A singular limit for rotating compressible fluids

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Rotating compressible fluid

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon}(\mathbf{g} \times \varrho \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}$$

$$\mathbb{S} = \mu \Big(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \Big), \ \mu > 0, \ \mathbf{g} = [0, 0, 1]$$

Boundary conditions

$$\Omega = R^2 \times [0,1]$$

$$\mathbf{u} \cdot \mathbf{n} = u_3|_{\partial\Omega} = 0, \ [\mathbb{S}\mathbf{n}] \times \mathbf{n} = [S_{2,3}, -S_{1,3}, 0]|_{\partial\Omega} = 0$$

Ill-prepared initial data

$$\varrho_\varepsilon(0,\cdot) = \overline{\varrho} + \varepsilon r_{0,\varepsilon},\ \{r_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^2 \cap L^\infty(\Omega)$$

$$\mathbf{u}_\varepsilon(0,\cdot) = \mathbf{u}_{0,\varepsilon},\ \{\mathbf{u}_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^2 \cap L^\infty(\Omega; R^3)$$

Asymptotic limit

Suppose

$$p(\varrho) \approx \varrho^{\gamma}, \ \gamma > 3/2$$

$$r_{0,\varepsilon} \to r_0 \text{ weakly in } L^2(\Omega)$$

$$\mathbf{u}_{0,\varepsilon} \to \mathbf{U}_0 \text{ weakly in } L^2(\Omega; R^3)$$

Then

$$r_\varepsilon \equiv \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \to r \text{ weakly-(*) in } L^\infty(0,T; L^2 \oplus L^\gamma(\Omega))$$

$$\mathbf{u}_\varepsilon \to \ \mathbf{U} \text{ weakly in } L^2(0,T; W^{1,2}(\Omega; R^3))$$

and strongly in $L^2((0,T) \times K; R^3)$ for any compact $K \subset \Omega$

Moreover

$$\mathbf{g} \times \mathbf{U} + \frac{p'(\overline{\varrho})}{\overline{\varrho}} \nabla_x r = 0, \ r = r(x_1, x_2), \ \mathbf{u} = [\mathbf{U}_h, 0], \ \mathbf{U}_h = \mathbf{U}_h(x_1, x_2)$$

$$\partial_t \left(\Delta_h r - \frac{1}{p'(\overline{\varrho})} r \right) + \nabla_x^\perp r \cdot \nabla_h (\Delta_h r) = \frac{\mu}{\overline{\varrho}} \Delta_h^2 r$$

$$r(0, \cdot) = \tilde{r}, \ -\Delta_h \tilde{r} + \frac{1}{p'(\overline{\varrho})} \tilde{r} = \overline{\varrho} \int_0^1 \mathrm{curl}_h \mathbf{U}_{0,h} \ \mathrm{d}x_3 + \int_0^1 r_0 \mathrm{d}x_3$$

Related results

- **Incompressible case:**
  J.-Y.Chemin, B.Desjardins, I.Gallagher, E.Grenier: *Basic of mathematical geophysiscs*, Oxford University Press 2006
- **Compressible case:**
  D.Bresch, B.Desjardins, D.Gérard-Varet: Rotating dluis in a cylinder, *Disc. Cont. Dynamical Syst.* **11** (2004), pages 1133-1176

Energy inequality

$$\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{1}{\varepsilon^2} E(\varrho, \overline{\varrho}) \right) (\tau, \cdot) \ \mathrm{d}x + \int_0^\tau \int_\Omega \mathbb{S} : \nabla_x \mathbf{u} \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \int_\Omega \left(\frac{1}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2} E(\varrho_{0,\varepsilon}, \overline{\varrho}) \right) \ \mathrm{d}x$$

$$E(\varrho, \overline{\varrho}) = H(\varrho) - H'(\overline{\varrho})(\varrho - \overline{\varrho}) - H(\overline{\varrho})$$

$$H(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \ \mathrm{d}z$$

Acoustic equation

$$p'(\overline{\varrho}) = 1$$

$$\varepsilon \partial_t r_\varepsilon + \mathrm{div}_x \mathbf{V}_\varepsilon = 0$$

$$\varepsilon \partial_t \mathbf{V}_\varepsilon + \Big(\mathbf{g} \times \mathbf{V}_\varepsilon + \nabla_x r_\varepsilon \Big) = \varepsilon \mathbf{f}_\varepsilon$$

$$r_\varepsilon = \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}, \ \mathbf{V}_\varepsilon = \varrho_\varepsilon \mathbf{u}_\varepsilon$$

$$\mathbf{f}_\varepsilon = \mathrm{div}_x \mathbb{S}_\varepsilon - \mathrm{div}_x (\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) - \frac{1}{\varepsilon^2} \nabla_x \Big(p(\varrho_\varepsilon) - p'(\overline{\varrho})(\varrho_\varepsilon - \overline{\varrho}) - p(\overline{\varrho}) \Big)$$

Spectral analysis

$$\mathcal{B}\left[\begin{array}{c} r \\ \mathbf{V} \end{array}\right] = \left[\begin{array}{c} \mathrm{div}_x \mathbf{V} \\ \mathbf{g} \times \mathbf{V} + \nabla_x r \end{array}\right]$$

Dispersion relation (in Fourier variables)

$$\lambda^2 = -\mu$$

$$\mu = \frac{1 + |\xi|^2 + k^2 \pm \sqrt{(1 + |\xi|^2 + k^2)^2 - 4k^2}}{2}$$

$$\mathrm{Ker}[\mathcal{B} = \{[r, \mathbf{V} \mid r = r(x_1, x_2)$$

$$\nabla_h \cdot \mathbf{V}_h = 0,\ \nabla_h r = [V_2, -V_1]\}$$

Variation-of-constants formula

$$\left[\begin{array}{c} r_\varepsilon \\ \mathbf{V}_\varepsilon \end{array} \right](t)$$

$$= \exp \left(-\mathcal{B} \frac{t}{\varepsilon} \right) \left[\begin{array}{c} r_{0,\varepsilon} \\ \mathbf{V}_{0,\varepsilon} \end{array} \right] + \int_0^T \exp \left(-\mathcal{B} \frac{t-s}{\varepsilon} \right) \left[\begin{array}{c} 0 \\ \mathbf{f}_\varepsilon \end{array} \right] \ {\rm d}s$$

RAGE theorem

Theorem

Let H be a Hilbert space, $A : \mathcal{D}(A) \subset H \to H$ a self-adjoint operator, $C : H \to H$ a compact operator, and P_c the orthogonal projection onto H_c, where

$$H = H_c \oplus \mathrm{cl}_H \Big\{ \mathrm{span} \{ w \in H \mid w \textit{ an eigenvector of } A \} \Big\}.$$

Then

$$\left\| \frac{1}{\tau} \int_0^\tau \exp(-\mathrm{i}tA) C P_c \exp(\mathrm{i}tA) \ \mathrm{d}t \right\|_{\mathcal{L}(H)} \to 0 \textit{ for } \tau \to \infty.$$

RAGE theorem

C non-negative selfadjoint:

$$\frac{1}{T}\int_0^T \left\langle \exp\left(-\mathrm{i}\frac{t}{\varepsilon}A\right) C \exp\left(\mathrm{i}\frac{t}{\varepsilon}A\right) P_c X, Y \right\rangle_H \mathrm{d}t \le h(\varepsilon)\|X\|_H\|Y\|_H$$

$$Y = P_c X$$

$$\frac{1}{T}\int_0^T \left\| \sqrt{C} \exp\left(\mathrm{i}\frac{t}{\varepsilon}A\right) P_c X \right\|_H^2 \mathrm{d}t \le h(\varepsilon)\|X\|_H^2$$

$$h(\varepsilon) \to 0 \text{ as } \varepsilon \to 0$$

RAGE theorem

$$\frac{1}{T^2}\left\| \sqrt{C} P_c \int_0^t \exp\left(\mathrm{i}\frac{t-s}{\varepsilon} A \right) X(s)\ \mathrm{d}s \right\|^2_{L^2(0,T;H)}$$

$$\leq \frac{1}{T}\int_0^T \int_0^T \left\| \sqrt{C} \exp\left(\mathrm{i}\frac{t-s}{\varepsilon} A \right) P_c X(s) \right\|^2_H \ \mathrm{d}t\ \mathrm{d}s$$

$$\leq h(\varepsilon)\int_0^T \left\| \exp\left(-\mathrm{i}\frac{s}{\varepsilon} A \right) X(s) \right\|^2_H \ \mathrm{d}s = h(\varepsilon)\int_0^T \|X(s)\|^2\ \mathrm{d}s$$

RAGE theorem

Apply RAGE theorem to:

$$A = \mathrm{i}\mathcal{B},\ C = P_M \circ \chi$$

$$P_M : L^2(\Omega) \times L^2(\Omega; R^3) \to H_M$$

$$H_M = \{[r, \mathbf{V}] \mid \tilde{r}(\xi_h, k) = 0,\ \tilde{\mathbf{V}}(\xi_h, k) = 0$$
$$\text{for } |\xi_h| + |k| > M\}$$

$\tilde{r}$, $\tilde{\mathbf{V}}$ denote the Fourier transform