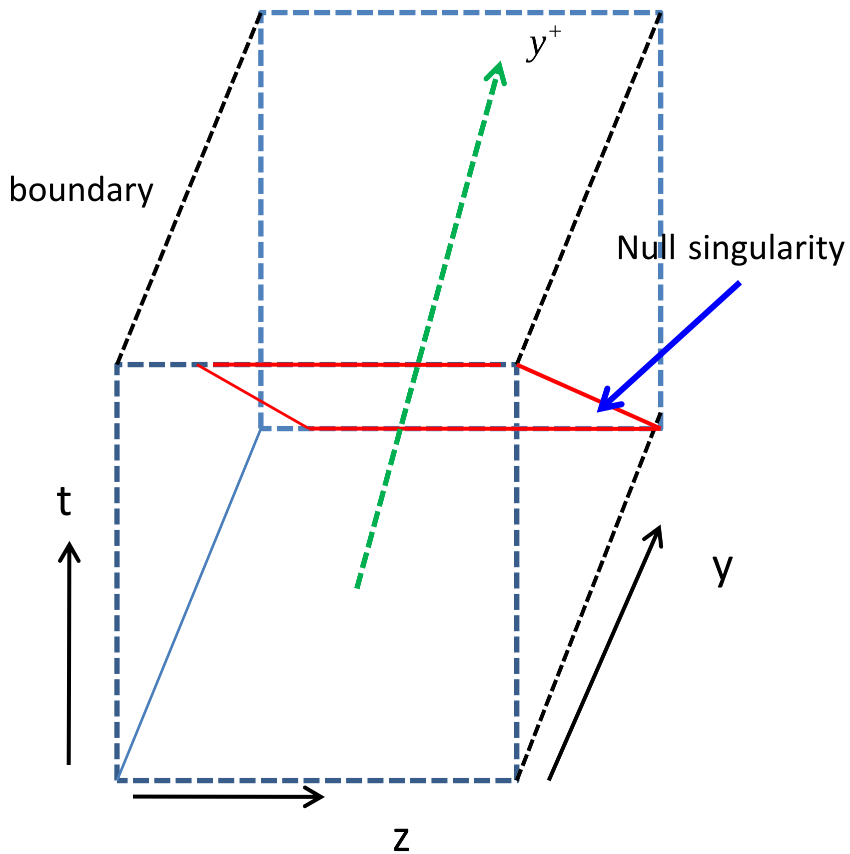
- Null geodesics can reach this point in a finite affine parameter.
- Tidal forces between such geodesics diverge.
- However constant w surfaces, in particular the boundary at w=0 are FLAT
- The only source in the gauge theory is a y^+ dependent coupling.



- These solutions are in fact related to

$$ds^2 = \frac{1}{z^2}\left[dz^2 + e^{f(x^+)}(-2dx^+dx^- + d\vec{x}^2)\right]$$

where $\quad F(x^+) = e^{-f(x^+)/2} \qquad\qquad \frac{F''}{F} = \frac{1}{4}(\Phi')^2$

by coordinate transformations

$$z \;=\; w\; e^{f(y^+)/2} \qquad\qquad x^- \;=\; y^- - \frac{1}{4}w^2(\partial_+ f)$$

- This is an example of the general fact that a Weyl transformation on the boundary is equivalent to a special class of coordinate transformations in the bulk - the Penrose-Brown-Hanneaux (PBH) transformations.

A more general class

- In fact there is a more general class of solutions of the following form

$$ds^2 = \frac{1}{z^2}\left[dz^2 + \tilde{g}_{\mu\nu}(x)dx^\mu dx^\nu\right] + d\Omega_5^2$$

- The 4d metric $\tilde{g}_{\mu\nu}(x)$ and the dilaton $\Phi(x)$ are functions of the four coordinates x^ν and the 5-form field strength is standard.
- This is a solution if $\tilde{R}_{\mu\nu} = \frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi$ and $\nabla^2\Phi = 0$.
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- Thus a solution of 3+1 dimensional dilaton gravity may be lifted to be a solution of 10d IIB supergravity with fluxes.

- We will consider solutions of this type where the 4d metric is conformal to flat space

$$\tilde{g}_{\mu\nu} = e^{h(x)} \ \eta_{\mu\nu}$$

The connection between Weyl transformations on the boundary and PBH transformations then ensures that there is a different foliation of the AdS space-time in which the boundary is flat – and all we have is a nontrivial dilaton.

- We will always define the dual gauge theory to live on this flat boundary.

Kasner-like Solutions

- The easiest form of time dependent solution is the lift of a usual 4d Kasner universe

$$ds^2 = \frac{1}{z^2}\left[dz^2 - dt^2 + \sum_{i=1}^{3} t^{2p_i} dx^i dx^i\right]$$

$$e^{\Phi(t)} = |t|^{\sqrt{2(1-\sum p_i^2)}} \qquad \sum_{i=1}^{3} p_i = 1$$

- This has a spacelike curvature singularity at t=0.
- The effective string coupling vanishes here – as required.
- However the coupling diverges at infinite past and future.

- Nevertheless it is instructive to see what the dual gauge theory looks like. This can be explicitly worked out for

$$p_1 = p_2 = p_3 = \frac{1}{3}$$

- In this case the 4d metric is conformal to flat space

$$ds^2 = \frac{1}{z^2}\left[dz^2 + \frac{2t}{3}\left(-dt^2 + (dx^1)^2 + \cdots (dx^3)^2\right)\right]$$

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- The exact PBH transformation may be written down and the metric which has a flat boundary is

$$ds^2 = \frac{1}{w^2}\left[dw^2 - \frac{(16T^2 - 5w^2)^2}{256T^4}dT^2 + \frac{(16T^2 - w^2)^{\frac{4}{3}}(16T^2 + 5w^2)^{\frac{2}{3}}}{256T^4}\left((dx^1)^2 + \cdots (dx^3)^2\right)\right]$$

$$e^{\Phi} = \left(T\;\left[\frac{16T^2 + 5w^2}{16T^2 - w^2}\right]^{2/3}\right)^{\sqrt{3}}$$

Time dependence with bounded couplings

- We need to obtain solutions which approach the Kasner solution near a space-like singularity, but also asymptote to standard anti-de-Sitter with constant dilaton at early times – so that the coupling of the dual gauge theory is always bounded.
- One such solution

$$ds^2 = \frac{dz^2}{z^2} + \frac{1}{z^2}\left(1 - \frac{1}{\tau^4}\right)\left[-d\tau^2 + \tau^2[dr^2 + \sinh^2 r \; d{\Omega_2}^2]\right]$$

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Flat space-time in Milne coordinates

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$$\Phi(\tau) = \sqrt{3} \, \ln \left[\frac{\tau^2 - 1}{\tau^2 + 1}\right]$$

- Singularity is at $\tau = 1$

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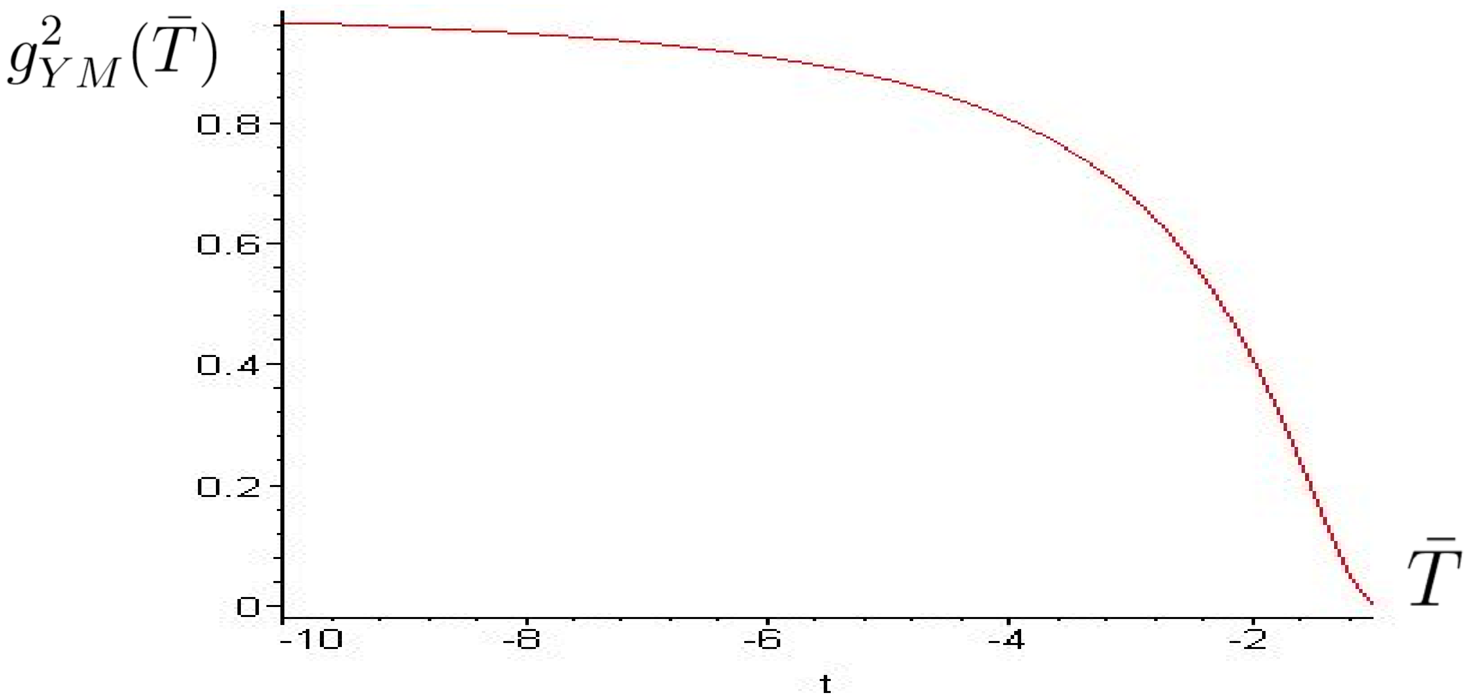
- In this case, we have not been able to determine the exact form of the transformations. However all one needs is the transformation near the boundary. We will, therefore, determine this in an expansion around the boundary.
- The metric near the boundary becomes

$$
\begin{aligned}
ds^2 &= [\frac{1}{\rho^2} + O(\rho^4)]\, d\rho^2 - [\frac{1}{\rho^2} - \frac{10\,\bar{T}^2}{(\bar{T}^4-1)^2} + \frac{25\,\bar{T}^4}{(\bar{T}^4-1)^4}\,\rho^2 + O(\rho^4)]\, d\bar{T}^2 \\
&+ [\frac{1}{\rho^2} + \frac{2\,\bar{T}^2}{(\bar{T}^4-1)^2} + \frac{(\bar{T}^4-8)}{(\bar{T}^4-1)^4}\,\rho^2 + O(\rho^4)]\left[\frac{\bar{T}^2}{r^2}\,dr^2 + \frac{\bar{T}^2}{4}\,(r-\frac{1}{r})^2\, d\Omega_2^2\right]
\end{aligned}
$$

- The boundary metric is now explicitly flat.

- In terms of these new coordinates the spacelike singularity appears at $\bar{T} = -1$ and the asymptotic past is $\bar{T} \to -\infty$
- The effective string coupling is bounded, decreasing from a finite value in the past to a zero value at the singularity.
- Therefore, the dual gauge theory lives on flat space and has a time dependent coupling constant which vanishes at some finite time $\bar{T} = -1$



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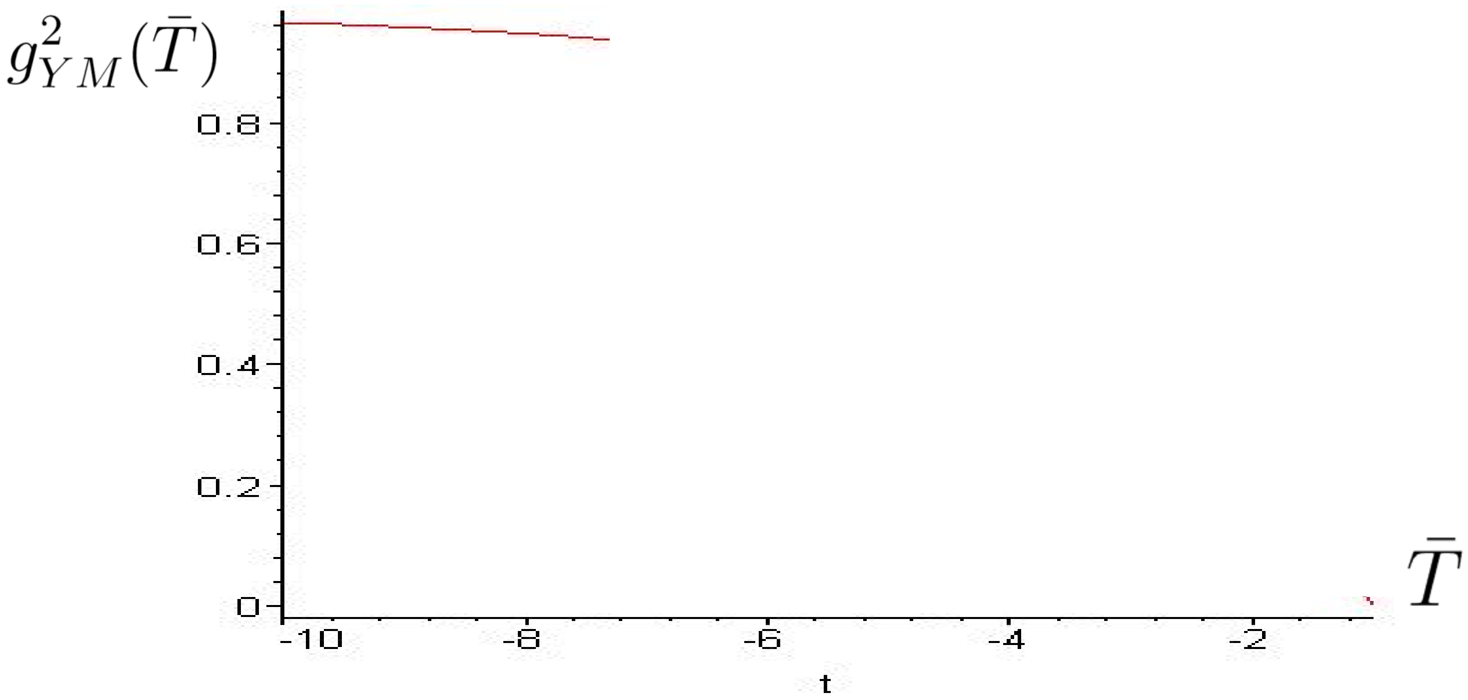
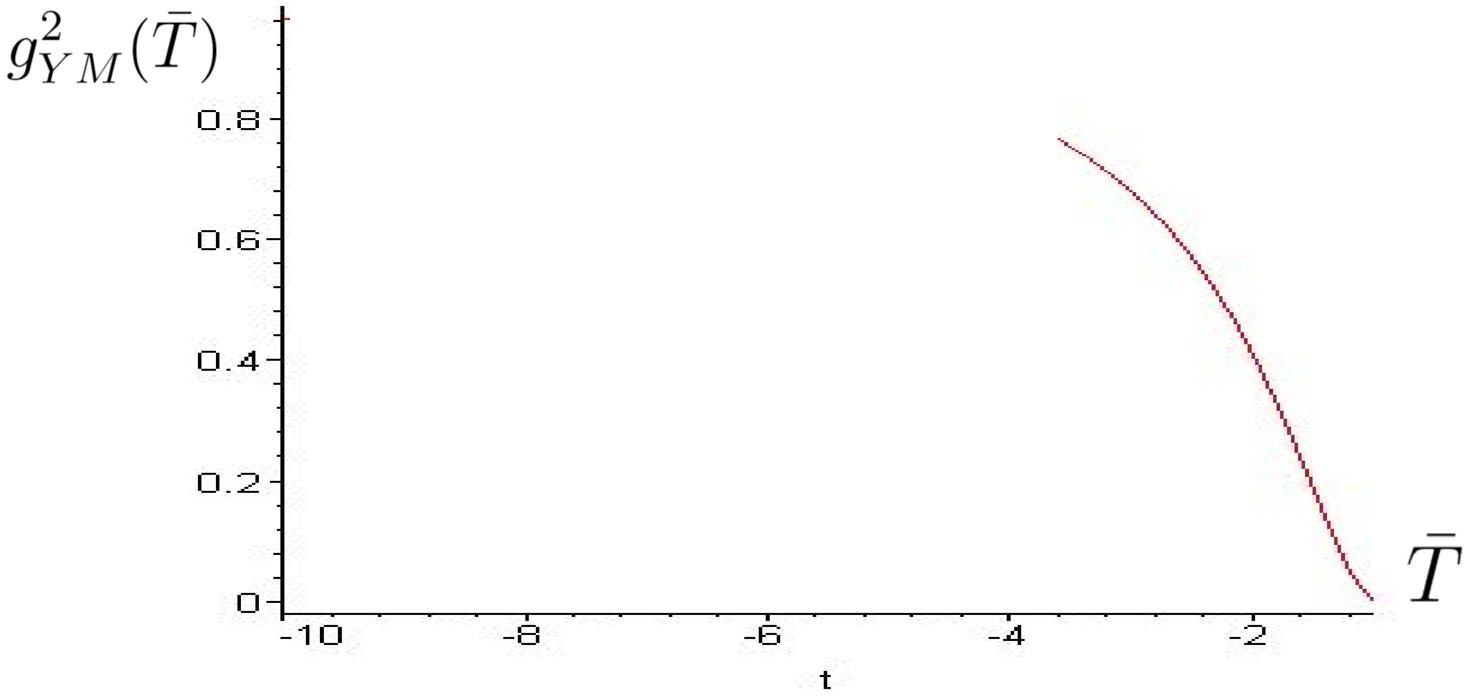
- At early times the ‘t Hooft coupling is large and supergravity can be trusted.



- In terms of these new coordinates the spacelike singularity appears at $\bar{T} = -1$ and the asymptotic past is $\bar{T} \to -\infty$
- The effective string coupling is bounded, decreasing from a finite value in the past to a zero value at the singularity.
- At late times the 't Hooft coupling becomes small and supergravity is meaningless. This is when the singularity appears in the bulk.



Behavior near singularity

- The solution we have

$$ds^2 = \frac{dz^2}{z^2} + \frac{1}{z^2}\,(1 - \frac{1}{\tau^4})\,\left[-d\tau^2 + \tau^2[dr^2 + \sinh^2 r\ d{\Omega_2}^2]\right]$$

$$\Phi(\tau) = \sqrt{3}\,\ln\left[\frac{\tau^2 - 1}{\tau^2 + 1}\right]$$

approaches the Kasner solution near the singularity at $\tau = 1$

- We are in fact considering dilaton driven cosmology in 3+1 dimensions and can use the standard BKL analysis
- For a large class of initial conditions the dilaton always approaches the Kasner value.

The Energy-Momentum Tensors

- We want to verify that the state we start out with is indeed the vacuum of the gauge theory.
- One way to do this is to calculate the holographic stress tensor.
- In the regime where supergravity is reliable, this evaluates the energy momentum tensor of the dual gauge theory.
- We will compute this using the method of covariant counterterms.

  (Henningson and Skenderis ; Balasubramanian and Kraus)

  We will compute this in the foliation where the boundary is flat.

- Consider a 5d metric of the form

$$ds^2 = \frac{1}{z^2}\left[dz^2 + g_{\mu\nu}(x,z)dx^\mu dx^\nu\right]$$

- The cutoff boundary is taken to be at $z = z_0$
- With appropriate counterterms, the holographic stress tensor is given by

$$T^{\mu\nu} = \frac{1}{8\pi G_5}\left[\Theta^{\mu\nu} - \Theta\, h^{\mu\nu} - 3\, h^{\mu\nu} + \frac{1}{2}G^{\mu\nu} - \frac{1}{4}\nabla^\mu\,\Phi\nabla^\nu\,\Phi + \frac{1}{8}h^{\mu\nu}(\nabla\,\Phi)^2\right]$$

where $\Theta^{\mu\nu}$ is the extrinsic curvature of the boundary, $h_{\mu\nu}(x)$ is the induced metric

$$h_{\mu\nu}(x) = \frac{1}{z_0^2}g_{\mu\nu}(x, z_0)$$

and $G^{\mu\nu}$ is the Einstein tensor computed from the induced metric.

EM Tensor : Null Solutions

- For the null solutions, the energy momentum tensor vanishes for any choice of the dilaton profile.
- This ensures that the initial state is vacuum
- The fact that the answer continues to vanish for any light front time is a reflection of the absence of particle production in the presence of a null isometry.

EM Tensor : Time dependent solutions

- For the time dependent solutions which have bounded couplings, the energy momentum tensor of the gauge theory evaluated by holographic methods is

$$< T_\mu^\nu >= \frac{N^2}{2\,\pi^2\,(\bar{T}^4-1)^4}\,\mathrm{diag}\,\left(12-3\,\bar{T}^4, 4+9\,\bar{T}^4, 4+9\,\bar{T}^4, 4+9\,\bar{T}^4\right)$$

- At early times $\bar{T} \to -\infty$ this vanishes fast

$$< T_\mu^\nu > \to \frac{N^2}{2\pi^2\;\bar{T}^{12}}\mathrm{diag}\;(-3, 9, 9, 9)$$

- Near the singularity at $\bar{T} \to 1$ this diverges

$$< T_\mu^\nu > \to \frac{N^2}{512\pi^2\;(\bar{T}-1)^4}\,\mathrm{diag}\;(9, 13, 13, 13)$$

- However here the holographic calculation is not valid

Properties of the gauge theory

- Even though the theory lives on flat space, the dilaton factor is in front of the kinetic term and diverges at the time of bulk singularity.

$$\frac{1}{4}\int d^4x \ \frac{1}{e^{\Phi}} \ \mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]$$

  Normally one would absorb the coupling factor by a field redefinition so that only nonlinear terms involve the coupling,

$$A_\mu(x) \rightarrow e^{\Phi(x)/2} \ A_\mu(x)$$

- The field redefinition will introduce extra stuff involving the derivative of the dilaton

- This could lead to singular pieces in terms which are quadratic in the fields.
- This is because in such terms $\nabla\Phi$ appear without any accompanying factor of e^{Φ}
- The nonlinear terms can involve only factors of $(\nabla^n e^{\Phi})$
- While $(\nabla^n e^{\Phi})$ can be arranged to be finite at the singularity, since e^{Φ} vanishes (or becomes small) here, $\nabla\Phi$ is necessarily large or even infinite.

- The quadratic part of the gauge field lagrangian becomes

$$-F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\left[[(\nabla\Phi)^2 + \nabla^2\Phi]A_\nu A^\nu - (\partial_\mu\Phi)(\partial_\nu\Phi)A^\mu A^\nu + 2\partial_\nu\Phi A^\mu\partial_\mu A^\nu\right]$$

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- We expect that perturbation theory works near the singularity – as far as we can see there is no problem in loop diagrams for correlations near $y^+ \to 0$
- More significantly there is no particle production, because of a null isometry.
- It appears that the gauge theory provides a smooth evolution in the light front time.

- This does not mean that we would be able to calculate in a reliable fashion the nature of the bulk at later times
- In particular it remains to be seen whether there is a good classical space-time at late times.

TIME DEPENDENT BACKGROUNDS

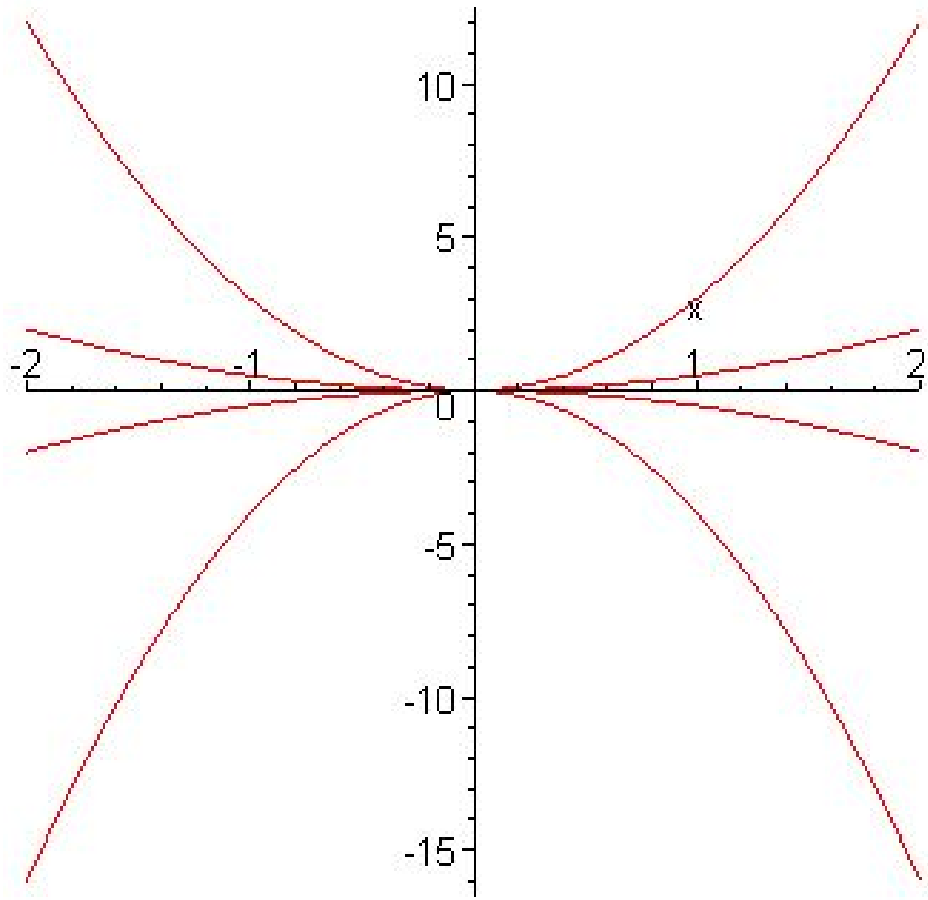
For time dependent dilaton we similarly fix a gauge $A_0 = 0$
However now these terms lead to a tachyonic time dependent mass term for the transverse components.

$$M^2(t) = \frac{1}{4}\left[-(\partial_t \Phi)^2 + 2\partial_t^2 \Phi\right]$$

Since for $t \to 0$ the dilaton is of the form $\Phi = 2\alpha \log(t)$ with $\alpha = \sqrt{3}/2$, we will study the problem for arbitrary α

In this case, the tachyonic mass term increases without bound as we approach the singularity :

$$M^2(t) = -\frac{\alpha^2 + \alpha}{t^2}$$



t = -100

t = -1

- At the classical level this means that the redefined field is driven to infinitely large values in a finite time for generic initial conditions.
- The mode decomposition for a (redefined) transverse gauge field component χ

$$\chi(x,t) = \int \frac{d^3k}{(2\pi)^3} \frac{\sqrt{\pi t}}{2} \left[a \; H^{(2)}_{\alpha+\frac{1}{2}}(kt) \; e^{-ik\cdot x} + c.c. \right]$$

- At early times $t \to -\infty$ we have usual oscillatory behavior

$$\chi(k,t) \to \frac{1}{2k} e^{-ikt}$$

- While near $t \to 0$ the field blows up

$$\chi(k,t) \to t^{-\alpha}$$

- Unless we have very fine tuned initial conditions

$$a = a*$$

- We need to analyze the problem in terms of the original variable $\tilde{\chi} = e^{\Phi/2}\,\chi$ which has a finite limit as $t \to 0$
- Now the modes with momentum may be now written as

$$\begin{aligned}\tilde{\chi}_k(t) &= -\frac{1}{2|k|}\left[a_k\ (-|k|t)^\nu\ H_\nu^{(1)}(-|k|t) + h.c.\right] \quad t<0 \\ &= \frac{1}{2|k|}\left[b_k\ (|k|t)^\nu\ H_\nu^{(2)}(|k|t) + h.c.\right] \quad t>0\end{aligned}$$

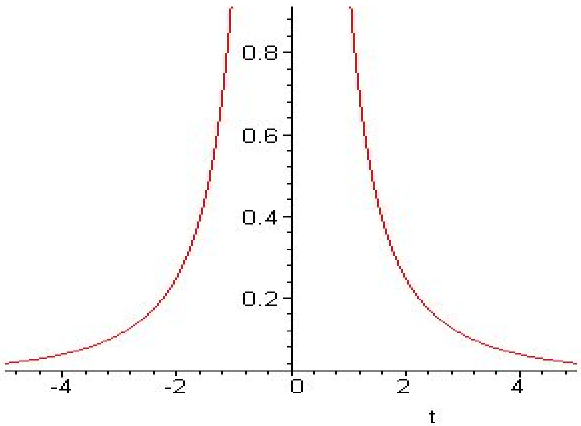
$$\nu \equiv \alpha + \frac{1}{2}$$

- Suppose we consider the state which behaves like the vacuum of the harmonic oscillator with frequency $|k|$ at early times

$$a_k|in>=0$$

- We need to ask if this state is also killed by b_k

- Thinking of -t as a spatial coordinate the problem is identical to that of quantum mechanics in a potential



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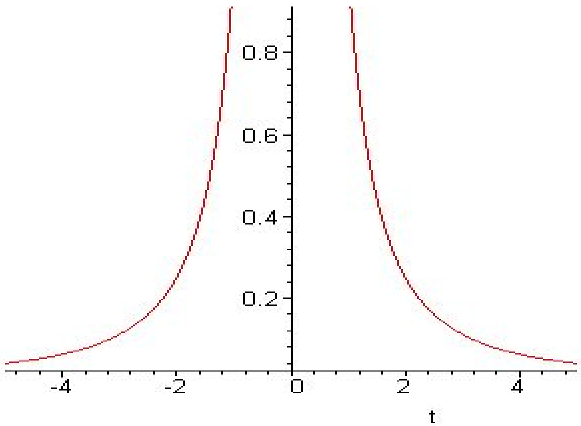

- Particle production in our problem happens if there is a nontrivial reflection from this potential.

- Consider the mode functions

$$
\begin{aligned}
\psi_+(t) &= -(-|k|t)^\nu \; H_\nu^{(1)}(-|k|t) \qquad (t<0) \\
\psi_+(t) &= (|k|t)^\nu \left[A \; H_\nu^{(2)}(|k|t) + B \; H_\nu^{(2)}(|k|t)\right] \qquad t>0
\end{aligned}
$$

- We want to require

$$
\begin{aligned}
\mathrm{Lim}_{\epsilon\to 0}\left[\psi_+(\epsilon) - \psi_-(-\epsilon)\right] &= 0 \\
\mathrm{Lim}_{\epsilon\to 0}\left[\partial_t\psi_+(\epsilon) - \partial_t\psi_-(-\epsilon)\right] &= 0
\end{aligned}
$$

- This determines A and B . Using the relationship

$$
\frac{\partial}{\partial x}\left[x^\nu \; Z_\nu(x)\right] = x^\nu \; Z_{\nu-1}(x)
$$

Where $Z_\nu(x)$ is any of the Bessel functions

- The expression for the coefficient B is

$$B = -i\frac{1}{\sin(\pi\nu)}\frac{J_\nu(|k|\epsilon)J_{\nu-1}(|k|\epsilon) - J_{-\nu}(|k|\epsilon)J_{-\nu+1}(|k|\epsilon)}{J_\nu(|k|\epsilon)J_{-\nu+1}(|k|\epsilon) - J_{-\nu}(|k|\epsilon)J_{\nu-1}(|k|\epsilon)} - i\frac{\cos(\pi\nu)}{\sin(\pi\nu)}$$

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- For $\alpha < 1/2$, $B \rightarrow -i\cot(\pi\nu)$. This means that there is a possible continuation and there is a finite amount of particle production.

- A more careful calculation may be performed by modifying the behavior of the dilaton near $t = 0$

$$
\begin{aligned}
e^{\Phi} &= |t|^{2\alpha} \quad |t| > \epsilon \\
e^{\Phi} &= |\epsilon|^{2\alpha} \quad |t| < \epsilon
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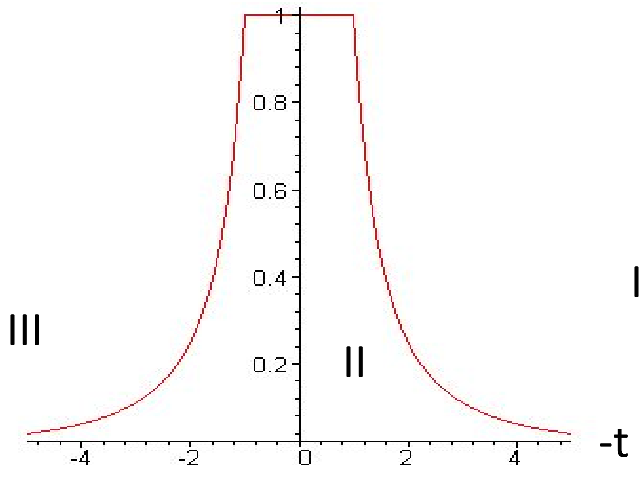
  The analog quantum mechanics problem then has a bounded potential and the scattering problem is well defined



- The solutions to the mode equations are

$$
\begin{aligned}
\tilde{\chi}_I(t) &= (-kt)^\nu \; H_\nu^{(1)}(-kt) \\
\tilde{\chi}_{II}(t) &= A \,\exp[i\frac{kt^{2\alpha+1}}{\epsilon^{2\alpha}\;(2\alpha+1)}] + B \,\exp[-i\frac{kt^{2\alpha+1}}{\epsilon^{2\alpha}\;(2\alpha+1)}] \\
\tilde{\chi}_{III}(t) &= (kt)^\nu \;\left[C \; H_\nu^{(1)}(kt) + D \; H_\nu^{(2)}(kt)\right]
\end{aligned}
$$

- As usual, the Bogoliubov coefficients in region II and III are to be determined by matching the solution and its first derivative across the junctions. Finally we will take the limit $\epsilon \longrightarrow 0$.
- A nonzero value of C will signify particle production

- The final result for C and D are

$$
\begin{aligned}
C &= \frac{\pi i}{4}(k\epsilon)\{\cos(\frac{2k\epsilon}{2\alpha+1})\left[H_{\nu}^{(1)}(k\epsilon)H_{\nu-1}^{(2)}(k\epsilon) + H_{\nu}^{(2)}(k\epsilon)H_{\nu-1}^{(1)}(k\epsilon)\right] \\
&- \sin(\frac{2k\epsilon}{2\alpha+1})\left[H_{\nu-1}^{(1)}(k\epsilon)H_{\nu-1}^{(2)}(k\epsilon) - H_{\nu}^{(2)}(k\epsilon)H_{\nu}^{(1)}(k\epsilon)\right]\}
\end{aligned}
$$

$$
\begin{aligned}
D &= -\frac{\pi i}{4}(k\epsilon)\{2\cos(\frac{2k\epsilon}{2\alpha+1})H_{\nu}^{(1)}(k\epsilon)H_{\nu-1}^{(1)}(k\epsilon) \\
&- \sin(\frac{2k\epsilon}{2\alpha+1})\left[(H_{\nu-1}^{(1)}(k\epsilon))^2 - (H_{\nu}^{(1)}(k\epsilon))^2\right]\}
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\end{aligned}
$$

- The unitarity relation

$$|C|^2-|D|^2=-1$$

may be explicitly checked.

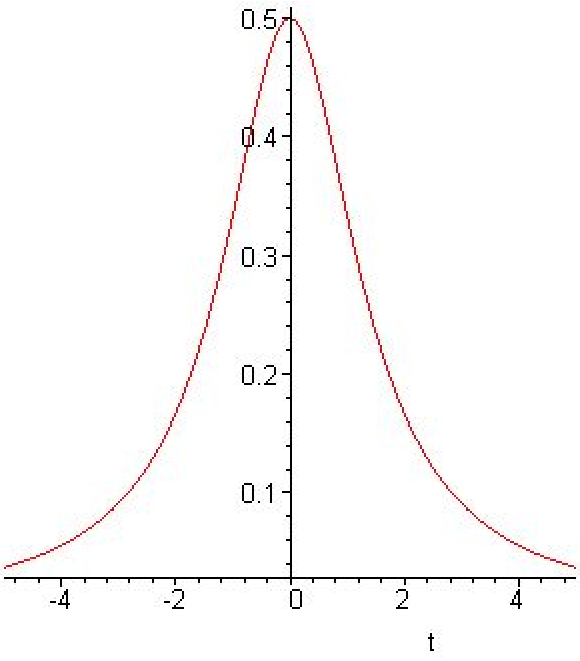
- In the limit $\epsilon \to 0$ there is a big difference between $\alpha < 1/2$ and for $\alpha > 1/2$
- For $\alpha < 1/2$ the coefficients have finite limits

$$C \to -i\frac{\cos(\pi\nu)}{\sin(\pi\nu)} \qquad D \to -i\frac{e^{-i\pi\nu}}{\sin(\pi\nu)}$$

While for $\alpha > 1/2$ both C and D diverge (of course maintaining the unitarity relation).

Thus – at the level of the free theory it appears that there is some meaningful continuation past t=0 for $\alpha < 1/2$,at the cost of FINITE amount of particle production.

- In other regularizations, the Bogoliubov coefficients cannot be computed exactly – however a WKB analysis leads to identical conclusions.



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  Here supergravity is not reliable anyway – the dual is some stringy background : we do not know how to describe that. Since this modification is only near the singularity, we can keep it unchanged at early times – hence maintain our overall scenario.

Schrodinger Picture and Interactions

- To get a better idea and to incorporate the effect of interactions, it is better to switch to the Schrodinger picture and look at the behavior of the wave functional as we approach $t \longrightarrow 0$
- For the state defined earlier,

$$a_k|in >= 0$$

- The wave functional may be easily calculated :

$$\Psi[\tilde{\chi}_k, t] = \prod_k \frac{C}{|f_k(t)| \ |t|^\alpha} \exp \ \left[i|t|^{-2\alpha} \ \frac{\partial_t f_k(t)}{f_k(t)} \ \frac{\tilde{\chi}_k \tilde{\chi}_{-k}}{2} \right]$$

- Where $\quad f_k(t) = (-kt)^\nu \ H_\nu^{(2)}(-kt)$

- At early times, $t \rightarrow -\infty$

$$\frac{\partial_t f_k(t)}{f_k(t)} \sim i\frac{|k|\ |t|^{2\alpha}}{2}$$

- Thus the wavefunctional behaves as

$$\Psi[\tilde{\chi}_k, t] \sim \prod_k \exp\ [-\frac{1}{2}|k|\ \tilde{\chi}_k \tilde{\chi}_{-k}]$$

- This is the standard harmonic oscillator ground state wave function.

- It is straightforward to check that as $t \to 0$

$$\frac{\partial_t f_k(t)}{f_k(t)} \sim 2c\ |t|(1 + O(t^2))$$

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- Therefore for $\alpha < 1/2$, the phase of the wave functional has a smooth limit, whereas for $\alpha > 1/2$ the phase oscillates wildly as we approach the limit $t \to 0$. This prevents a meaningful continuation of the wave functional beyond $t = 0$, reinforcing the conclusion we reached in the Heisenberg picture.

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- Wave functionals for coherent states have an identical behavior.

- The wild oscillations of the phase of the wavefunctional results in diverging expectation values for physical quantities.
- For example the expectation value of the momentum conjugate to $\tilde{\chi}_k$ behaves as

$$\langle \hat{P}^2 \rangle \sim (-t)^{2(1-2\alpha)}$$

- So far we have ignored the interactions. In the presence of interactions we cannot solve for the wave-functional exactly. However we can estimate the behavior near $t = 0$
- The Schrodinger equation is

$$\left[-\frac{1}{2}e^{\Phi(t)}\ \frac{\delta^2}{\delta\tilde{\chi}^2} + e^{-\Phi(t)}\ V(\tilde{\chi}) - i\frac{\partial}{\partial t}\right]\Psi[\tilde{\chi}, t] = 0$$

- Where $$V(\tilde{\chi}) = \frac{1}{2}k^2\tilde{\chi}^2 + \lambda\tilde{\chi}^4$$

- Recall that $e^{\Phi} = |t|^{2\alpha}$. Thus as $t \to 0$ the potential term dominates over the kinetic term. Thus the leading answer is

$$\Psi(\tilde{\chi}, t) \sim \exp\left[iV(\tilde{\chi})\frac{(-t)^{1-2\alpha}}{1-2\alpha}\right]\Psi_0(\tilde{\chi})$$

- When $V(\tilde{\chi})$ is quadratic, this is precisely what we had earlier.
- This answer may be used to check the self-consistency of our approximation.

- However now we realize that in order to obtain the leading behavior, the effect of interactions can be easily incorporated.
- All we need to do is to use the appropriate
- This does not change the power of t in the phase.
- The conclusions remain unchanged : for $\alpha > 1/2$ there is no meaningful evolution past $t = 0$ while for $\alpha < 1/2$ there is.

What does this mean ?

- The particle production calculation looks at the state at late times as compared to the initial state, without bothering about what happens in between.
- However it turns out that regardless of the value of α , the expecation value of the hamiltonian in a generic state becomes very large near $t = 0$, and in fact diverges when we let the dilaton really vanish here.

- This simply follows from the form of the hamiltonian

$$\mathcal{H} = -\frac{1}{2}e^{\Phi(t)} \ \frac{\delta^2}{\delta\tilde{\chi}^2} + e^{-\Phi(t)} \ V(\tilde{\chi})$$

- Since the potential term dominates near $t=0$ we have

$$\langle\mathcal{H}\rangle \sim e^{-\Phi(t)}\langle V(\tilde{\chi})\rangle$$

  In $\langle V(\tilde{\chi})\rangle$ the phase factor of the wave function cancels, and the time dependence is basically governed by the factor $e^{-\Phi(t)}$

- For a generic state $\langle V(\tilde{\chi})\rangle$ is nonzero and $\langle\mathcal{H}\rangle$ diverges.
- Taking into account the fact that there are N^2 degrees of freedom we have typically

$$\langle\mathcal{H}\rangle \sim \frac{N^2}{|t|^{2\alpha}}$$

- This is slower than the supergravity answer we found,

$$\langle\rho\rangle \sim \frac{N^2}{|t|^4}$$

- There are some special states in which $\langle V(\tilde{\chi})\rangle$ vanishes, and the contribution to $\langle \mathcal{H} \rangle$ comes from the kinetic term.
- It turns out that for $\alpha < 1$ these subdominant terms are finite - otherwise they diverge.
- Furthermore for large N the fluctuations $\frac{\langle \mathcal{H}^2 \rangle}{\langle H \rangle^2}$ are suppressed.

- To make sense of the time evolution we need to modify the dilaton profile near $t = 0$, e.g. in ways discussed above.
- This makes time evolution non-singular, and for $\alpha < 1/2$ the amount of particle production is finite and small.
- The time dependent dilaton has resulted in pumping in a lot of energy into the system when $t < 0$ and extracting out a lot of energy for $t > 0$ - leaving with a finite amount of energy.
- One would expect that interactions will lead to thermalization.
- From the bulk point of view this would mean that we have a black hole in the future.
- However calculating the details of the process requires a much more detailed knowledge of the gauge theory at strong coupling.

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- In the regions where the bulk solution becomes singular – and therefore cannot be trusted, the gauge theory dual becomes weakly coupled and therefore is not expected to have a gravity dual in any case
- For null singularities, it appears that the gauge theory evolution may indeed be well defined.
- For space-like singularities, some cases may lead to a smooth evolution, but possibily leads to a black hole.