

Time-dependent AdS/CFT Duality & Null Singularity

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[hep-th/0602054](#) and [0710.2640](#) [hep-th]

Outline

- 1 I. Motivation
- 2 II. Constructing the duality
- 3 III. Quantum SYM
- 4 IV. Holographic duality and singularity
- 5 V. Conclusions

- Our work was motivated by the big crunch cosmology proposal of Hertog-Horowitz (2005) and the matrix string cosmology proposal of Craps-Sethi-Verlinde (2005).
- The big crunch cosmological background is non-supersymmetric. In matrix string cosmology proposal, the matrix string action is not supersymmetric due to the introduction of lightcone momentum p^+ .
- In AdS/CFT, the existence of supersymmetry allows us a better control over the string/supergravity background, and over the quantum and nonperturbative behaviour of the field theory. Therefore our aim was to construct a supersymmetric time-dependent AdS/CFT duality
- Some overlap with works of Das, Michelson, Narayan, Trivedi: 0602107, 0610053; Awad, Das, Narayan, Trivedi: 0711.2994 in the construction of the null SUGRA background and some Yang-Mills analysis.
- I will be focusing on the YM analysis in this talk.

Construction of the time-dependent background

- Consider ansatz

$$ds^2 = \frac{R^2}{u^2} \left(-2dx^+ dx^- + h(u, x^+, x^i)(dx^+)^2 + (dx^i)^2 + du^2 \right) + R^2 d\Omega_5^2,$$

$$F_{\mu\nu\rho\lambda\sigma} = \frac{1}{R} \varepsilon_{\mu\nu\rho\lambda\sigma}, \quad F_{abcde} = \frac{1}{R} \varepsilon_{abcde}, \quad \text{for the } AdS_5\text{-like and } S^5 \text{ part resp.,}$$

$$\phi = \phi(x^+), \quad \chi = \chi(x^+),$$

Background of these kind without dilaton and axion and their AdS/CFT correspondence has been considered before (Brecher, Chamblin, Reall 01). For our purpose it is crucial to turn on ϕ and χ .

- (++)-component of the Einstein equation gives the constraint:

$$R_{++} = \frac{1}{2}(\phi')^2 + \frac{1}{2}e^{2\phi}(\chi')^2 = -h_{22} - h_{33} + \Omega.$$

Here

$$h = h_{ij}x^i x^j + \frac{1}{2}\Omega u^2.$$

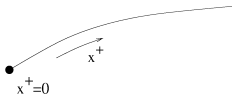
Properties of our solution/ Symmetries

- 1. The soln preserves 8 SUSY, i.e. 1/4 BPS.
- 2. Our supergravity solution is invariant under a scaling transformation :

$$u \rightarrow \lambda u, \quad x^+ \rightarrow x^+, \quad x^- \rightarrow \lambda^2 x^-, \quad x^i \rightarrow \lambda x^i.$$

- 3. All curvature invariants constructed out of the curvature tensor and metric are regular. However there can be geodesic singularity. For example, the curve $\{y^+ = \lambda, \quad y^M = \text{constant}, \quad \forall M \neq +\}$ is a geodesic. On this geodesic, we have the invariant

$$R_{++} = R_{MN} \frac{dy^M}{d\lambda} \frac{dy^N}{d\lambda}$$



Thus a divergence of R_{++} implies that the geodesic has to be terminated there, and signify geodesic incompleteness.

- A simple example is:

$$ds^2 = \frac{R^2}{u^2} \left(-dy^+ dy^- + \sum_i (dy^i)^2 M_i^2(y^+) + du^2 \right) + R^2 d\Omega_5^2.$$

with

$$\phi = \pm 2\sqrt{\alpha(1-\alpha)} \log(y^+), \quad \chi = 0, \quad M_2 = M_3 = (y^+)^\alpha,$$

where $y^+ > 0$ and $0 < \alpha < 1$ in both cases and ϕ_0 is a constant.

- We have a singularity at $y^+ = 0$

$$R_{++} = \frac{2\alpha(1-\alpha)}{(y^+)^2}.$$

Note that there is a vanishing scale factor resembling the big bang.

Construction of the dual SYM

- It is convenient to write the ordinary $\mathcal{N} = 4$ SYM in 10d notation,

$$\mathcal{L}_B = \frac{1}{4g_{\text{YM}}^2} \text{Tr} \left([Y_M, Y_N][Y^M, Y^N] \right), \quad \mathcal{L}_\psi = \frac{1}{2g_{\text{YM}}^2} \text{Tr} \left(\bar{\psi} \Gamma^M [Y_M, \psi] \right),$$

fields: $Y_\mu \equiv -iD_\mu = -i\partial_\mu + A_\mu$, Y^a , and Majorana spinor ψ .
 $\mu = 0, 1, 2, 3$; $a = 4, \dots, 9$.

- Under the SUSY transformation ($M, N = 0, 1, \dots, 9$),

$$\delta Y_\mu = \delta A_\mu = -i\bar{\epsilon} \Gamma_\mu \psi, \quad \delta Y^a = -i\bar{\epsilon} \Gamma^a \psi, \quad \delta \psi = \frac{i}{2} [Y_M, Y_N] \Gamma^{MN} \epsilon,$$

where $\epsilon = \epsilon(x^+)$. We have

$$\delta \mathcal{L}_B + \delta \mathcal{L}_\psi \simeq \frac{2\bar{\epsilon}'}{g_{\text{YM}}^2} \text{Tr} \left(\Gamma^M \psi [Y_M, Y^+] \right) + \text{terms vanish if } \Gamma^+ \epsilon = 0.$$

- Thus we can take ϵ to be constant and satisfies $\Gamma^+ \epsilon = 0$. And the total action is SUSY invariant.

- In general the SUGRA background has a nontrivial axion $\chi(x^+)$, thus the term

$$\mathcal{L}_{\chi B} = \hat{\chi}(x^+) \text{Tr} \left(\frac{1}{4} \varepsilon^{\mu\nu\alpha\beta} [Y_\mu, Y_\nu] [Y_\alpha, Y_\beta] \right),$$

is no longer topological and is not supersymmetric by itself.

- Turns out its superpartner is

$$\mathcal{L}_{\chi F} = \hat{\chi}'(x^+) \text{Tr} \left(\frac{i}{4} \bar{\psi} \Gamma^2 \Gamma^3 \Gamma^+ \psi \right)$$

and the variation of the axionic terms cancels those of the YM terms if $\frac{2}{g_{\text{YM}}^2} \epsilon' = -\hat{\chi}'(x^+) \Gamma^2 \Gamma^3 \epsilon$. That is

$$\epsilon(x^+) = \exp \left(\frac{-1}{2} \int_0^{x^+} dy^+ \hat{\chi}'(y^+) g_{\text{YM}}^2(y^+) \Gamma_{23} \right) \epsilon_0,$$

for a constant spinor $\epsilon_0 : \Gamma^+ \epsilon_0 = 0$.

Duality Map

- We propose that the time-dependent SYM theory is dual to the quantum gravity on the time dependent bulk geometry, with the matching:

$$e^{\phi} = g_{YM}^2, \quad \chi = \chi_{YM}$$

and

$$R^4 = 16\pi N \langle g_s^{-1} \rangle^{-1} l_s^4, \quad \text{where} \quad \langle g_s^{-1} \rangle := \int dx^+ e^{-\phi} / \int dx^+$$

The second relation makes sense provided that $\langle g_s^{-1} \rangle$ is well defined.

- The SUGRA solution can be obtained as a near horizon limit of the SUGRA solution for a stack of D3-branes with pp-wave on it. Consider a stack of N such D3-branes. The relations in the first line can be justified by a Born-Infeld analysis. The second line can be obtained if one equates the mass and charge of the D-branes.

- With the SYM, one can try to understand the spacetime singularity from the SYM point of view.
- In our first paper, we computed the field theory two point functions and find that it is regular at the free level. This is in contrast with the two-point function as determined from the bulk-to-boundary approach, where the bulk singularity appears.
- An interpretation is that one may try to say that the SUGRA singularity is corrected by α' corrections and becomes regular. However this is too naive.
It is more convincing if one could reproduce the spacetime singularity first and then have an understanding on what kind of gauge theory effects may resolve the singularity.

To do this, one needs to be able to detect the properties of the bulk spacetime from the boundary theory.

- Holographic RG flow (de Boer, Verlinde, Verlinde; Skenderis)

One can reconstruct bulk metric as a series expansion from the boundary out of the CFT data by solving the Einstein equation.

- Not helpful here for problems involving spacetime singularity, where the **regularity assumption and Einstein are both questionable**. A different approach is needed here.

- The UV/IR relation is our guidance:

The singularity of the bulk is manifested in the u^2 term of the metric

$$ds^2 = \frac{R^2}{u^2} \left(-2dx^+ dx^- + \frac{1}{2} \Omega(x^+) u^2 (dx^+)^2 + (dx^i)^2 + du^2 \right) + R^2 d\Omega_5^2,$$

To see the interior $u > 0$, it suggests one to introduce a cutoff Λ in the gauge theory since $u \sim 1/\Lambda$.

Setting up the perturbation theory

- Define $A_\mu = g_{\text{YM}} \mathcal{A}_\mu$, $Y^a = g_{\text{YM}} Z^a$, $\psi = g_{\text{YM}} \lambda$, the scalar and fermion action simply becomes

$$S_X = \int d^4x \text{Tr} \left[-\frac{1}{2} \mathcal{D}^\mu Z^a \mathcal{D}_\mu Z^a + \frac{g_{\text{YM}}^2}{4} [Z^a, Z^b]^2 \right],$$

$$S_\Psi = \frac{1}{2} \int d^4x \text{Tr} \left[\bar{\lambda} \Gamma^\mu [-i \mathcal{D}_\mu, \lambda] + g_{\text{YM}} \bar{\lambda} \Gamma^a [Z^a, \lambda] \right],$$

where

$$\mathcal{D}_\mu Z^a = \partial_\mu Z^a + i g_{\text{YM}} [\mathcal{A}_\mu, Z^a] \quad \text{and} \quad \mathcal{D}_\mu \lambda = \partial_\mu \lambda + i g_{\text{YM}} [\mathcal{A}_\mu, \lambda].$$

- The YM term becomes

$$S_{\text{YM}} = \int d^4x \text{Tr} \left[\frac{-1}{4} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} + \frac{g'_{\text{YM}}}{g_{\text{YM}}} \partial_\mu \mathcal{A}^\mu \mathcal{A}_- + a \mathcal{A}_-^2 \right], \quad a := \frac{g'_{\text{YM}}{}^2}{2g_{\text{YM}}^2} - \left(\frac{g'_{\text{YM}}}{g_{\text{YM}}} \right)'$$

- Generalized Lorentz gauge: $\partial_\mu \mathcal{A}^\mu + f(x^+) \mathcal{A}_- = 0$.

$$\text{Gauge fixing term } S_{\text{g.f.}} = \int d^4x \text{Tr} \left[-\frac{1}{2\xi} (\partial_\mu \mathcal{A}^\mu + f(x^+) \mathcal{A}_-)^2 \right].$$

$$\begin{aligned} S_{\text{YM}} + S_{\text{g.f.}} &= \int d^4x \text{Tr} \left[\frac{1}{2} \mathcal{A}_\mu \partial^2 \mathcal{A}^\mu + \frac{1}{2} \left(1 - \frac{1}{\xi}\right) (\partial_\mu \mathcal{A}^\mu)^2 + \left(a - \frac{f^2}{2\xi}\right) \mathcal{A}_-^2 \right. \\ &\quad \left. + \left(\frac{g'_{\text{YM}}}{g_{\text{YM}}} - \frac{f}{\xi}\right) (\partial_\mu \mathcal{A}^\mu) \mathcal{A}_- \right] + \text{cubic and quartic terms.} \end{aligned}$$

- A particular simple gauge choice is $\xi = 1$, $f = g'_{\text{YM}}/g_{\text{YM}}$. In this case,

$$S_{\text{YM}} + S_{\text{g.f.}} = \int d^4x \text{Tr} \left[\frac{1}{2} \mathcal{A}_\mu \partial^2 \mathcal{A}^\mu + \tilde{a} \mathcal{A}_-^2 \right] + \text{cubic and quartic terms,}$$

$$\text{where } \tilde{a} := -\left(\frac{g'_{\text{YM}}}{g_{\text{YM}}}\right)'$$

- Finally the axionic coupling terms become

$$S_{\chi^B} = \int d^4x g_{\text{YM}}^2 \hat{\chi}' \text{Tr} \left[\mathcal{A}_- (\partial_2 \mathcal{A}_3 - \partial_3 \mathcal{A}_2) - \mathcal{A}_2 \partial_- \mathcal{A}_3 + i g_{\text{YM}} \mathcal{A}_- [\mathcal{A}_2, \mathcal{A}_3] \right],$$

$$S_{\chi^F} = \int d^4x g_{\text{YM}}^2 \hat{\chi}' \text{Tr} \left[\frac{-i}{4} \bar{\lambda} \Gamma^2 \Gamma^3 \Gamma_- \lambda \right].$$

These give rise to correction to the propagators of λ and \mathcal{A}_μ and a vertex involving $\mathcal{A}_-, \mathcal{A}_2, \mathcal{A}_3$.

Feynman rule

At the free level, the action is identical to the usual $\mathcal{N} = 4$ SYM, and hence the correlation functions are unmodified and identical to the $\mathcal{N} = 4$ case at the free level.

- The propagators are

$$K_{ab}(x) = \int \frac{d^4 p}{(2\pi)^4} \frac{-i\delta_{ab}}{p^2} e^{ipx}, \quad K_{\mu\nu}(x) = \int \frac{d^4 p}{(2\pi)^4} \frac{-i\eta_{\mu\nu}}{p^2} e^{ipx}, \quad D(x) = \int \frac{d^4 p}{(2\pi)^4} \frac{i\not{p}}{p^2} e^{ipx}.$$

- Most of the vertices are the same as in the usual $\mathcal{N} = 4$ SYM except now the coupling constant is replaced by $g_{\text{YM}}(x^+)$.
- The term

$$\int d^4 x \text{Tr} [\tilde{a} \mathcal{A}_-^2]$$

and axionic terms:

$$\mathcal{L}_2 = \int dx g_{\text{YM}}^2 \hat{\chi}' \text{Tr} (\mathcal{A}_- \partial_2 \mathcal{A}_3 - \mathcal{A}_- \partial_3 \mathcal{A}_2 - \mathcal{A}_2 \partial_- \mathcal{A}_3),$$

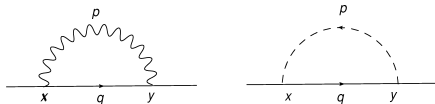
$$\mathcal{L}_3 = \int dx \frac{-i}{4} g_{\text{YM}}^2 \hat{\chi}' \text{Tr} \bar{\lambda} \Gamma_2 \Gamma_3 \Gamma_- \lambda$$

constitute corrections to the propagator.

Sample calculation: Wilsonian action

Let's compute the Wilsonian effective action at one-loop. In particular look at the fermion kinetic term.

- The fermion propagator receives 1-loop contribution from the Feynman diagrams ($\chi = 0$)



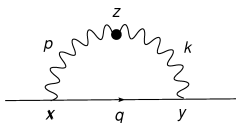
Both diagrams are planar.

- Diagram 1 gives

$$\begin{aligned}
 I_1 &= N \int_{x,y} \text{Tr} i g_{\text{YM}}(x^+) \bar{\lambda}(x) \Gamma^M D_{xy} \Gamma^N i g_{\text{YM}}(y^+) \lambda(y) K_{MN,xy} \\
 &= 4N \int_{x,y,p} \text{Tr} g_{\text{YM}}(x^+) \bar{\lambda}(x) F(p) \not{p} g_{\text{YM}}(y^+) \lambda(y) e^{ip(x-y)}
 \end{aligned}$$

where

$$F(p) := \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(q - p/2)^2 (q + p/2)^2}.$$



- Diagram 2 gives

$$\begin{aligned}
 I_2 &= N \int_{x,y,z} \text{Tr} i g_{\text{YM}}(x^+) \bar{\lambda}(x) \Gamma^+ D_{xy} \Gamma^+ i g_{\text{YM}}(y^+) \lambda(y) K_{+-,xz} 2i \tilde{a}(z^+) K_{+-,zy} \\
 &= 4N \int_{x,y,z,p,k} \text{Tr} g_{\text{YM}}(x^+) \bar{\lambda}(x) \Gamma^+ G(p,k) g_{\text{YM}}(y^+) \lambda(y) \tilde{a}(z^+) e^{ip(x-y)} e^{ik(z-y)},
 \end{aligned}$$

where

$$G(p,k) := \int \frac{d^4 q}{(2\pi)^4} \frac{(q+p/2)_-}{(q+p/2)^2 (q-p/2)^2 (k-q+p/2)^2}.$$

- To obtain Wilsonian effective action, integrate out modes with momentum above the cutoff scale Λ and replace their contribution to low mom modes by introducing new interaction vertices in the effective action.

- The separation of modes into low and high momentum modes is not gauge invariant. There is an issue on gauge invariance of the Wilsonian action.
- We adopt the procedure to impose a cutoff in the loop mom. (Bilal)
All non-gauge invariant terms cancel in the case of supersymmetric gauge theories.
- This gives, e.g.,

$$F_W(p) := \int_0^1 d\alpha \int_{\Lambda}^{\infty} \frac{d^4 q}{(2\pi)^4} \frac{1}{(q^2 + \alpha(1-\alpha)p^2)^2},$$

and

$$\Gamma_{\text{eff}} = \frac{iN}{2\pi^2\Lambda^2} \int d^4 x g_{\text{YM}}'^2 \text{Tr} \bar{\lambda} \gamma^+ \partial^+ \lambda = i \int d^4 x \frac{Ne^\phi}{2\pi\Lambda^2} \phi'^2 \text{Tr} \bar{\lambda} \Gamma^+ \partial^+ \lambda.$$

Holographic reconstruction of bulk metric

- UV/IR relations: (Susskind, Witten; Peet, Polchinski)

$$u \sim \frac{1}{\Lambda}, \quad (\text{supergravity modes})$$

$$u \sim \frac{g_{\text{YM}} N^{1/2}}{\Lambda}. \quad (\text{D-brane probe})$$

- If we use the D-brane probe relation,

$$u = \frac{g_{\text{YM}}(x^+) N^{1/2}}{\Lambda} \frac{1}{\pi},$$

then the Wilsonian effective action becomes

$$\Gamma_{\text{eff}} \approx i \int d^4x \frac{u^2}{8} \phi'^2 \text{Tr} \bar{\lambda} \Gamma^+ \partial^+ \lambda.$$

- On the other hand, a D3-brane placed at arbitrary $u(x^+)$ is half supersymmetric for our background. The induced metric on the worldvolume

$$ds^2 = -2dx^+ dx^- + dx_i^2 + \left(\frac{1}{4} \phi'^2 + u'^2/u^2 \right) u^2 dx^{+2}.$$

- The worldvolume action, considered as an expansion in derivatives, will have the term

$$-\frac{i}{2} \bar{\lambda} \gamma^\mu \partial_\mu \lambda = -\frac{i}{2} \bar{\lambda} \Gamma^a \partial_a \lambda + \frac{i}{4} \hat{g}_{+++} \bar{\lambda} \Gamma^+ \partial^+ \lambda, \quad \hat{g}_{+++} = \phi'^2/2,$$

where γ_μ are the curved space gamma matrices, and Γ^a are the flat space ones:

$$\gamma^- = \Gamma^- + \frac{1}{2} \hat{g}_{+++} \Gamma^+, \quad \gamma^+ = \Gamma^+, \quad \gamma_i = \Gamma_i.$$

- The additional term

$$\frac{i}{4} \hat{g}_{+++} \text{Tr} \bar{\lambda} \Gamma^+ \partial^+ \lambda$$

is precisely equal to the Wilsonian effective action!

- It is interesting that the kinetic part of the one-loop Wilsonian action coincides with the brane probe action.
- In general, the D3-brane probe action cannot be identified with the Wilsonian action. However supersymmetry can sometimes protect certain loop amplitudes. In particular it seems to be the case for the kinetic term of the Wilsonian effective action.
- This leads us to the proposal to identify the metric of bulk spacetime from the kinetic term of the Wilsonian action.

At one loop, this reads

$$g_{++}^{\text{Bulk}} = g_{++}^{\text{YM}}$$

- A similar calculation may be performed at the higher loops.
For example, the metric component generally takes the form:

$$\hat{g}_{++} = a(\phi, u)(\phi')^2,$$

where we have used the UV/IR relation to replace the $1/\Lambda$ dependence with u -dependence.

- In general, following the above procedure, the higher loop corrections in gauge theory gives a 5d metric:

$$ds^2 = \frac{R^2}{u^2} (g_{\mu\nu}^{\text{YM}}(u, x^+) dx^\mu dx^\nu + du^2).$$

Due to the complicated u -dependence, the metric will generally not satisfy the Einstein equation.

The dual supersymmetric Yang-Mills thus provides a framework for computing the quantum corrections to the supergravity action in a systematic manner.

Q. Could singularity be resolved by the higher loops quantum effects ?

Conclusions

1. We proposed a gauge/string duality relation which includes a wide varieties of spacetime, both regular and singular spacetime (with big band or big crunch like singularity). The proposed duality is supersymmetric on both sides.
 2. Spacetime metric can be holographically reconstructed from the kinetic term of the SYM Wilsonian effective action at the one loop order.
 3. Higher order corrections to the kinetic term of the Wilsonian action include quantum corrections (both α' and g_s) to the SUGRA equations of motion. It is not clear yet whether the singularity in these class of background could be resolved.
- It will be nice to learn some generic lesson about cosmology.