Global Behavior of Solutions to Brinkman-Forchheimer Equations

V.K. Kalantarov
Koç University, Istanbul
(joint work with S.Zelik)

Workshop on Dissipative PDE's on Bounded and Unbounded Domains, ICMS, Edinburgh, 24.09.10

$$\begin{cases} \partial_t u - \gamma\Delta u + f(u) + \nabla p = g(x), \ \nabla \cdot u = 0, \\ u(x,t) = 0, x \in \partial\Omega, \\ u(x,0) = u_0(x), x \in \Omega \end{cases} \tag{1}$$

Here $\Omega \subset R^3$ is a bounded domain with C^2 boundary $\partial\Omega$, $g = (g_1, g_2, g_3)$ is a given function, $u = (u_1, u_2, u_3)$ is the fluid velocity vector p is the pressure and f is a given nonlinearity. Nonlinear term $f \in C^3 \to C^3$ satisfies the conditions

$$(f'(u)v) \cdot v \geq -K|v|^2 + k|v|^2|u|^{r-1},$$

$$|f'(u)| \leq C(1 + |u|^{r-1}).$$

The typical example $f(u) = au + |u|^{r-1}u$.

- Global Existence and Uniqueness

- Global Existence and Uniqueness
- Regularity of Solutions

- Global Existence and Uniqueness
- Regularity of Solutions
- Existence of a Global Attractor

- Global Existence and Uniqueness
- Regularity of Solutions
- Existence of a Global Attractor
- Structural Stability

- Global Existence and Uniqueness
- Regularity of Solutions
- Existence of a Global Attractor
- Structural Stability
- Blow up problem

Number of papers is devoted to the study of structural stability of for BF equations: continuous dependence on changes in Brinkman and Forchheimer coefficients and convergence of solutions of BF equations to the solution of Forchheimer equation

$$\partial_t u + f(u) + \nabla p = g(x), \ \nabla \cdot u = 0,$$

as the Brinkman coefficient tends to zero.
Payne and Straughan, 1999

$$\partial_t u - \gamma \Delta u + au + b|u|u + \nabla p = h(x), \ \nabla \cdot u = 0,$$

$$\|U(t)\|_{L^2}^2 \leq \frac{\|F(t)\|_{L^2}^2 e^{-2at}}{4\gamma_1\gamma_2}\sigma^2$$

BF equations describe fluid flow in a saturated porous medium.u is the average fluid velocity, γ is Brinkman coefficient, b is the Forchheimer coefficient.

Çelebi, K. , Uğurlu, 2006

$$\partial_t u - \gamma \Delta u + au + b|u|^{r-1} u + \nabla p = h(x), \ \nabla \cdot u = 0,$$

Continuous dependence on parameters in H^1 sense. Blow-up of solutions for $b > 0$.

Gordeev (1973) ; Likhtarnikov (1979)

$$\begin{cases} \partial_t u - \Delta u + Ku + \phi(u) + \nabla p = h(x), \\ \partial_t p + \nabla \cdot (D(x)u) = 0, \end{cases}$$

where u is the velocity of the tidal flow and p is the deflection of the level of the see.

B.A.Kagan *Hydrodynamical Models of Tidal Motions in the Sea*, 1968

Çelebi, K. , Uğurlu, 2006

$$\partial_t u - \gamma \Delta u + au + b|u|^{r-1}u + \nabla p = h(x), \ \nabla \cdot u = 0,$$

Continuous dependence on parameters in H^1 sense. Blow-up of solutions for $b > 0$.
Uğurlu (2008), Ouyang and Qin (2008), Wang and Lin (2008)

Seta, Takegoshi, Kitano, Okui (2006)

$$\partial_t u - \gamma \Delta u + au + b|u|u + (u, \nabla)u + \nabla p = h(x)$$

As¡usual we set $\mathcal{V} = \{v \in C_0^\infty(\Omega)^3 : (\nabla, v) = 0\}$ and denote by H and $H^1 = V$ the closure of $\mathcal{V}$ in $L^2(\Omega)$ and $H^1(\Omega)$

The next Lemma gives the usual energy estimate for BF equations:

Lemma1. Let (u, p) be a sufficiently smooth solution of the problem . Then

$$\|u(t)\|^2 + \int_t^{t+1} \left[\|u(s)\|_{L^2}^2 + \|u(s)\|_{L^{r+1}}^{r+1}\right] ds \leq C\|u_0\|^2 e^{-\alpha t} + C(1 + \|g\|^2) \quad (2)$$

Multiplying the equation by u and using $f(u) \cdot u \geq -C + \kappa |u|^{r+1}$ we have

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \alpha\|u(t)\|_{H^1}^2 + \alpha\|u(t)\|_{L^{r+1}}^{r+1} \leq C(1 + \|g\|_{L^2}^2)$$

The inequality we obtain by using the Gronwall inequality.

Corollary Let (u,p) be a smooth solution of the problem (1). Then the following estimate holds:

$$\|\partial_t u\|_{L^1([t,t+1],H^{-2})} \leq Q(\|u_0\|_{L^2})e^{-\alpha t} + Q(\|g\|),$$

where the monotone function Q and the constant C are independent of t and u.

$$\partial_t u = Au - \Pi f(u) + \Pi g \tag{3}$$

$$\|f(u)\|_{L^{r^*}([t,t+1]L^{r^*})} \leq C\|u(0)\|^2_{L^2}e^{-\alpha t} + C(1 + \|g\|^2_{L^2})$$

The function $v = \partial_t u$ solves

$$\begin{cases} \partial_t v - \Delta v + f'(u)v + \nabla q = 0, \ \nabla \cdot v = 0 \\ v(0) = Au_0 - \Pi(f(0)) + \Pi g. \end{cases} \tag{4}$$

Moreover

$$\|v(0)\| \leq Q(\|u_0\|_{H^2}) + \|g\|_{L^2}$$

and the solution of the problem satisfies

$$\|v(t)\|_{L^2}^2 + \int_t^{t+1} \|v(s)\|_{H^1}^2 ds \leq Q(\|u_0\|_{H^2})e^{Kt} + Q(\|g\|_{L^2})$$

Corollary. A regular solution of the problem (1) satisfies:

$$\|u(t)\|_{H^2} + \|\nabla p(t)\|_{L^2} \le Q(\|u(0)\|_{H^2})e^{Kt} + Q(\|g\|_{L^2}) \qquad (5)$$

for some positive constant K and monotone function Q independent of t and u_0.

Indeed, due to the control of $\|v(t)\|_{L^2}$, we may rewrite equation (1) as an elliptic boundary value problem

$$\Delta w(t) - f(w(t)) + \nabla p(t) = g_u(t) := -g + \partial_t u(t) \qquad (6)$$

and apply the maximal regularity result to that equation. Together with (**??**) this gives indeed estimate (5) and proves the corollary.

Lemma 2. If (u, p) is a sufficiently regular solution of the problem (1) then

$$\|\partial_t u(t)\|_{L^2} \le \frac{1+t^3}{t^3} Q(\|u_0\|_{L^2}) e^{-\alpha t} + Q(\|g\|_{L^2}), \ \ t > 0. \qquad (7)$$

Theorem 1.Let (u, p) be a sufficiently regular solution of (1), then

$$\|u(t)\|_{H^2} + \|p(t)\|_{H^1} \le Q(\|u_0\|_{H^2}) e^{-\alpha t} + Q(\|g\|_{L^2}), \ \ t > 0. \qquad (8)$$

Moreover

$$\|u(t)\|_{H^2} + \|p(t)\|_{H^1} \le Q(\frac{1+t^3}{t^3}\|u_0\|_{L^2}) e^{-\alpha t} + Q(\|g\|_{L^2}), \ \ t > 0. \qquad (9)$$

These estimates allow us to prove the existence and uniqueness of a solution of the problem (1) as well as to establish the existence of a global attractor for the associated semigroup.

Definition. A function

$$u \in C([0,\infty), H) \cap L^2_{loc}([0,\infty), H^1) \cap L^{r+1}_{loc}([0,\infty), L^{r+1}(\Omega)) \quad (10)$$

is called a weak solution of (1) if it satisfies (1) in the sense of distributions, i.e.,

$$-\int_{\mathbb{R}} (u(t), \partial_t \varphi(t))\, dt = \\ -\int_{\mathbb{R}} (\nabla u(t), \nabla \varphi(t)) - (f(u(t)), \varphi(t)) + (g, \varphi(t))\, dt$$

for all $\varphi \in C_0^\infty(\mathbb{R}_+ \times \Omega)$ such that $(\nabla, \varphi(t)) \equiv 0$.
The next lemma establishes the uniqueness of a weak solution.

$$\begin{cases} -\Delta w + f(w) + \nabla p = g(x), \ \nabla \cdot w = 0, \ x \in \Omega \\ w = 0, x \in \partial\Omega, \quad \int_\Omega p(x)dx = 0 \end{cases} \tag{11}$$

Lemma 3. If $g \in L^q(\Omega)$ then the problem (11) has a unique solution

$$(w,p) \in \mathcal{F}_q := \{(w,p) : w \in W^{2,q}(\Omega) \cap L^{r+1}(\Omega), \ p \in W^{1,q}(\Omega)\}$$

and the following estimate holds true

$$\|w\|_{W^{2,q}} + \|p\|_{L^{r+1}}^{r+1} \leq C(1 + \|g\|_{L^q}) \tag{12}$$

Theorem 2. Let w be an energy solution of the problem (11), $g \in L^2$ and the assumptions on f hold. Then $w \in H^2(\Omega)$ and

$$\|w\|_{H^2} + \|p\|_{H^1} \leq Q\left(\|(w,p)\|_{\mathcal{F}_q}\right)\left(1 + \|g\|_{L^2}^{2-k}\right)$$

for some increasing function Q and some positive $k = k(r)$.

$$\partial_t u - \Delta u + (u \cdot \nabla)u + f(u) + \nabla p = h(x), \;\; \nabla \cdot u = 0, \qquad (13)$$

The assumption $r \geq 3$ guarantees that

$$(u, \nabla)u \in L^{4/3} \subset L^q, q := (r+1)^* \leq 4/3.$$

If $r > 3$ and $g \in L^2$ then for every $u_0 \in H$ the problem (13) has a unique weak solution u that satisfies the energy estimate (2).

$$\partial_t v - \Delta v + (v, \nabla)u_1 + (u_2, \nabla)v + \nabla q = f(u_2) - f(u_1), \;\; (\nabla, v) = 0,$$

$$\frac{d}{dt}\|v\|_{L^2}^2 + 2\|\nabla v\|_{L^2}^2 + \alpha(|u_1|^{r-1} + |u_2|^{r-1}, |v|^2) \leq$$

$$C\|v\|_{L^2}^2 + 2|((v, \nabla u)u_1, v)|.$$

Her we have used

$$(f(u_1)-f(u_2),u_1-u_2) \geq -C\|u_1-u_2\|^2_{L^2}+\alpha(|u_1|^{r-1}+|u_2|^{r-1},|v|^2)$$

$r-1>2$

$$\begin{aligned} 2|((v,\nabla u)u_1,v)| \leq 2(|u_1|\cdot|v|,|\nabla v|) \leq \\ \|\nabla v\|^2_{L^2}+C(|u_1|^2,|v|^2) \leq \\ \|\nabla v\|^2_{L^2}+\alpha(|u_1|^{r-1}+|u_2|^{r-1},|v|^2)+C\|v\|^2_{L^2}. \end{aligned}$$

KVBF equations

$$\begin{cases} \partial_t u - \Delta u - \alpha \Delta \partial_t u + (u \cdot \nabla) u + au + b|u|^{r-1} u + \nabla p = h(x), \\ \nabla \cdot u = 0, \end{cases} \tag{14}$$

[CKU] V.K. Kalantarov, A.O.Çelebi and D.UğurluOn continuous dependence on coefficients of the Brinkman-Forchheimer equations. Appl. Math. Lett., 19 (2006), no. 8, 801–807.
[Go] R. G. Gordeev, The existence of a periodic solution in a certain problem of tidal dynamics. In "Problems of mathematical analysis, No. 4: Integral and differential operators. Differential equations", pp. 3–9, 142–143, Leningrad. Univ., Leningrad, 1973.
[Li]A. L. Likhtarnikov, Existence and stability of bounded and periodic solutions in a nonlinear problem of tidal dynamics, In "The direct method in the theory of stability and its application" (Irkutsk, 1979), pp. 83–91, 276, Nauka, Novosibirsk, 1981.
[PS] L. E. Payne and B. Straugham, Convergence and continuous dependence for the Brinkman-Forchheimer equations. Studies in Applied Mathematics **10**(1999), 419-439.
[WL]B. Wang and S. Lin, Existence of global attractors for the three-dimensional Brinkman-Forchheimer equation. Mathematical Methods in the Applied Sciences, Math. Meth. Appl. Sci. (2008)

Thank you fro the attention