On the Weighted Energy-Dissipation principle for gradient flows in metric spaces

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Gradient flows: challenges and new directions

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Outline

- Gradient flows and WED functionals in Hilbert spaces
- Recaps on gradient flows in metric spaces
- The WED approach in metric spaces: difficulties
- A different viewpoint
- Convergence of the WED approximation in the convex case
- Outlook to the nonconvex case

Gradient flows in Hilbert spaces

Setup:

- $(\mathcal{H}, \|\cdot\|)$ (sep.) Hilbert
- $\varphi : \mathcal{H} \to (-\infty, +\infty]$ (proper), l.s.c., convex functional
- $\bar{u} \in \mathrm{dom}(\varphi)$

Gradient flow equation **in the autonomous case**

$$\begin{cases} u'(t) + \partial\varphi(u(t)) \ni 0 & \text{in } \mathcal{H}, \quad t \in (0,T), \\ u(0) = \bar{u} \end{cases} \tag{P}$$

with $\partial\varphi$ **convex-analysis subdifferential of φ:**

$$u \in D(\varphi),\ \xi \in \partial\varphi(u) \ \Leftrightarrow\ \varphi(w) - \varphi(u) \geq \langle \xi, w - u \rangle \quad \forall\, w \in \mathcal{H}$$

Classical results on (P):

Existence, uniqueness, approximation of solutions: **[Kōmura'67, Crandall/Pazy'69, Brézis'73]**

The WED functional for the gradient flow

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For fixed $\varepsilon > 0$, $I_\varepsilon : H^1(0,T;\mathcal{H}) \to (-\infty, \infty]$

$$I_\varepsilon(v) := \int_0^T \mathrm{e}^{-t/\varepsilon} \left(\frac{1}{2}\|v'(t)\|^2 + \frac{1}{\varepsilon}\varphi(v(t)) \right) \mathrm{d}t$$

is the **WED=Weighted Energy Dissipation** functional for (P)

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is the **WED=Weighted Energy Dissipation** functional for (P)

Every $u_\varepsilon \in \mathrm{Argmin}\{I_\varepsilon(v) \ : \ v \in H^1(0,T;\mathcal{H}),\ v(0) = \bar{u}\}$ satisfies

$$\begin{cases} -\varepsilon {u_\varepsilon}''(t) + u_\varepsilon'(t) + \partial\varphi(u_\varepsilon(t)) \ni 0 \text{ in } \mathcal{H},\ t \in (0,T) & \text{(Eul.-Lagr. equation)}, \\ u_\varepsilon(0) = \bar{u} & \text{(initial condition)}, \\ u_\varepsilon'(T) = 0 & \text{(final condition)} \end{cases}$$

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¿¿ **For** $\varepsilon \searrow 0$ $u_\varepsilon \to u$, sol. of (P)?? **YES**, [Mielke/Stefanelli, ESAIM-COCV 2010].

Sketch of the proof (I)

$$\boxed{\min_{v \in H^1(0,T;\mathcal{H}),\ v(0)=\bar{u}} \int_0^T \mathrm{e}^{-t/\varepsilon} \left(\frac{1}{2}\|v'(t)\|^2 + \frac{1}{\varepsilon}\varphi(v(t)) \right) \mathrm{d}t} \Rightarrow -\varepsilon u_\varepsilon''(t) + u_\varepsilon'(t) + \partial\varphi(u_\varepsilon(t)) \ni 0$$

Facts:

- φ convex & l.s.c. $\Rightarrow$ I_ε is (uniformly) convex on $H^1(0,T;\mathcal{H})$ & l.s.c. $\Rightarrow$ I_ε has a minimizer u_ε

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- Crucial estimate on $(u_\varepsilon)_\varepsilon$:

$$\exists\, C>0\ :\ \varepsilon\|u_\varepsilon''\|_{L^2(0,T;\mathcal{H})}+\varepsilon^{1/2}\|u_\varepsilon'\|_{L^\infty(0,T;\mathcal{H})}+\|u_\varepsilon'\|_{L^2(0,T;\mathcal{H})}+\|\partial\varphi(u_\varepsilon)\|_{L^2(0,T;\mathcal{H})} \leq C$$

Proof: Use that $\int_0^T \|-\varepsilon u_\varepsilon'' + u_\varepsilon' + \partial\varphi(u_\varepsilon)\|^2\,\mathrm{d}t = 0$, hence

$$\int_0^T \varepsilon^2\|u_\varepsilon''\|^2\,\mathrm{d}t + \int_0^T \|u_\varepsilon'\|^2\,\mathrm{d}t + \int_0^T \|\partial\varphi(u_\varepsilon)\|^2\,\mathrm{d}t =$$

$$+2\int_0^T(\varepsilon u_\varepsilon'', u_\varepsilon')\,\mathrm{d}t - 2\int_0^T(u_\varepsilon', \partial\varphi(u_\varepsilon))\,\mathrm{d}t + 2\int_0^T(\varepsilon u_\varepsilon'', \partial\varphi(u_\varepsilon))\,\mathrm{d}t$$

handled by arguments relying on the Hilbert structure

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$$\boxed{\min_{v\in H^1(0,T;\mathcal{H}),\ v(0)=\bar u} \int_0^T \mathrm{e}^{-t/\varepsilon}\left(\frac{1}{2}\|v'(t)\|^2 + \frac{1}{\varepsilon}\varphi(v(t))\right)\,\mathrm{d}t} \Rightarrow -\varepsilon u_\varepsilon''(t)+u_\varepsilon'(t)+\partial\varphi(u_\varepsilon(t)) \ni 0$$

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- $(u_\varepsilon)_\varepsilon$ is a **Cauchy sequence in** $\mathrm{C}^0([0,T];\mathcal{H})$
- **passage to the limit in as** $\varepsilon\searrow 0$

Outcome and applications of the results (in the Hilbert case)

The convergence of the WED minimizers (u_ε) to u (unique) solution of the gradient flow is interesting

- ✓ **alternative method for existence** for gradient flow
- ✓ towards **relaxation**: convergence of **approximate WED minimizers**, cf. also **[Conti/Ortiz '08]**
- ✓ **new regularity results** (via interpolation)
- ✓ **WED approximation and time discretization** $\rightsquigarrow$ e.g. **[Mielke/Stefanelli '08]**

WED literature

- Pioneers: [Lions '63/'65], [Oleinik '64], [Kohn/Nirenberg '65], [Lions/Magenes '72]
- Brakke mean curvature flow of varifolds: [Ilmanen '94]
- Existence of periodic solutions to specific gradient flows: [Hirano '94]
- De Giorgi's conjecture for semilinear waves: [De Giorgi '96] $\rightsquigarrow$ [Stefanelli '11], [Serra/Tilli '12, '16], [Liero/Stefanelli '13], [Tentarelli/Tilli '17]
- WED approach to rate-independent evolution: [Mielke/Ortiz 2008], [Mielke/Stefanelli '08]
- WED approach to gradient flows: [Mielke/Stefanelli 2010]: $\rightsquigarrow$ doubly nonlinear equations/nonconvex energies [Spadaro/Stefanelli '11], [Akagi/Stefanelli '10, '11, '14, '16, '17]......
- General parabolic systems/gradient flow of the TV-functional: [Bögelein/Duzaar/Marcellini '14, '15, '17]

♣ **General theoretical frame**: approach via minimization of functionals on spaces of trajectories: [Brézis-Ekeland-Nayroles principle '76], [Ghoussoub 2008], [Stefanelli 2008–, Visintin 2008–]...

The WED approach to metric gradient flows

♣ **Gradient flows in metric spaces:**

- **[De Giorgi, Marino, Saccon, Tosques, Degiovanni, Ambrosio '80 $\sim$ '90]...[Ambrosio/Gigli/Savaré '05]** $\rightsquigarrow$ theory of **Curves of Maximal Slope** and **Minimizing Movements**
- **[Benamou, Brenier, Jordan/Kinderlehrer/Otto, Otto, McCann, Carrillo, Sturm, Villani.........]** $\rightsquigarrow$ gradient flows in Wasserstein spaces, heat flow in metric-measure spaces...

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Our motivation for the WED approach:

alternative proof of existence of curves of maximal slope

♣ With Savaré/Segatti/Stefanelli, **A different functional: WED on** $(0,+\infty)$

$$u_\varepsilon \in \mathrm{Argmin}\left\{ \int_0^{+\infty} \mathrm{e}^{-t/\varepsilon} \left(\frac{1}{2}\|v'(t)\|^2 + \frac{1}{\varepsilon}\varphi(v(t)) \right) \mathrm{d}t \right\} \rightsquigarrow \text{converg. as } \varepsilon \downarrow 0$$

with derivatives replaced by **scalar "surrogates"**. **WED functional has a "scalar character"**, too, suited to analysis in metric spaces.

⇝ links with the theory of **optimal control**, dynamic programming principle

Heuristics for the metric formulation of gradient flows

Suppose $\varphi : \mathcal{H} \to (-\infty, +\infty]$ **smooth**

$$u'(t) + \mathrm{D}\varphi(u(t)) = 0 \Leftrightarrow \|u'(t) + \mathrm{D}\varphi(u(t))\|^2 = 0$$

$$\Leftrightarrow \|u'(t)\|^2 + \|\mathrm{D}\varphi(u(t))\|^2 + \underbrace{2(u'(t), \mathrm{D}\varphi(u(t)))}_{\text{chain rule}} = 0$$

$$\Leftrightarrow \|u'(t)\|^2 + \|\mathrm{D}\varphi(u(t))\|^2 + 2\frac{\mathrm{d}}{\mathrm{d}t}\varphi(u(t)) = 0$$

$\Rightarrow$ **equivalent formulation** as energy-dissipation equality:

$$\frac{d}{dt}\varphi(u(t)) = -\frac{1}{2}\|u'(t)\|^2 - \frac{1}{2}\|\mathrm{D}\varphi(u(t))\|^2$$

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$$

involving **norms of derivatives,** rather than derivatives!! $\rightsquigarrow$ OK for metric spaces, with **“surrogate notions” of (norms of) derivatives**.

"Derivatives" in metric spaces (I)

- **Ambient:** (X, d) complete

Metric derivative & geodesics

Given $u : [0, T] \to X$ absolutely continuous, its metric derivative is

$$|u'|(t) := \lim_{h\to 0} \frac{d(u(t), u(t+h))}{|h|} \quad \text{for a.a. } t \in (0, T),$$

$$\boxed{\|u'(t)\| \ \rightsquigarrow \ |u'|(t)}$$

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A curve $\gamma : [0, 1] \to X$ is a (constant-speed) **geodesic** if:

$$\frac{d(\gamma(s), \gamma(t))}{|t - s|} = d(\gamma(0), \gamma(1)) \doteq |\gamma'| \quad \forall\, s, t \in [0, 1].$$

"Derivatives" in metric spaces (II)

- **Ambient:** $\boxed{(X, d) \text{ complete}}$

Local slope & Chain rule

- Given $\varphi : X \to (-\infty, +\infty]$ (proper), and $u \in D(\varphi)$, the **local slope** of φ at u is

$$|\partial\varphi|(u) := \limsup_{v \to u} \frac{(\varphi(u) - \varphi(v))^+}{d(u, v)} \quad u \in D(\varphi)$$

$$\boxed{\| - \mathrm{D}\varphi(u)\| \rightsquigarrow |\partial\varphi|(u)}$$

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- We say that $\varphi : X \to (-\infty, +\infty]$ satisfies the **chain rule** w.r.t. $|\partial\varphi|$ if
  - $\forall\, v \in \mathrm{AC}([0, T]; D(\varphi))$, the map $t \mapsto \varphi(v(t))$ is **absolutely continuous**,
  - and

$$-\frac{\mathrm{d}}{\mathrm{d}t}\varphi(v(t)) \leq |v'|(t)\; |\partial\varphi|(v(t)) \quad \text{for a.a. } t \in (0, T).$$

Metric formulation of gradient flows

Curves of Maximal Slope

An **absolutely continuous** curve $u : [0, T] \to X$ is a **Curve of Maximal Slope** for φ (w.r.t. the local slope) if

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\varphi(u(t)) = -\frac{1}{2}|u'|^2(t) - \frac{1}{2}|\partial\varphi|^2(u(t)) \quad \text{a.e. in } (0, T).} \tag{CMS}$$

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- In view of chain rule $\boxed{-\frac{\mathrm{d}}{\mathrm{d}t}\varphi\circ v \leq |v'|\,|\partial\varphi(v)|}$, (CMS) is equivalent to the

inequality

$$\frac{d}{dt}\varphi(u(t)) \leq -\frac{1}{2}|u'|^2(t) - \frac{1}{2}|\partial\varphi|^2(u(t)) \quad \text{a.e. in } (0,T).$$

Existence for Curves of Maximal Slope

Theorem **[Ambrosio/Gigli/Savaré '05]**

IF $\varphi : X \to (-\infty, +\infty]$ (proper, l.s.c)

1. is "coercive" $\sim$ φ has compact sublevels

2. the local slope $u \mapsto |\partial\varphi|(u)$ is l.s.c.

3. φ satisfies the chain rule w.r.t. $|\partial\varphi|$

THEN $\forall\ \bar{u} \in D(\phi)$ there **exists** a Curve of Maximal Slope u for φ, with $u(0) = \bar{u}$.

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THEN $\forall\ \bar{u} \in D(\phi)$ there **exists** a Curve of Maximal Slope u for φ, with $u(0) = \bar{u}$.

A sufficient condition for properties (2) & (3) is

$$\boxed{\varphi \text{ is } \lambda\text{-geodesically convex for } \lambda \in \mathbb{R}}$$

i.e. for all u_0, $u_1 \in D(\varphi)$ there exists a geodesic $\gamma : [0,1] \to X$ such that $\gamma(0) = u_0$, $\gamma(1) = u_1$ and

$$\varphi(\gamma(\theta)) \leq (1-\theta)\varphi(\gamma(0)) + \theta\varphi(\gamma(1)) - \frac{1}{2}\lambda\theta(1-\theta)d^2(u_0, u_1) \quad \forall\, \theta \in [0,1].$$

WED functionals on $(0, +\infty)$

Basic assumption on $\varphi : X \to (-\infty, +\infty]$

φ is l.s.c., bounded below, φ has compact sublevels

♣ For fixed $\varepsilon > 0$, consider $\mathcal{I}_\varepsilon : \mathrm{AC}(0, +\infty; X) \to (-\infty, \infty]$

$$\mathcal{I}_\varepsilon(v) := \int_0^{+\infty} \mathrm{e}^{-t/\varepsilon} \left(\frac{1}{2}|v'|^2(t) + \frac{1}{\varepsilon}\varphi(v(t)) \right) \mathrm{d}t$$

with $|v'|$ **metric derivative** of v.

Fact (I)

$$\min \{\mathcal{I}_\varepsilon(v) \; : \; v \in \mathrm{AC}(0, +\infty; X), \; v(0) = \bar{u}\} \text{ has at least a solution } u_\varepsilon$$

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Indeed, let (v_n) be an infimizing sequence:

- (integral) estimate for $\varphi(v_n)$ + coercivity of $\varphi \Rightarrow$ "compactness" for (v_n)
- (integral) estimate for $|v_n'| \Rightarrow$ "equicontinuity" for (v_n)

$\Rightarrow$ Ascoli-Arzelà type thm. $\rightsquigarrow$ up to a subseq., $v_n \to u_\varepsilon$ WED-min.

The metric inner variation equation

$$u_\varepsilon \in \mathrm{Argmin}\left\{ \int_0^{+\infty} \mathrm{e}^{-t/\varepsilon} \left(\frac{1}{2}|v'|^2(t) + \frac{1}{\varepsilon}\varphi(v(t)) \right) \mathrm{d}t \ : \ v \in \mathrm{AC}(0,+\infty;X), \ v(0) = \bar{u} \right\}$$

- Euler-Lagrange equation in the Hilbert case:

$$-\varepsilon {u_\varepsilon}''(t) + u_\varepsilon'(t) + \partial\varphi(u_\varepsilon(t)) \ni 0 \text{ in } \mathcal{H}, \ t \in (0,+\infty) \qquad \text{(EUL)}$$

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Fact (II)

u_ε fulfils the **metric inner variation** equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\varphi(u(t)) - \frac{\varepsilon}{2}|u_\varepsilon'|^2(t) \right) + |u_\varepsilon'|^2(t) = 0 \quad \text{for a.a. } t \in (0,+\infty)$$

viz. the **metric version** of (EUL).

The metric inner variation equation

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- Euler-Lagrange equation in the Hilbert case:

$$\underbrace{-\varepsilon u_\varepsilon''(t) \times u_\varepsilon'(t)}_{\frac{\mathrm{d}}{\mathrm{d}t}(-\varepsilon \|u_\varepsilon'(t)\|^2)} + u_\varepsilon'(t) \times u_\varepsilon'(t) + \underbrace{\partial\varphi(u_\varepsilon(t)) \times u_\varepsilon'(t)}_{\frac{\mathrm{d}}{\mathrm{d}t}\varphi(u_\varepsilon(t))} = 0 \quad t \in (0,+\infty) \tag{EUL}$$

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viz. the **metric version** of (EUL).

Analytical difficulties (I)

From $\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\varphi(u(t)) - \frac{\varepsilon}{2}|u'|^2(t)\right) + |u'|^2(t) = 0}$ we get the *energy inequality*

$$\varphi(u_\varepsilon(t)) - \frac{\varepsilon}{2}|u_\varepsilon'|^2(t) + \int_0^t |u_\varepsilon'|^2(s)\,\mathrm{d}s \le \varphi(\bar{u}) \quad \text{for all } t > 0 \qquad \text{(ENE)}$$

which is **NOT sufficient** for taking the limit $\varepsilon \searrow 0$!!

♠ Lack of

- **pointwise** (*or integral*) estimates for $\varphi(u_\varepsilon)$ $\rightsquigarrow$ "compactness"
- **integral** estimates for $|u_\varepsilon'|$ $\rightsquigarrow$ "equicontinuity"

& estimates in [Mielke/Stefanelli, ESAIM-COCV 2010] not doable here!!

♢ **Problem 1:** **further estimates for u_ε**

Analytical difficulties (II)

$$\varphi(u_\varepsilon(t)) - \frac{\varepsilon}{2}|u_\varepsilon'|^2(t) + \int_0^t |u_\varepsilon'|^2(s)\,\mathrm{d}s \leq \varphi(\bar{u}) \quad \text{per ogni } t > 0 \qquad \text{(ENE)}$$

♠ (ENE) does not have the "right structure": to conclude that $u_\varepsilon \to$ **Curve of Maximal Slope** for φ, we should

$$\varphi(u_\varepsilon(t)) - \frac{\varepsilon}{2}|u_\varepsilon'|^2(t) + \int_0^t |u_\varepsilon'|^2(s)\,\mathrm{d}s \leq \varphi(\bar{u})$$

$$\downarrow \quad \text{as } \varepsilon \searrow 0$$

$$\varphi(u(t)) + \frac{1}{2}\int_0^t |u'|^2(s)\,\mathrm{d}s + \frac{1}{2}\int_0^t |\partial\varphi|^2(u(s))\,\mathrm{d}s \leq \varphi(\bar{u})$$

BUT (ENE) does not contain information on $|\partial\varphi|(u)$!!

♢ **Problem 2:** deduce **another energy inequality** where pass to lim. $\varepsilon \searrow 0$

A new viewpoint

Study the **Value Functional**

$$V_\varepsilon(\bar{u}) := \min\left\{\mathcal{J}_\varepsilon(v) \ : \ v \in \mathrm{AC}(0,+\infty;X), \ v(0)=\bar{u}\right\}$$
$$= \min\left\{\int_0^{+\infty} \mathrm{e}^{-t/\varepsilon}\left(\frac{1}{2}|v'|^2(t) + \frac{1}{\varepsilon}\varphi(v(t))\right)\mathrm{d}t \ : \ v \in \mathrm{AC}(0,+\infty;X), \ v(0)=\bar{u}\right\}$$

♣ **Remark**: $\boxed{V_\varepsilon(\bar{u}) \uparrow \varphi(\bar{u}) \ \text{ as } \varepsilon \searrow 0}$

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$$V_\varepsilon(\bar{u}) := \min\left\{\mathcal{J}_\varepsilon(v)\ :\ v \in \mathrm{AC}(0,+\infty;X),\ v(0)=\bar{u}\right\}$$
$$= \min\left\{\int_0^{+\infty} \mathrm{e}^{-t/\varepsilon}\left(\frac{1}{2}|v'|^2(t) + \frac{1}{\varepsilon}\varphi(v(t))\right)\mathrm{d}t\ :\ v \in \mathrm{AC}(0,+\infty;X),\ v(0)=\bar{u}\right\}$$

♣ **Remark**: $V_\varepsilon(\bar{u}) \uparrow \varphi(\bar{u})$ as $\varepsilon \searrow 0$

A new argument for taking the limit as $\varepsilon \searrow 0$

- Use the **Dynamic Programming Principle** for V_ε and deduce that for all $\varepsilon > 0$

  u_ε is a Curve of Maximal Slope for V_ε

  if **φ is λ-geodesically convex**

- take the limit as $\varepsilon \searrow 0$ in **in the gradient flow equation of V_ε** instead of the metric inner variation equation for the WED minimization

The Dynamic Programming Principle in our context

♣ It can be shown that for all $t > 0$

$$V_\varepsilon(\bar{u}) = \min_{v \in \mathrm{AC}(0,+\infty;X),\ v(0)=\bar{u}} \left[\int_0^t \mathrm{e}^{-s/\varepsilon} \left(\frac{1}{2}|v'|^2(s) + \frac{1}{\varepsilon}\varphi(v(s)) \right) \mathrm{d}s + V_\varepsilon(v(t))\mathrm{e}^{-t/\varepsilon} \right]$$

viz., to achieve the minimum cost $V_\varepsilon(\bar{u})$ it is necessary & sufficient to:

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viz., to achieve the minimum cost $V_\varepsilon(\bar{u})$ it is necessary & sufficient to:

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viz., to achieve the minimum cost $V_\varepsilon(\bar{u})$ it is necessary & sufficient to:

1. let the system evolve on a (arbitrary) time-interval $[0, t]$, along a **(arbitrary) trajectory** v
2. pay the **corresponding cost**

The Dynamic Programming Principle in our context

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The Dynamic Programming Principle in our context

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1. let the system evolve on a (arbitrary) time-interval $[0, t]$, along a **(arbitrary) trajectory** v
2. pay the **corresponding cost**
3. pay what remains to pay in an **optimal way**
4. **minimize** over all possible trajectories

The Dynamic Programming Principle in our context

From

$$\boxed{V_\varepsilon(\bar{u}) = \min_{v \in \mathrm{AC}(0,+\infty;X),\ v(0)=\bar{u}} \left[\int_0^t \mathrm{e}^{-s/\varepsilon} \left(\frac{1}{2}|v'|^2(s) + \frac{1}{\varepsilon}\varphi(v(s)) \right) \mathrm{d}s + V_\varepsilon(v(t))\mathrm{e}^{-t/\varepsilon} \right]}$$

one deduces that $u_\varepsilon \in \mathrm{Argmin}_{v(0)=\bar{u}} \mathcal{I}_\varepsilon(v)$ fulfils

$$V_\varepsilon(u_\varepsilon(0)) - V_\varepsilon(u_\varepsilon(t))\mathrm{e}^{-t/\varepsilon} = \int_0^t \mathrm{e}^{-s/\varepsilon} \left(\frac{1}{2}|u_\varepsilon'|^2(s) + \frac{1}{\varepsilon}\varphi(u_\varepsilon(s)) \right) \mathrm{d}s \quad \text{for all } t > 0$$

The Dynamic Programming Principle in our context

From

$$\boxed{V_\varepsilon(\bar{u}) = \min_{v \in \mathrm{AC}(0,+\infty;X),\ v(0)=\bar{u}} \left[\int_0^t \mathrm{e}^{-s/\varepsilon} \left(\frac{1}{2}|v'|^2(s) + \frac{1}{\varepsilon}\varphi(v(s)) \right) \mathrm{d}s + V_\varepsilon(v(t))\mathrm{e}^{-t/\varepsilon} \right]}$$

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whence the **fundamental identity**

$$\boxed{-\frac{\mathrm{d}}{\mathrm{d}t} V_\varepsilon(u_\varepsilon(t)) = \frac{1}{2}|u_\varepsilon'|^2(s) + \frac{1}{\varepsilon}\varphi(u_\varepsilon(t)) - \frac{1}{\varepsilon}V_\varepsilon(u_\varepsilon(t)) \quad \text{for a.a. } t \in (0,T).}$$

.... almost the definition of curve of maximal slope for V_ε....

A Hamilton-Jacobi type equation in a metric context

For all $\varepsilon > 0$ there holds

$$\boxed{\frac{1}{\varepsilon}\varphi(\bar{u}) - \frac{1}{\varepsilon}V_\varepsilon(\bar{u}) = \frac{1}{2}|\partial V_\varepsilon|^2(\bar{u}) \quad \text{for all } \bar{u} \in \mathrm{D}(\varphi)} \tag{H-J}$$

- Why *Hamilton-Jacobi*? It's in the form

$$\frac{1}{\varepsilon}V_\varepsilon(x) + \mathcal{H}(x, |\partial V_\varepsilon|(x)) = 0 \quad \text{with the Hamiltornian } \mathcal{H}(x,p) := \frac{1}{2}|p|^2 - \frac{1}{\varepsilon}\varphi(x)$$

- Just a 'scalar relation', **while [Ambrosio/Feng 2014], [Gangbo/Świech 2015]**: viscosity solutions for metric HJ

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- Just a 'scalar relation', **while [Ambrosio/Feng 2014], [Gangbo/Świech 2015]**: viscosity solutions for metric HJ

♣ The proof of $\boxed{\leq}$ only uses lower semicontinuity and coercivity of φ & the Dynamic Programming Principle

♠ The proof of $\boxed{\geq}$ instead relies on

$$\boxed{\varphi \ \lambda\text{-geodesically convex on } X}$$

The gradient flow of the WED minimizers

♣ Let $u_\varepsilon \in \mathrm{Argmin}_{v(0)=\bar{u}} \mathcal{J}_\varepsilon(v)$: Combining

$$-\frac{\mathrm{d}}{\mathrm{d}t} V_\varepsilon(u_\varepsilon(t)) = \frac{1}{2}|u_\varepsilon'|^2(t) + \frac{1}{\varepsilon}\varphi(u_\varepsilon(t)) - \frac{1}{\varepsilon} V_\varepsilon(u_\varepsilon(t)) \quad \text{for a.a. } t \in (0,T).$$

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we conclude

$$\boxed{-\frac{\mathrm{d}}{\mathrm{d}t} V_\varepsilon(u_\varepsilon(t)) = \frac{1}{2}|u_\varepsilon'|^2(t) + \frac{1}{2}|\partial V_\varepsilon|^2(u_\varepsilon(t)) \quad \text{for a.a. } t \in (0,T)} \qquad (\mathrm{EQV})$$

viz. for all $\varepsilon > 0$, u_ε **is a Curve of Maximal Slope for the Value functional** V_ε.

✓ It's in (EQV) that one has to take the limit as $\varepsilon \searrow 0$!!!

Convergence in the convex case

Assumptions on φ

- $\varphi : X \to (-\infty, +\infty]$ bounded below, l.s.c., **coercive**,
- $\varphi : X \to (-\infty, +\infty]$ λ-geodesically **convex**, $\lambda \in \mathbb{R}$

TO DOs:

- a priori estimates on seq. $(u_\varepsilon)_\varepsilon$ of WED minimizers
- limit passage as $\varepsilon \downarrow 0$ in

$$\boxed{-\frac{\mathrm{d}}{\mathrm{d}t} V_\varepsilon(u_\varepsilon(t)) = \frac{1}{2}|u'_\varepsilon|^2(t) + \frac{1}{2}|\partial V_\varepsilon|^2(u_\varepsilon(t)) \quad \text{for a.a. } t \in (0,T)} \qquad (\mathrm{EQV})$$

From the gradient flow of V_ε to the gradient flow of φ

- **Passage to the limit as $\varepsilon \searrow 0$** in

$$\frac{1}{2}\int_0^t |u_\varepsilon'|^2(s)\,\mathrm{d}s + \frac{1}{2}\int_0^t |\partial V_\varepsilon|^2(u_\varepsilon(s))\,\mathrm{d}s + V_\varepsilon(u_\varepsilon(t)) = V_\varepsilon(\bar{u})$$

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A priori estimates:

1. **integral equicontinuity**

$$\int_0^t |u_\varepsilon'|^2(s)\,\mathrm{d}s \leq C$$

From the gradient flow of V_ε to the gradient flow of φ

- **Passage to the limit as $\varepsilon \searrow 0$** in

$$\frac{1}{2}\int_0^t |u_\varepsilon'|^2(s)\,\mathrm{d}s + \frac{1}{2}\int_0^t |\partial V_\varepsilon|^2(u_\varepsilon(s))\,\mathrm{d}s + V_\varepsilon(u_\varepsilon(t)) = V_\varepsilon(\bar{u}) \leq \varphi(\bar{u}) \leq C$$

A priori estimates:

1. **integral equicontinuity**

$$\int_0^t |u_\varepsilon'|^2(s)\,\mathrm{d}s \leq C$$

2. From $\int_0^t |u_\varepsilon'|^2(s)\,\mathrm{d}s \leq C$ & $\frac{1}{\varepsilon}(\phi(u_\varepsilon) - V_\varepsilon(u_\varepsilon)) = \frac{1}{2}|u_\varepsilon'|^2$ we get

$$\int_0^t \phi(u_\varepsilon(s))\,\mathrm{d}s \leq C \qquad \leftarrow \quad \textbf{integral compactness}$$

From the gradient flow of V_ε to the gradient flow of φ

- **Passage to the limit as $\varepsilon \searrow 0$** in

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3. a version of **Ascoli-Arzelà Theorem** (in metric spaces):

$$\exists\,(u_{\varepsilon_k})_k,\ \exists\, u \in \mathrm{AC}(0,+\infty;X)\ :\quad u_{\varepsilon_k}(t) \to u(t) \quad \forall\, t \in (0,+\infty).$$

From the gradient flow of V_ε to the gradient flow of φ

lim. $\varepsilon \searrow 0$ of $\boxed{\frac{1}{2}\int_0^t |u_\varepsilon'|^2(s)\,\mathrm{d}s + \frac{1}{2}\int_0^t |\partial V_\varepsilon|^2(u_\varepsilon(s))\,\mathrm{d}s + V_\varepsilon(u_\varepsilon(t)) = V_\varepsilon(\bar{u})}$

♣ It can be checked that

$$\begin{aligned}
\frac{1}{2}\int_0^t |u'|^2(s)\,\mathrm{d}s &\leq \liminf_{k\to\infty} \frac{1}{2}\int_0^t |u_{\varepsilon_k}'|^2(s)\,\mathrm{d}s \\
\varphi(u(t)) &\leq \liminf_{k\to\infty} V_{\varepsilon_k}(u_{\varepsilon_k}(t)), \quad V_{\varepsilon_k}(\bar{u}) \to \varphi(\bar{u}) \\
\frac{1}{2}\int_0^t |\partial\varphi|^2(u(s))\,\mathrm{d}s &\leq \liminf_{k\to\infty} \frac{1}{2}\int_0^t |\partial V_\varepsilon|^2(u_{\varepsilon_k}(s))\,\mathrm{d}s
\end{aligned}$$

From the gradient flow of V_ε to the gradient flow of φ

lim. $\varepsilon \searrow 0$ of
$$\boxed{\frac{1}{2}\int_0^t |u_\varepsilon'|^2(s)\,\mathrm{d}s + \frac{1}{2}\int_0^t |\partial V_\varepsilon|^2(u_\varepsilon(s))\,\mathrm{d}s + V_\varepsilon(u_\varepsilon(t)) = V_\varepsilon(\bar{u})}$$

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\end{aligned}$$

- Hence $u \in \mathrm{AC}(0,+\infty;X)$ fulfils for all $t \in (0,+\infty)$

$$\frac{1}{2}\int_0^t |u'|^2(s)\,\mathrm{d}s + \frac{1}{2}\int_0^t |\partial\varphi|^2(u(s))\,\mathrm{d}s + \varphi(u(t)) \leq \varphi(\bar{u})$$

equivalent (via chain rule) to
$$\boxed{-\frac{\mathrm{d}}{\mathrm{d}t}\varphi(u(t)) = \frac{1}{2}|u'|^2(t) + \frac{1}{2}|\partial\varphi|^2(u(t)) \quad \text{a.e. in } (0,+\infty)}$$

The main result in the (geodesically) convex xase

Theorem 1 [R./Savaré/Segatti/Stefanelli, JMPA 2018]

Suppose that $\varphi : X \to (-\infty, +\infty]$ is

- bounded below, l.s.c., **coercive**,
- λ-geodesically **convex**, $\lambda \in \mathbb{R}$

let $\bar{u} \in D(\varphi)$.

Then,

any sequence of WED minimizers (u_ε) with $u_\varepsilon(0) = \bar{u}$
has a subsequence (u_{ε_k}) which converges as $\varepsilon_k \searrow 0$ to
$u \in \mathrm{AC}(0, +\infty; X)$, Curve of Maximal Slope for φ such that $u(0) = \bar{u}$.

Can we do without (geodesic-)convexity?

Geodesic convexity used for $\boxed{\geq}$ in metric Hamilton-Jacobi:

$$\frac{1}{\varepsilon}\varphi(\bar{u}) - \frac{1}{\varepsilon}V_\varepsilon(\bar{u}) = \frac{1}{2}|\partial V_\varepsilon|^2(\bar{u}) \quad \text{for all } \bar{u} \in X$$

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Without convexity, via Dynamic Programming Principle we only have

$$\boxed{-\frac{\mathrm{d}}{\mathrm{d}t}V_\varepsilon(u_\varepsilon(t)) = \frac{1}{2}|u'_\varepsilon|^2(s) + \frac{1}{\varepsilon}\varphi(u_\varepsilon(t)) - \frac{1}{\varepsilon}V_\varepsilon(u_\varepsilon(t)) \quad \text{for a.a. } t \in (0,T)}$$

whence

$$\int_0^t \frac{1}{2}|u'_\varepsilon|^2(s)\,\mathrm{d}s + \int_0^t \frac{1}{\varepsilon}\left(\varphi(u_\varepsilon(s)) - V_\varepsilon(u_\varepsilon(s))\right)\mathrm{d}s + V_\varepsilon(u_\varepsilon(t)) = V_\varepsilon(u_\varepsilon(0)) \qquad (\star)$$

¿¿ Pass to the limit $\varepsilon \to 0$ directly in $(\star)$??

Passage to the limit without (geodesic-)convexity

lim. $\varepsilon \searrow 0$ of $\boxed{\int_0^t \frac{1}{2}|u_\varepsilon'|^2(s)\,\mathrm{d}s + \int_0^t \frac{1}{\varepsilon}\left(\varphi(u_\varepsilon(s)) - V_\varepsilon(u_\varepsilon(s))\right)\mathrm{d}s + V_\varepsilon(u_\varepsilon(t)) = V_\varepsilon(\bar{u})}$

It can be shown that

$$\frac{1}{2}\int_0^t |\partial^-\varphi|^2(u(s))\,\mathrm{d}s \leq \liminf_{k\to\infty} \int_0^t \frac{1}{\varepsilon_k}\left(\varphi(u_{\varepsilon_k}(s)) - V_\varepsilon(u_{\varepsilon_k}(s))\right)\,\mathrm{d}s$$

with the **relaxed slope**

$$|\partial^-\varphi|(u) := \inf\{\liminf_{n\to\infty} |\partial\varphi|(u_n) \ : \ u_n \to u, \ \sup_n \varphi(u_n) < \infty\}$$

Passage to the limit without (geodesic-)convexity

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- Hence $u \in \mathrm{AC}(0,+\infty;X)$ fulfils for all $t \in (0,+\infty)$

$$\frac{1}{2}\int_0^t |u'|^2(s)\,\mathrm{d}s + \frac{1}{2}\int_0^t |\partial^-\varphi|^2(u(s))\,\mathrm{d}s + \varphi(u(t)) \leq \varphi(\bar{u})$$

equiv. (**IF** chain rule w.r.t. $|\partial^-\varphi|$) to $\boxed{-\frac{\mathrm{d}}{\mathrm{d}t}\varphi(u(t)) = \frac{1}{2}|u'|^2(t) + \frac{1}{2}|\partial^-\varphi|^2(u(t))}$

A result for nonconvex energies

Theorem 2 **[R./Savaré/Segatti/Stefanelli, JMPA 2018]**

Suppose that $\varphi : X \to (-\infty, +\infty]$ is

- bounded below, l.s.c., **coercive**,
- **chain rule w.r.t.** $|\partial^- \varphi|$

let $\bar{u} \in D(\varphi)$.

Then,

any sequence of WED minimizers (u_ε) with $u_\varepsilon(0) = \bar{u}$

has a subsequence (u_{ε_k}) which converges as $\varepsilon_k \searrow 0$ to

$u \in \mathrm{AC}(0, +\infty; X)$, Curve of Maximal Slope for φ **w.r.t.** $|\partial^- \varphi|$, such that $u(0) = \bar{u}$.

Summary and conclusions

♡ Convergence of WED minimizers to a curve of maximal slope by passing to the limit in

$$\boxed{-\frac{\mathrm{d}}{\mathrm{d}t}V_\varepsilon(u_\varepsilon(t)) = \frac{1}{2}|u_\varepsilon'|^2(s) + \frac{1}{\varepsilon}\varphi(u_\varepsilon(t)) - \frac{1}{\varepsilon}V_\varepsilon(u_\varepsilon(t)) \quad \text{for a.a. } t \in (0,T)}$$

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- **gradient flow** of the value functional V_ε if ϕ **λ-geodesically convex**

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$$-\frac{\mathrm{d}}{\mathrm{d}t} V_\varepsilon(u_\varepsilon(t)) = \frac{1}{2}|u_\varepsilon'|^2(s) + \frac{1}{\varepsilon}\varphi(u_\varepsilon(t)) - \frac{1}{\varepsilon}V_\varepsilon(u_\varepsilon(t)) \quad \text{for a.a. } t \in (0,T)$$

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- **without** geodesic convexity $\rightsquigarrow$ $\frac{1}{\varepsilon}(\varphi - V_\varepsilon)$ interpretable as **approx. of** $\frac{1}{2}|\partial^- \varphi|^2$

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♣ **Applications:** (a class of) gradient flows of nonconvex energies in Banach spaces; gradient flows in Wasserstein spaces as in **[Ambrosio/Gigli/Savaré '05]**...

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♣ **Applications:** (a class of) gradient flows of nonconvex energies in Banach spaces; gradient flows in Wasserstein spaces as in **[Ambrosio/Gigli/Savaré '05]**...

♠ WED approach is **alternative** to Minimizing Movement scheme but **not independent**: convergence proof uses knowledge that a curve of maximal slope for φ exists!