Recent advances in Input-to-State Stability theory from global to almost global

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Stabilization of dynamical systems and processes
Outline

• Definition of Input-to-State Stability (ISS)
• Lyapunov-like characterizations
• Motivations for almost global ISS
• Almost global ISS
• Density functions and dual Lyapunov
• Hyperbolic saddles
• Robustness of almost global stability
Input to State Stability

[Sontag ‘89]

\[ \dot{x} = f(x, d) \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \]

\[ |x(t, x_0, d(.))| \leq \max\{ \beta(|x_0|, t), \gamma(\|d\|_\infty) \} \]

\[ \forall t \geq 0 \quad \forall x_0 \in \mathbb{R}^n \quad \forall d(.) \]
ISS tools & results

• Equivalent formulations:
  - Lyapunov-like characterizations
    \[ \alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \quad \alpha_1, \alpha_2 \in \mathcal{K}_\infty \]
    \[ \frac{\partial V}{\partial x}(x) \cdot f(x,d) \leq -\alpha(|x|) + \delta(|d|) \quad \alpha, \delta \in \mathcal{K}_\infty \]
  - Asymptotic gain characterization:
    \[ \limsup_{t \to +\infty} |x(t, x_0, d(.))| \leq \gamma(\|d\|_\infty) \]
ISS tools & results (2)

• Preservation of ISS under cascades:
  
  ![Diagram](image)

  ISS subsystem → ISS subsystem

• Small gain results:

  ![Diagram](image)

  ISS subsystem → ISS subsystem → ISS subsystems
Why almost global?

- Systems on manifolds:
  \[
  \dot{\theta} = -\sin(\theta) - \dot{\theta} + d \\
  [\theta, \dot{\theta}] \in S \times R
  \]

- Complex attractors:
  Van der Pol oscillator:
  \[
  \ddot{y} - \mu(1 - y^2) \dot{y} + y = d
  \]
Almost GAS

• Stickiness effect at unstable invariant sets

\[ |x(t, \cdot)| \leq \beta(|x_0|, t) \quad \forall a.a. \quad x_0 \in \mathbb{R}^n \]

• A.S. & almost global attractivity

\[ \limsup_{t \to +\infty} |x(t, x_0)| = 0 \quad \forall a.a. \quad x_0 \in \mathbb{R}^n \]
Almost Global ISS

• Asymptotic stability and Asymptotic Gain

\[ \forall d(.) \ \exists B_d : \mu(B_d) = 0 : \forall x_0 \in M \setminus B_d \]

\[ \limsup_{t \to +\infty} |x(t, x_0, d(.))| \leq \gamma(\|d\|_\infty) \]

• Problem: how to test the property?
An example: system on a circle

• Constant disturbances:

\[ \dot{\theta} = -\sin(\theta) + d \]

• Square wave disturbance with positive measure bad set?
Density functions

[Rantzer ‘01]

• Assume existence of $C^1$ function:

$$\rho(x) : M \setminus \{0\} \rightarrow [0, +\infty)$$

• **Finite volume outside a neighborhood of 0**

$$\int_{M \setminus U_0} \rho(x) dx < +\infty$$

• Expanding along solutions:

$$\text{div}_x \rho(x) f(x) > 0 \quad \text{a.e.}$$

• Then: Almost global attractivity to 0
Dual Lyapunov and ISS

- **Existence of** $C^1$ **function:**
  \[ \rho(x) : M \setminus \{0\} \to [0, +\infty) \]

- **Finite volume outside neighborhood of 0:**
  \[ \int_{M \setminus U_0} \rho(x)dx < +\infty \]

- **Expansion along solutions for large states:**
  \[ |x| > \gamma(|d|) \implies \text{div}_x \rho(x) f(x, d) > 0 \quad \text{a.e.} \]

- **Weak almost ISS:**
  \[ \forall d(.) \quad \exists B_d : \mu(B_d) = 0 : \forall x_0 \in M \setminus B_d \]
  \[ \liminf_{t \to +\infty} |x(t, x_0, d(.))| \leq \gamma(\|d\|_{\infty}) \]
Primal & Dual Lyapunov

\[ \rho(\theta) = \frac{1}{\sin^2(\theta/2)} \]

\[ \text{div}_\theta \left( -\sin(\theta) + d \frac{\sin^2(\theta)}{2} \right) = \sin^2(\theta/2)[1 - 2d \cos^2(\theta/2)] \]

WEAK ALMOST ISS

\[ V(\theta) = \sin^2(\theta/2) \]

\[ \dot{V} = -\frac{1}{2} \sin^2(\theta) + d \frac{\sin(\theta)}{2} \]

REGIONAL ISS
Saddle points

- Consider a saddle point $x_e$
- Assume $\text{div}_x f(x_e) < 0$
- $\text{div}_x \varrho(x) \cdot f(x) > 0 \quad a.e. \Rightarrow \varrho(x) = 0$

on stable manifold

Problem:
no closed form expression
A second order example

• Is the mathematical pendulum with friction **Almost Input-to-State Stable**?

\[ \ddot{\theta} = -\sin(\theta) - \dot{\theta} + d \]

\[ [\theta, \dot{\theta}] \in S \times R \]

• **Equilibria**: [0,0] and [\(\pi\),0]

• [\(\pi\),0] is a saddle point of negative divergence
Almost ISS of systems with exponentially unstable equilibria

Consider a $C^1$ system:

$$\dot{x} = f(x, d) \quad f : M \times D \to TM$$

- $M$ is a $C^2$ manifold without boundary
- Existence of $V : M \to [0, +\infty)$, of class $C^1$
  $$L_{f(x,0)}V(x) < 0 \quad \forall x : f(x,0) \neq 0$$
- Finite set of equilibria

Unstable equilibria are **exponentially** unstable

- A compact set $C \subseteq M$
Main result
[Praly 2011, TAC]

There exists $\Delta > 0$ and $\gamma$ of class $K$, such that for all $d(.)$ smaller than $\Delta$, almost all solutions with $x(0)$ in $C$ fulfil:

$$\limsup_{t \to +\infty} \left| x(t) \right|_{E_s} \leq \gamma\left(\left\| d \right\|_{\infty}\right)$$

Where $E_s$ is the set of asymptotically stable equilibria and $\left| . \right|_{E_s}$ denotes the point to set distance in $M$.

The set of “bad” points is obviously input dependent.
Sketch of the proof

- Locally Lipschitz stable manifold under small input disturbances
- Strict decrease of $V$ except around equilibria
- Solutions eventually close to a single equilibrium
- If asymptotically stable then $K$ function bound applies
- If not initial condition is in a zero measure set
Corollary

Consider a $C^1$ system:

$$\dot{x} = f(x, d) \quad f : M \times D \to TM$$

- $M$ is a $C^2$ manifold without boundary
- Existence of $V : M \to [0, +\infty)$, of class $C^1$
  $$L_{f(x,0)}V(x) < 0 \quad \forall x : f(x,0) \neq 0$$
- Finite set of equilibria
  Unstable equilibria are exponentially unstable
- Ultimate boundedness

Then Almost global ISS holds
Pendulum Equations:

\[ \ddot{\theta} = -\dot{\theta} - \sin(\theta) + d \]

Mechanical Energy:

\[ W(\theta, \dot{\theta}) = \frac{\dot{\theta}^2}{2} - \cos(\theta) \]

Deriving along solutions

\[ \dot{W} = -\dot{\theta}^2 + \dot{\theta}d \leq -\frac{\dot{\theta}^2}{2} + \frac{d^2}{2} \leq -\frac{W}{2} + \frac{1 + d^2}{2} \]

Ultimate boundedness
moreover

- Lyapunov function
  \[ V(\theta, \dot{\theta}) = \frac{\dot{\theta}^2}{2} - \cos(\theta) + \varepsilon \dot{\theta} \sin(\theta) \]

- Strict decrease outside equilibria
  \[ \dot{V} \bigg|_{d=0} = -\dot{\theta}^2 - \varepsilon \sin^2(\theta) - \varepsilon \dot{\theta} \sin(\theta) + \varepsilon \dot{\theta}^2 \cos(\theta) < 0 \]

- Hyperbolic saddle in \([\pi,0]\)

- **Almost global ISS of pendulum with friction**
ISS of oscillators

- Lur’e systems with stiffening nonlinearity

![Diagram of Lur’e system with feedback loop and nonlinearity]

- Ultimate boundedness [Teel & Arcak]
- Strict Lyapunov function outside periodic solution & equilibrium at the origin
- Exponentially unstable equilibrium in 0
- Almost global ISS of limit cycle follows
Small gain results?

- Cascades of Almost ISS systems are ok
- Circular argument in small gain theorem
- Additional conditions to rule out bad sets of non-zero measure?
Conclusions

• New tools for ISS on manifolds
• Complementary to Hybrid Systems approach
• Patchy arguments needed
• Theory still in its infancy

Thank you