Robustness of funnel control in the gap metric

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joint work with Christoph Hackl\textsuperscript{1}, Norman Hopfe\textsuperscript{2}, Achim Ilchmann\textsuperscript{3} and Stephan Trenn\textsuperscript{4}

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An example

### Linear ODE & adaptive/funnel controllers

\[ \dot{y} = \alpha y + u \]

\( \alpha \in \mathbb{R} \) unknown

**\( \lambda \)-tracker**

\[
\begin{align*}
  u(t) &= -k(t) y(t) \\
  \dot{k}(t) &= \text{dist}(y(t), [-\lambda, \lambda]) \|y(t)\|
\end{align*}
\]

- + simple tracking
- + adaptive
- - strictly increasing \( k \)

**funnel controller**

\[
\begin{align*}
  u(t) &= -k(t) y(t) \\
  k(t) &= \frac{1}{\psi(t)-\|y(t)\|}
\end{align*}
\]

- + \( k \) only large if required
- + tracking in finite time
An example

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\( \alpha \in \mathbb{R} \) unknown

+ simple tracking
+ adaptive
- strictly increasing \( k \)

funnel controller

\[ u(t) = -k(t) y(t) \]
\[ k(t) = \psi(t) \frac{1}{\| y(t) \|} \]

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Linear ODE & adaptive/funnel controllers

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**funnel controller**

\[ u(t) = -k(t) y(t) \]
\[ k(t) = \frac{1}{\psi(t) - \| y(t) \|} \]
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\[ + \text{ tracking in finite time} \]
1. Recall: Funnel control for relative degree one systems
2. Funnel control for systems with relative degree two
3. Example: stiffly coupled machines
4. Gap metric and robust stabilization
5. Robustness of funnel control
Outline

1. Recall: Funnel control for relative degree one systems
2. Funnel control for systems with relative degree two
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Byrnes–Isidori normal form for \((A, b, c)\)

\[
\begin{align*}
\frac{d}{dt} \begin{pmatrix} \xi \\ \eta \end{pmatrix} &= \begin{bmatrix} a_1 & a_2 \\ a_3 & Q \end{bmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{pmatrix} cb \\ 0 \end{pmatrix} u, \\
\begin{pmatrix} \xi \\ \eta \end{pmatrix}(0) &= \begin{pmatrix} \xi^0 \\ \eta^0 \end{pmatrix},
\end{align*}
\]

\[y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad a_1 \in \mathbb{R}, \ a_2, a_3^\top \in \mathbb{R}^{1 \times (n-1)}, \]

\[Q \in \mathbb{R}^{(n-1) \times (n-1)}.\]

- \(M_{n,m}\): class of all \(n\)-dim. systems with relative degree \(m\), positive high-frequency gain \((cb > 0\) for rel. deg. 1) and minimum phase \((Q\) stable),

- reference signal \(y_{ref}\),

input disturbance \(u_d\).
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- $\mathcal{M}_{n,m}$: class of all $n$-dim. systems with relative degree $m$, positive high-frequency gain ($cb > 0$ for rel. deg. 1) and minimum phase ($Q$ stable),
- reference signal $y_{\text{ref}}$, input disturbance $u_d$. 
Funnel controller $C_{\mathcal{F}}(\varphi)$

\[ u(t) = -k(t) e(t) + u_d \]

\[ k(t) = \frac{\varphi(t)}{1 - \varphi(t)|e(t)| \left( \frac{1}{\psi(t) - |e(t)|} \right) \geq 1} \]

\[ \frac{1}{\varphi(t)} = \psi(t) \]

\[ |e(t)| = |y_{ref}(t) - y(t)| \]
Funnel control

Theorem

\[(A, b, c) \in \mathcal{M}_{n,1} \quad \& \quad \text{funnel controller} \quad C_F(\varphi) \quad \& \quad u_d = u - w, \quad y_{ref} = y - e\]

gives, for \((u_d, y_{ref}) \in L^\infty \times W^{1,\infty}\) and initial value \(\begin{pmatrix} y^0 \\ \eta^0 \end{pmatrix} \in \mathbb{R}^n:\)

\[\forall t \geq 0 : (t, e(t)) \in F_\varphi = \{(t, \xi) \mid \varphi(t)|\xi| < 1\}\]
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\[\forall t \geq 0 : (t, e(t)) \in \{(t, \xi) \mid \varphi(t)|\xi| < 1 - \varepsilon\}.

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Funnel control: sketch of a proof

ODE theory and properties of the right-hand side of the closed-loop system assure existence and uniqueness of a maximal solution $(e, \eta): [0, \omega) \rightarrow \mathbb{R}^n$ of $[(A, b, c), C_F(\varphi)]$. 
Funnel control: sketch of a proof

To show: \( \forall t \in [0, \omega) : \psi(t) - |e(t)| \geq \varepsilon \) for some \( \varepsilon > 0 \);
\( \varepsilon \) depends on system entries & funnel properties, e.g. \( \varepsilon \leq \lambda/2 \).

Seeking a contradiction: suppose \( \exists t_1 \in [0, \omega) : \psi(t_1) - |e(t_1)| < \varepsilon \),

then \( \exists t_0 > 0 : t_0 = \max\{t \in [0, t_1) : |\psi(t) - |e(t)|| = \varepsilon\} \),

thus, for all \( t \in [t_0, t_1] \)

\[
\psi(t) - |e(t)| \leq \varepsilon \\
\land |e(t)| \geq \psi(t) - \varepsilon \geq \lambda/2
\]

and, together,

\[
\frac{|e(t)|}{\psi(t) - |e(t)|} \geq \frac{\lambda}{2\varepsilon}.
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$$\psi(t) - |e(t)| \leq \varepsilon$$

$$\wedge \ |e(t)| \geq \psi(t) - \varepsilon \geq \lambda/2$$

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and, together,

\[
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\]
In view of

\[ \dot{e}(t) = -a_1(y_{\text{ref}}(t) - e(t)) + a_2 \int_0^t e^{Q(t-s)} a_3 (y_{\text{ref}}(s) - e(s)) \, ds \]

\[ - a_2 e^{Qt} \eta^0 + \dot{y}_{\text{ref}}(t) - cb u_d(t) + cb \frac{-1}{\psi(t) - |e(t)|} e(t), \]

it follows for some \( L > 0 \) (Lipschitz constant of \( \psi \)), that for all \( t \in [t_0, t_1] \), \( e(t)\dot{e}(t) \leq -L|e(t)| \) which gives

\[ |e(t_1)| - |e(t_0)| = \int_{t_0}^{t_1} \frac{e(\tau)\dot{e}(\tau)}{|e(\tau)|} \, d\tau \leq -L(t_1 - t_0) \]

\[ \leq -|\psi(t_1) - \psi(t_0)| \leq \psi(t_1) - \psi(t_0), \]

whence the contradiction

\[ \varepsilon = \psi(t_0) - |e(t_0)| \leq \psi(t_1) - |e(t_1)| < \varepsilon. \]
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\]
Simulations for $\dot{y} = \alpha y + u$ & $C_F(\varphi)$

Figure: $P_{\alpha=1; x^0=1}$ & funnel controller $C_F(\varphi)$; with $u_d(\cdot) = \sin(2\cdot)$, the funnel boundary is bounded away from 0: $1/\varphi(t) \geq \lambda = 0.1$
### Outline

1. Recall: Funnel control for relative degree one systems
2. Funnel control for systems with relative degree two
3. Example: stiffly coupled machines
4. Gap metric and robust stabilization
5. Robustness of funnel control
Figure: Closed-loop system \([(A, b, c), \mathcal{C}_F(\varphi_0, \varphi_1)]\) subject to input disturbances \(u_d\) and reference signal \(y_{\text{ref}}\)
Funnel controller $C_F(\varphi_0, \varphi_1)$

$$u(t) = -k_0(t)^2 e(t) - k_1(t) \dot{e}(t) + u_d(t), \quad e(t) = y(t) - y_{\text{ref}}(t)$$

$$k_i(t) = \frac{\varphi_i(t)}{1 - \varphi_i(t)|e^{(i)}(t)|}, \quad i = 0, 1,$$
Funnel control for systems with relative degree two

**Theorem**

\[(A, b, c) \in \mathcal{M}_{n,2} \quad \& \quad \text{funnel controller} \quad C_{\mathcal{F}}(\varphi_0, \varphi_1) \quad \& \quad u_d = u - w, \quad y_{\text{ref}} = y - e\]

Given, for \((u_d, y_{\text{ref}}) \in L^\infty \times W^{2,\infty}\) and initial value \(x^0 \in \mathbb{R}^n\) with \(\varphi_0(0)|y_{\text{ref}}(0) - cx^0| < 1\) and \(\varphi_1(0)|\dot{y}_{\text{ref}}(0) - cAx^0| < 1\):

\[\forall t \geq 0 : (t, e(t)) \in \mathcal{F}_{\varphi_0} \quad \text{and} \quad (t, \dot{e}(t)) \in \mathcal{F}_{\varphi_1}\]
Funnel control for systems with relative degree two

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gives, for \((u_d, y_{\text{ref}}) \in L^\infty \times W^{2,\infty}\) and initial value \(x^0 \in \mathbb{R}^n\) with

\[\varphi_0(0) |y_{\text{ref}}(0) - cx^0| < 1 \quad \text{and} \quad \varphi_1(0) |\dot{y}_{\text{ref}}(0) - cAx^0| < 1:\]

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Funnel control for systems with relative degree two

Theorem

\[(A, b, c) \in \mathcal{M}_{n,2} \quad \& \quad \text{funnel controller } \mathcal{C}_\mathcal{F}(\varphi_0, \varphi_1) \quad \& \quad u_d = u - w, \quad y_{\text{ref}} = y - e\]

gives, for \((u_d, y_{\text{ref}}) \in L^\infty \times W^{2,\infty} \) and initial value \(x^0 \in \mathbb{R}^n\) with \(\varphi_0(0) |y_{\text{ref}}(0) - cx^0| < 1 \) and \(\varphi_1(0) |\dot{y}_{\text{ref}}(0) - cAx^0| < 1\):

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Funnel control for systems with relative degree two

Theorem

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gives, for \((u_d, y_{\text{ref}}) \in L^\infty \times \mathcal{W}^{2,\infty}\) and initial value \(x^0 \in \mathbb{R}^n\) with 
\[
\varphi_0(0) |y_{\text{ref}}(0) - cx^0| < 1 \text{ and } \varphi_1(0) |\dot{y}_{\text{ref}}(0) - cAx^0| < 1:
\]

\[\forall i \in \{0, 1\} \exists \varepsilon_i > 0 \forall t \geq 0 : 1/\varphi_i(t) - |e^{(i)}(t)| \geq \varepsilon_i\]
The idea of two funnels

- If the error $e$ evolves within the funnel $\mathcal{F}_{\varphi_0}$, then the derivative of the error eventually has to fulfill

\[
\dot{e}(t) < \frac{d}{dt}(1/\varphi_0)(t) \quad \text{or} \quad \dot{e}(t) > -\frac{d}{dt}(1/\varphi_0)(t),
\]

i.e. at some time the error must decrease faster than the upper funnel boundary gets smaller, or the error must increase faster than the lower funnel boundary grows.

- Thus, the derivative funnel must be large enough to allow the error to follow the funnel boundaries; therefore we make the following assumption to the funnels

\[
\exists \delta > 0 \ \forall \ t > 0 : \frac{1}{\varphi_1(t)} \geq \delta - \frac{d}{dt}(1/\varphi_0)(t).
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Thus, the derivative funnel must be large enough to allow the error to follow the funnel boundaries; therefore we make the following assumption to the funnels

$$\exists \delta > 0 \ \forall \ t > 0 : \ 1/\varphi_1(t) \geq \delta - \frac{d}{dt}(1/\varphi_0)(t).$$
Corollary

The funnel controller $C_F(\varphi_0, \varphi_1)$ achieves stabilization/tracking when applied

- subject to input saturation;
- to nonlinear systems with relative degree two (and bounded nonlinearity);
- to systems with relative degree one;
- to infinite dimensional systems/functional differential equations with relative degree two.
Corollary

The funnel controller $C_F(\varphi_0, \varphi_1)$ achieves stabilization/tracking when applied

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Outline

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3. Example: stiffly coupled machines
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Application on position control problem

Figure: Laboratory setup of rotatory system: stiffly coupled machines (drive and load)
**Mathematical model**

Rotary system with actuator for position control

\[
\begin{align*}
\dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \left[ \text{sat}_{\hat{u}_A}(u(t) + u_A(t)) \right] \\
& \quad - u_L(t) - (T_{\vartheta_0} x_2)(t), \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t), \\
\end{align*}
\]

where the state variable \( x(t) = (\phi(t), \Omega(t))^\top \) represents angle \( \phi(t) \) and angular velocity \( \Omega(t) = \dot{\phi}(t) \) at time \( t \geq 0 \) in \([\text{rad}]\) and \([\text{rad/s}]\), resp.
Measurements for funnel controlled system

Figure: $y(\cdot) + n(\cdot) [\text{rad}]$ and $y_{\text{ref}}(\cdot) [\text{rad}]$
Measurements for funnel controlled system

Figure: $e(\cdot) \text{ [rad]}$ and $\pm \psi_0(\cdot) \text{ [rad]}$
Measurements for funnel controlled system

Figure: $\dot{y}(\cdot) + \dot{n}(\cdot) \left[ \frac{\text{rad}}{s} \right]$ and $\dot{y}_{\text{ref}}(\cdot) \left[ \frac{\text{rad}}{s} \right]$
Measurements for funnel controlled system

Figure: $\dot{e}(\cdot) \left[ \frac{\text{rad}}{s} \right]$ and $\pm \psi_1(\cdot) \left[ \frac{\text{rad}}{s} \right]$
Measurements for funnel controlled system

Figure: $u(\cdot) + u_A(\cdot) \,[Nm]$ and $u_L(\cdot) \,[Nm]$
Measurements for funnel controlled system

Figure: $k_0(\cdot)^2 \left[ \frac{Nm}{rad} \right]$ and $k_1(\cdot) \left[ \frac{Nms}{rad} \right]$
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What is robustness?

Does funnel control work for systems which are disturbed, e.g.

- do not have a positive high-gain $cAb$;
- are not minimum phase;
- do not have relative degree 2?

Idea

Answer can be found utilizing the gap metric.
Robustness

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Answer can be found utilizing the gap metric.
What is robustness?

Does funnel control work for systems which are disturbed, e.g.
- do not have a positive high-gain $cAb$;
- are not minimum phase;
- do not have relative degree 2?

Idea

Answer can be found utilizing the **gap metric**.
Closed-loop system

\[ \begin{align*}
(P, C) : & \quad y = Pu \\
& \quad w = Ce \\
& \quad u_d = u - w \\
& \quad y_{ref} = y - e
\end{align*} \]

Funnel controller \( C_F(\varphi_0, \varphi_1) \)
Definitions: operators and graphs . . .

Operators
Let $\mathcal{U}, \mathcal{Y}$ be normed vector spaces.
- $P : \mathcal{U} \to \mathcal{Y}, \; u \mapsto y = Pu$,
- $C : \mathcal{Y} \to \mathcal{U}, \; e \mapsto w = Ce$.

Graphs
- $G_P := \{(u, Pu) \mid u \in \mathcal{U}, Pu \in \mathcal{Y}\} \subset \mathcal{U} \times \mathcal{Y}$
- $G_C := \{(Cy, y) \mid y \in \mathcal{Y}, Cy \in \mathcal{U}\} \subset \mathcal{U} \times \mathcal{Y}$

are the graphs of $P$ and $C$, resp.
Definitions: operators and graphs ...

Operators

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are the graphs of $P$ and $C$, resp.
Directed gap (Kato, 66)

$$\bar{\delta}(P, \tilde{P}) := \bar{\delta}(G_P, G_{\tilde{P}}) := \sup_{w \in G_P, \|w\| = 1} \text{dist}(w, G_{\tilde{P}}).$$
Example: gap between two linear systems

### Linear systems

\[
P_{\alpha;x^0} : \dot{x} = \begin{bmatrix} 0 & 1 \\ -\alpha^2 & 2\alpha \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad x(0) = x^0 \in \mathbb{R}^2 \\
y = [1, 0] x,
\]

\[
P_{M,\alpha;\tilde{x}^0} : \dot{x} = \tilde{A} x + \tilde{b} u, \quad x(0) = \tilde{x}^0 \in \mathbb{R}^4 \\
y = \tilde{c} x
\]

where \((A, b, c)\) is a realization of \(\frac{1}{(s-\alpha)^2}\)

and \((\tilde{A}, \tilde{b}, \tilde{c})\) is a realization of \(\frac{-2M(s-M)}{(s-\alpha)^2(s+2M)(s+M)}\).

### Gap

\[
\limsup_{M \to \infty} \delta(P_{\alpha;0}, P_{M,\alpha;0}) = 0
\]
Example: gap between two linear systems

**Linear systems**

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P_{\alpha; x^0} : \dot{x} = \begin{bmatrix} 0 & 1 \\ -\alpha^2 & 2\alpha \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad x(0) = x^0 \in \mathbb{R}^2
\]
\[
y = [1, 0] x,
\]
\[
P_{M, \alpha; x^0} : \dot{x} = \tilde{A} x + \tilde{b} u, \quad x(0) = \tilde{x}^0 \in \mathbb{R}^4
\]
\[
y = \tilde{c} x
\]

where \((A, b, c)\) is a realization of \(\frac{1}{(s-\alpha)^2}\)

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**Gap**

\[
\limsup_{M \to \infty} \tilde{\delta}(P_{\alpha;0}, P_{M,\alpha;0}) = 0
\]

**Exeter University**

Markus Mueller

Robustness of funnel control in the gap metric
Robust stabilization

**Closed-loop**

\[ u_d \rightarrow + \rightarrow P \text{ or } \tilde{P} \]

\[ w \rightarrow + \rightarrow C \]

\[ e \rightarrow + \rightarrow y \rightarrow \rightarrow -y_{\text{ref}} \]

**Theorem (Georgiou & Smith, 97)**

\[ [P, C] \text{ is "gain"-stable} \]

\[ \text{and the "gap" } \tilde{\delta}(P, \tilde{P}) \ll 1 \]

\[ \Rightarrow \]

\[ [\tilde{P}, C] \text{ is "gain"-stable}. \]
Outline

1. Recall: Funnel control for relative degree one systems
2. Funnel control for systems with relative degree two
3. Example: stiffly coupled machines
4. Gap metric and robust stabilization
5. Robustness of funnel control
"New" general system class

Linear system

Let

\[ \mathcal{P}_q = \left\{ (\tilde{A}, \tilde{b}, \tilde{c}) \in \mathbb{R}^{q \times q} \times \mathbb{R}^{q \times 1} \times \mathbb{R}^{1 \times q} \mid (\tilde{A}, \tilde{b}, \tilde{c}) \text{ stabilizable and detectable} \right\} \]

where \((\tilde{A}, \tilde{b}, \tilde{c})\) is a linear system of form

\[
\begin{align*}
\dot{x} &= \tilde{A} x + \tilde{b} u, \quad x(0) = \tilde{x}^0 \in \mathbb{R}^q \\
y &= \tilde{c} x.
\end{align*}
\]
Theorem

Let $(\tilde{A}, \tilde{b}, \tilde{c}) \in \mathcal{P}_q$. If

- $\tilde{x}^0 \in \mathbb{R}^q$ and $(u_d, y_{\text{ref}}) \in L^\infty \times W^{2,\infty}$ are sufficiently small;
- $\delta(P(A,b,c), P(\tilde{A},\tilde{b},\tilde{c}))$ is sufficiently small, $(A, b, c) \in \mathcal{M}_{n,2}$;

then, for $e = y - y_{\text{ref}},$

$$u(t) = -k_0(t)^2 e(t) - k_1(t) e(t) + u_d(t)$$

$$k_i(t) = \frac{\varphi_i(t)}{1 - \varphi_i(t) \|e^{(i)}(t)\|}, \quad i = 0, 1$$

satisfies

$$\forall t \geq 0 : (t, e^{(i)}(t)) \in \mathcal{F}_{\varphi_i}, \quad i = 0, 1 \quad \& \quad k \in L^\infty \quad \& \quad x \in W^{2,\infty}$$
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\[
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\]

\[
    k_i(t) = \frac{\varphi_i(t)}{1 - \varphi_i(t)\|e(i)(t)\|}, \quad i = 0, 1
\]

satisfies

\[
    \forall t \geq 0 : (t, e(i)(t)) \in \mathcal{F}_{\varphi_i}, \quad i = 0, 1 \quad \text{and} \quad k \in L^\infty \quad \text{and} \quad x \in W^{2,\infty}
\]
Stability of \((\tilde{A}, \tilde{b}, \tilde{c}) \in \mathcal{P}_q\) & funnel controller

**Theorem**

Let \((\tilde{A}, \tilde{b}, \tilde{c}) \in \mathcal{P}_q\). If

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\[
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(\tilde{A}, \tilde{b}, \tilde{c}) & \quad \& \\
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k_i(t) = \frac{\varphi_i(t)}{1 - \varphi_i(t)\|e^{(i)}(t)\|}, \quad i = 0, 1
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\]
Stability of \((\tilde{A}, \tilde{b}, \tilde{c}) \in P_q \& \text{funnel controller}\)

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Conclusions and perspective

- **Problem**: stabilization via nonlinear “adaptive” feedback.
- **Solution**: funnel controller; drawback: limited system class.
- **New problem**: robustness (loss of classical assumptions).
- **Results using the nonlinear gap**: if
  - linear system is close to $\mathcal{M}_{n,2}$ and
  - initial values and disturbances are small
then funnel control is applicable.
  - The funnel controller is robust.
- **Perspectives**:
  - Evolve and apply and “gap metric” which allows to involve initial values directly.
  - Develop applicable “gap metric” for time-varying/nonlinear systems.

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Markus Mueller

Robustness of funnel control in the gap metric
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Funnel control 2: sketch of a proof

- There exists a maximal solution on $[0, \omega)$.
- In view of $e$ and $\dot{e}$ being bounded (evolve within bounded funnels) and the minimum phase condition, there exists $M > 0$ such that
  \[ \forall t \in [0, \omega) : \ddot{e}(t) < M + \gamma u(t). \]
  In particular, if $u(t) \ll 0$ then $\ddot{e}(t) \ll 0$.
- If $k_0(\cdot)^2 e(\cdot)$ is bounded, then $\dot{e}$ remains bounded away from the funnel boundary of $\mathcal{F}_{\varphi_1}$, because we may choose $\varepsilon_1 > 0$ such that, for all $t \in [0, \omega)$
  \[
  \begin{align*}
  \dot{e}(t) &= \psi_1(t) - \varepsilon_1 \quad \implies \quad \ddot{e}(t) < \dot{\psi}_1(t), \\
  \dot{e}(t) &= -\psi_1(t) + \varepsilon_1 \quad \implies \quad \ddot{e}(t) > -\dot{\psi}_1(t).
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Funnel control 2: sketch of a proof (cont.)

To show: \( \exists \varepsilon_0 > 0 \ \forall t \in [0, \omega) : e(t) \leq \psi(t) - \varepsilon_0. \)

For some “small” \( \varepsilon_0 > 0 \), consider \( t_0 \geq 0 \) such that, for some \( t < t_0 \),

\[ e(t_0) = \psi_0(t_0) - 2\varepsilon_0 \quad \text{and} \quad e(t) < \psi_0(t) - 2\varepsilon_0. \]

We show: \( \exists \tau(\varepsilon_0) > 0 \ \forall t > t_0 : e(t) \leq \psi_0(t) - 2\varepsilon_0 + \tau(\varepsilon_0) \)
and \( \tau(\varepsilon_0)/\varepsilon_0 \to 0 \) as \( \varepsilon_0 \to 0 \),

which implies: for sufficiently small \( \varepsilon_0 > 0 \) and all \( t \geq 0 \),

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Funnel control 2: sketch of a proof (cont.)

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• To show: \( \exists \varepsilon_0 > 0 \ \forall t \in [0, \omega) : e(t) \leq \psi(t) - \varepsilon_0. \)

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\[
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and \( \tau(\varepsilon_0)/\varepsilon_0 \to 0 \) as \( \varepsilon_0 \to 0 \),

• which implies: for sufficiently small \( \varepsilon_0 > 0 \) and all \( t \geq 0 \),

\[
e(t) \leq \psi_0(t) - \varepsilon_0.
\]
Funnel control 2: sketch of a proof (cont.)

- **Red phase:**
  \[(P) \dot{e}(t) \geq -\psi_1(t) + \delta/2,\]

- **Blue phase:**
  \[(L1) e(t) \leq \psi_0(t) - 2\varepsilon_0 + \tau(\varepsilon_0),\]
  \[(L2) \dot{e}(t) \leq -\psi_1(t) + \delta/2,\]

- **Both phases:**
  \[(PL) e(t) \geq \psi_0(t) - 2\varepsilon_0,\]

\[(PL) and (P), for 2\varepsilon_0 \leq \lambda_0/2:\]

\[u(t) < -\frac{1}{(2\varepsilon_0)^2} \frac{\lambda_0}{2} + \frac{1}{\delta/2} \|\psi_1\|_{\infty} + \|u_d\|_{\infty}.\]
Funnel control 2: sketch of a proof (cont.)

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  \((P)\) \(\dot{e}(t) \geq -\psi_1(t) + \delta/2,\)

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  \((L1)\) \(e(t) \leq \psi_0(t) - 2\varepsilon_0 + \tau(\varepsilon_0),\)
  
  \((L2)\) \(\dot{e}(t) \leq -\psi_1(t) + \delta/2,\)

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  \((PL)\) \(e(t) \geq \psi_0(t) - 2\varepsilon_0,\)

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  \[u(t) < -\frac{1}{(2\varepsilon_0)^2} \frac{\lambda_0}{2}\]
  
  \[+ \frac{1}{\delta/2} \|\psi_1\|_{\infty} + \|u_d\|_{\infty}.\]
Funnel control 2: sketch of a proof (cont.)

- **Red phase:**
  
  \[(P) \dot{e}(t) \geq -\psi_1(t) + \delta/2,\]

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\[
u(t) < -\frac{1}{(2\varepsilon_0)^2} \frac{\lambda_0}{2} + \frac{1}{\delta/2} \|\psi_1\|_\infty + \|u_d\|_\infty.\]
Funnel control 2: sketch of a proof (cont.)

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- \((PL)\) and \((P),\) for \(2\varepsilon_0 \leq \lambda_0/2: \)

\[
u(t) < -\frac{1}{(2\varepsilon_0)^2} \frac{\lambda_0}{2} + \frac{1}{\delta/2} \|\psi_1\|_{\infty} + \|u_d\|_{\infty}.
\]
Funnel control 2: sketch of a proof (cont.)

which (together with
\( \forall t \in [0, \omega) : \ddot{e}(t) < M + \gamma u(t) \))
yields:

\[
\forall t \in [t_0, t_1) : \ddot{e}(t) < -\overline{M}(\varepsilon_0)
\]

for some \( \overline{M}(\varepsilon_0) > 0 \) with
\( \overline{M}(\varepsilon_0) \to \infty \) as \( \varepsilon_0 \to 0 \).

Hence, the error is bounded by a parabola, for all \( t \in [t_0, t_1) \):

\[
e(t) < -\frac{\overline{M}(\varepsilon_0)}{2} (t - t_0)^2 + \ddot{e}(t_0) (t - t_0) + e(t_0) \leq \|\psi_1\|_{\infty} \leq \|\psi_0\|
\]
Funnel control 2: sketch of a proof (cont.)

which (together with \( \forall t \in [0, \omega) : \ddot{e}(t) < M + \gamma u(t) \)) yields:

\[
\forall t \in [t_0, t_1) : \quad \ddot{e}(t) < -\overline{M}(\varepsilon_0)
\]

for some \( \overline{M}(\varepsilon_0) > 0 \) with \( \overline{M}(\varepsilon_0) \to \infty \) as \( \varepsilon_0 \to 0 \).

Hence, the error is bounded by a parabola, for all \( t \in [t_0, t_1) \):

\[
e(t) < -\frac{\overline{M}(\varepsilon_0)}{2} (t - t_0)^2 + \dot{e}(t_0) (t - t_0) + e(t_0) \\
\leq \|\psi_1\|_{\infty} \\
\leq \|\psi_0\|
\]
In particular: there exists a maximal “overshoot” $\tau(\varepsilon_0)$ of the error starting at $\psi_0(t_0) - 2\varepsilon_0$, and we can show $\tau(\varepsilon_0)/\varepsilon_0 \to 0$ as $\varepsilon_0 \to 0$.

The red/parabolic phase is only active as long as (P) holds.

If (P) does not hold, then

$$\dot{e}(t) \leq -\psi_1(t) + \delta/2 < \dot{\psi}_0(t),$$

hence the distance between $e$ and the funnel boundary $\psi_0$ increases.
In particular: there exists a maximal “overshoot” $\tau(\varepsilon_0)$ of the error starting at $\psi_0(t_0) - 2\varepsilon_0$, and we can show $\tau(\varepsilon_0)/\varepsilon_0 \to 0$ as $\varepsilon_0 \to 0$.

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The distance gets bigger than $2\varepsilon_0$ either in the red phase or in the blue phase.

Once in the blue phase, we remain in it until $e(t) < \psi_0(t) - 2\varepsilon_0$, as required.
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