

On a fourth order degenerate parabolic equation

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**Dissipative PDEs in Bounded and Unbounded Domains
and Related Attractors**
Edinburgh, September 23, 2010

The model

Related results

Existence

Attractors

Separation from singularities

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$$u_t - \operatorname{div}(b(u)\nabla w) = 0, \quad (\text{eq1})$$

where $b(u)$ is a mobility coefficient and w is an auxiliary variable, given by

$$w = -\Delta u + f(u) + \delta u_t + \gamma(u) - g \quad (\text{eq2})$$

where f is a monotone function,

and we possibly have a **viscosity** term, if $\delta > 0$, as well as lower order terms.

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Physical motivation - I: thin films

- ▶ Setting $b(u) \sim u^s$, $s > 0$, $\delta = 0$, and $f(u) + \gamma(u) - g = 0$, then we obtain the standard **thin-film equation**.
- ▶ Here, u denotes the **height** of the film and it is expected to be **nonnegative**; w is the **pressure**.
- ▶ Recently, a variant has been proposed (cf. [DeGennes], [Oron-Davis-Bankoff] for a description of the underlying physics).
- ▶ In this situation one still has $b(u) \sim u^s$ for some $s > 0$. In addition, we now have the nonlinearity

$$f(u) \sim -\frac{1}{u^\kappa} + \frac{1}{u^h},$$

- ▶ With a little abuse, we will include the the term $1/u^h$ into γ .

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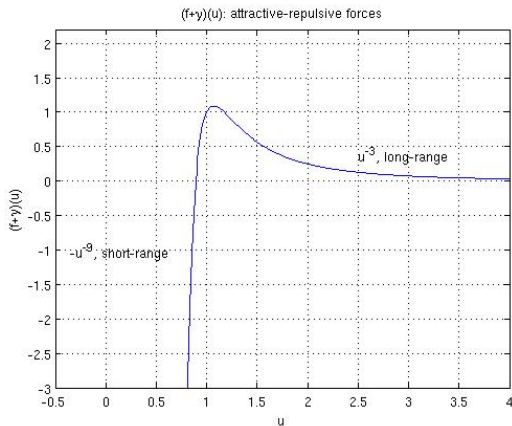
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The nonlinearity $(f + \gamma)(u) \sim -u^{-\kappa} + u^{-h}$

A model case is given by $h = 3, \kappa = 9$.



Energy and variational structure

- ▶ We note as F and Γ suitable antiderivatives of f and γ , respectively.
- ▶ Then, we can define the **energy** functional

$$\mathcal{E}(u) := \int_{\Omega} \left(\frac{|\nabla u|^2}{2} + F(u) + \Gamma(u) - gu \right). \quad (\text{energy})$$

- ▶ The sum $W := F + \Gamma$ represents the **configuration potential** of the film.
- ▶ Then, the system can be interpreted as the **gradient flow problem**

$$u_t = \operatorname{div} \mathcal{J}, \quad \mathcal{J} = b(u) \nabla w, \quad w = \partial \mathcal{E}$$

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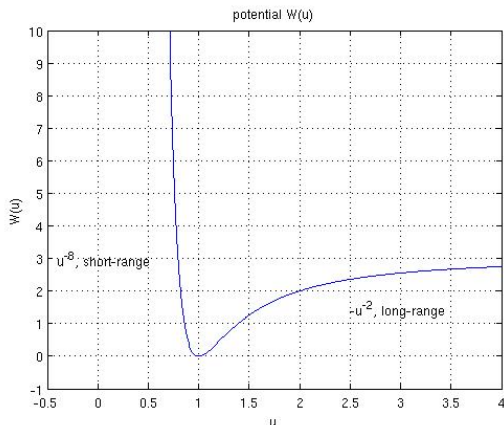
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The potential $W(u) \sim u^{-\kappa+1} - u^{-h+1}$

The potential W has the following form:



Physical effects of the potential

- ▶ By the **gradient-flow** structure, the system evolves towards the minimization of energy;
- ▶ We have **conservation** of total mass;
- ▶ The term $-u^{-9}$ in W' (or u^{-8} in W) describes **short-range attractive** forces, causing **wetting** (the film tends to spread on the substrate);
- ▶ The term u^{-3} in W' (or $-u^{-2}$ in W) describes **long-range attractive** forces, causing **de-wetting** (the film tends to retract).

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- ▶ System (eq1)-(eq2) can also represent the **Cahn-Hilliard** model, describing the evolution of the order parameter in binary materials.
- ▶ In particular, we can consider the case with nonlinear term behaving as

$$f(r) \sim (1 - r^2)^{-\kappa}, \quad r \in (-1, 1),$$

and mobility of the form $b(r) = (1 - r^2)^{\beta}$.

- ▶ In other words, we have **two** singularities at ± 1 , rather than **one** at 0
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The Grün approach (3D case)

- ▶ The **non-viscous** case is considered for $b(u) = u^s$, $0 < s < 4$.
- ▶ The function W' is assumed to satisfy $W'(u) \sim -u^{-\kappa} + u^{-h}$, with $\kappa > 2h + 1 > 1$.
- ▶ Then, existence of (at least) one weak solution is proved by means of an **entropy consistent finite element scheme**.
- ▶ A basic ingredient of the proof are the **energy** and **entropy** estimates.
- ▶ In Grün's proof, **strict positivity** is preserved at the **approximate** level.

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Existence Theorem

In what follows we will only consider the **3D case**.

We assume

$$s \in [0, 10), \quad \kappa > 1, \quad \kappa \geq s + 1, \quad \delta \geq 0, \quad (\kappa 1)$$

γ and g smooth and bounded,

$$u_0 > 0, \quad u_0^{1-\kappa} \in L^1(\Omega), \quad u_0 \in H^1(\Omega)$$

(energy space)

Then, (eq1)-(eq2) plus initial and **no-flux** b.c. has at least one weak solution.

Existence Theorem

In what follows we will only consider the **3D case**.

We assume

$$\mathbf{s} \in [0, 10), \quad \kappa > 1, \quad \kappa \geq \mathbf{s} + 1, \quad \delta \geq 0, \quad (\kappa 1)$$

γ and g smooth and bounded,

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Basic estimates

- ▶ **Energy estimate:** (eq1) $\times w$ + (eq2) $\times u_t$.

We obtain the **energy inequality**

$$\frac{d}{dt} \mathcal{E}(u) + \int_{\Omega} (b(u)|\nabla w|^2 + \delta|u_t|^2) \leq 0$$

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Do you see the idea?

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Phase-space; dissipativity

- ▶ The finiteness of the initial energy corresponds to asking $u_0 \in H^1(\Omega)$ with $u_0^{1-\kappa} \in L^1(\Omega)$.
- ▶ These conditions permit to individuate a **phase space** (“energy space” \mathcal{X}) that has a **complete metric structure**.
- ▶ Correspondingly, we can define a natural class of **weak solutions**.
- ▶ Thanks to the condition $\kappa \geq s + 1$ the entropy inequality does not require any further property on the initial datum.
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Weak trajectory attractor

- ▶ We consider the space of all weak solutions which satisfy the energy and entropy **inequalities**, namely

$$\mathcal{E}(u(T)) + \text{dissipative terms} \leq C_u e^{-\alpha t} + C_0,$$

where the constant C_0 is (linked to) the radius of the absorbing set and the constant C_u depends on the single trajectory u . At this level we cannot relate C_u to the “initial energy” $\mathcal{E}(u_0)$.

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Regularity of weak solutions

- ▶ From now on, all results will require $\delta > 0$.
- ▶ Can we have **equal signs** in the energy and entropy inequalities?
- ▶ If one looks at the **regularity properties** of the class of weak solutions, these are **too weak** to repeat the estimates at the “limit” level.
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An integration-by-parts formula

- ▶ We now know that $w, f(u) \in L^2$. So, they are now admissible test functions.
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$$(-\operatorname{div}(b(u)\nabla w), w) = \int_{\Omega} b(u)|\nabla w|^2$$

and that

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- ▶ Notice that we only know that $b(u) \in L^p$ for **some** $p > 1$ and that we do not know whether $\nabla w \in L^2$.
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- ▶ Let us assume that $\delta > 0$ and that (κ_2) holds.
- ▶ Then, thanks to the integration by parts formula we know that the **equal sign** holds both in the energy and in the entropy relations.
- ▶ Moreover, this is true for **all** solutions in the weak regularity class (in other words, the estimates need no longer to rely on an approximation argument).
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Our aim

- ▶ The bad properties of solutions are essentially due to the **degenerate** character of $b(u)$ (not to the **singular** character of $f(u)$).
- ▶ If we are able to show that solutions are "**strongly positive**" (i.e., larger than a strictly positive constant), then we get all the good properties we may want: **smooth regularity, uniqueness, ...**
- ▶ To prove this (**in the viscous case**), we need to restrict again the class of the admissible nonlinearities, asking

$$\kappa > 2s + 3 \quad (\text{in 3D}). \quad (\kappa 3)$$

- ▶ Then, we can prove that any weak solution starting from energy-regular initial data becomes **strongly positive for** $t > 0$.

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Idea of proof - (1)

- ▶ Basically, we use a modified Moser iteration argument.
- ▶ We rewrite, for $z := u^{-1}$, (eq2) as

$$\delta z_t + z^2 \Delta z^{-1} + z^{\kappa+2} = -z^2 w + \text{l.o.t.} \quad (\text{eq2-mod})$$

- ▶ Then, we test by $z^{\nu-1}$.
- ▶ The restriction ($\kappa 3$) on κ and the energy regularity ensure that the argument can be started for some small $\nu > 1$.

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$$\|z\|_{L^\infty(L^\nu)} + \|z\|_{L^{\nu+\kappa+1}} + \|z\|_{L^\nu(L^{3\nu})} \leq \text{r.h.s.}$$

- ▶ Then, for small ν we estimate the r.h.s. by the "potential" term. We can reach, in finitely many steps, any $\nu < 3(\kappa - s - 1)$.
- ▶ Then, we restart estimating the r.h.s. by the "time-derivative" term. Proceeding standardly, we arrive at an L^∞ -bound. But, to do this, we need to restart from $\nu > 3s + 2$.
- ▶ The restriction on κ comes from combining the two conditions.
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Further consequences of the separation property

- ▶ For $t > 0$ the system becomes at one time **nonsingular** and **nondegenerate**. Hence, we gain immediately
 - ▶ "Arbitrarily high" regularity of the solution;
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Thanks for your attention

