Energy dissipation, regularity and statistical dynamics for a nematic liquid crystal flow

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Complex fluids: Basic laws

- Conservation of mass:

$$\nabla \cdot u = 0 \tag{1}$$

where u is a vector valued function expressing the velocity of the fluid at a point in space.

- The balance of momentum is

$$\rho \left(\frac{\partial u}{\partial t} + (u \cdot \nabla) u \right) = \nabla \cdot T - \nabla p \tag{2}$$

where ρ is the density, T is the stress tensor and p is an isotropic pressure.

- The stress tensor T represents the forces which the material develops in response to being deformed.

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- The stress tensor T represents the forces which the material develops in response to being deformed.
- We need a constitutive relation relating T to the motion of the fluid.
- The constitutive law for the classical Newtonian fluid is

$$T = \nu \left(\nabla u + (\nabla u)^{tr} \right)$$

where ν is the viscosity. In this case the system (1), (2) becomes the celebrated Navier-Stokes system of equations.

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The additional stress tensor and energy dissipation

- In the case of non-Newtonian fluids containing suspensions of liquid crystal molecules the stress has an additional component representing forces due to the liquid crystal molecules.
- On the other hand one should have an equation for the liquid crystals, showing how the flow affects the orientation and distribution of the molecules.
- The additional stress tensor encodes the coupling between the flow and the molecules.
- The form of the additional stress tensor is directly related to energy dissipation. More precisely the "content" of the stress tensor should be such that the total energy of the fluid

$$\underbrace{E(t)}_{\text{total energy}} = \underbrace{\frac{1}{2}\int_{\mathbb{R}^d} |u(x,t)|^2\,dx}_{\text{kinetic energy of the flow}} + \underbrace{\mathcal{F}(t)}_{\text{free energy of the molecules}}$$

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The equations

The flow equations:

- $$\begin{cases} \partial_t u + u\nabla u = \nu\Delta u + \nabla p + \nabla\cdot\tau + \nabla\cdot\sigma \\ \nabla\cdot u = 0 \end{cases}$$

  where we have the symmetric part of the additional stress tensor:

  $$\tau = -\xi\left(Q + \frac{1}{3}Id\right)H - \xi H\left(Q + \frac{1}{3}Id\right)$$
  $$+2\xi(Q + \frac{1}{3}Id)QH - L\left(\nabla Q \odot \nabla Q + \frac{\mathrm{tr}(Q^2)}{3}Id\right)$$

  and an antisymmetric part $\sigma = QH - HQ$ where

  $$H = L\Delta Q - aQ + b[Q^2 - \frac{\mathrm{tr}(Q^2)}{3}Id] - cQ\mathrm{tr}(Q^2)$$

- The equation for the liquid crystal molecules, represented by functions with values in the space of Q-tensors (i.e. symmetric and traceless $d\times d$ matrices):

  $$(\partial_t + u\cdot\nabla)Q - S(\nabla u, Q) = \Gamma H$$

  with

  $$S(\nabla u, Q) \overset{def}{=} (\xi D + \Omega)(Q + \frac{1}{3}Id) + (Q + \frac{1}{3}Id)(\xi D - \Omega) - 2\xi(Q + \frac{1}{3}Id)\mathrm{tr}(Q\nabla u)$$

Energy dissipation and weak solutions-apriori bounds I

- The total energy

$$E(t) \stackrel{\text{def}}{=} \underbrace{\int_{\mathbb{R}^d} \frac{L}{2}|\nabla Q|^2 + \frac{a}{2}\mathrm{tr}(Q^2) - \frac{b}{3}\mathrm{tr}(Q^3) + \frac{c}{4}\mathrm{tr}^2(Q^2)\,dx}_{\text{free energy of the liquid crystal molecules}} + \underbrace{\frac{1}{2}\int_{\mathbb{R}^d} |u|^2(t,x)\,dx}_{\text{kinetic energy of the flow}}$$

  is decreasing $\frac{d}{dt}E(t) \leq 0$.
- Simple proof: multiply the first equation in the system to the right by $-H$, take the trace, integrate over $\mathbb{R}^d$ and by parts and sum with the second equation multiplied by u and integrated over $\mathbb{R}^d$ and by parts.
- In the process maximal derivatives are cancelled and you observe suprizing non-trivial cancellations

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Energy dissipation and weak solutions-apriori bounds II

- $$\frac{d}{dt}E(t) = -\nu \int_{\mathbb{R}^d} |\nabla u|^2\,dx$$
  $$-\Gamma \int_{\mathbb{R}^d} \mathrm{tr}\left(L\Delta Q - aQ + b[Q^2 - \frac{\mathrm{tr}(Q^2)}{d} Id] - cQ\mathrm{tr}(Q^2)\right)^2 dx \le 0$$
- Note that this does not readily provide L^p norm estimates.

Proposition

For $d = 2, 3$ there exists a weak solution (Q, u) of the coupled system, with restriction $c > 0$, subject to initial conditions

$$Q(0,x) = \bar{Q}(x) \in H^1(\mathbb{R}^d),\ u(0,x) = \bar{u}(x) \in L^2(\mathbb{R}^d), \nabla \cdot \bar{u} = 0 \ \ in\ \mathcal{D}'(\mathbb{R}^d) \quad (3)$$

The solution (Q, u) is such that $Q \in L^\infty_{loc}(\mathbb{R}_+; H^1) \cap L^2_{loc}(\mathbb{R}_+; H^2)$ and $u \in L^\infty_{loc}(\mathbb{R}_+; L^2) \cap L^2_{loc}(\mathbb{R}_+; H^1)$.

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Some other types of complex fluids

- Oldroyd-B:

$$\begin{cases} \partial_t u + u\nabla u - \nu\Delta u + \nabla p = \mu\nabla\cdot\rho \\ \partial_t\rho + u\nabla\rho + a\rho + \rho\Omega - \Omega\rho - b(D\rho+\rho D) = \mu_2 D \end{cases}$$

- The formal energy estimate is:

$$\frac{1}{2}\frac{d}{dt}(\mu_2\|u(t)\|_{L^2} + \mu_1\|\rho(t)\|_{L^2}) + \nu\mu_2\|\nabla u(t)\|_{L^2} + a\mu_1\|\rho(t)\|_{L^2} \leq |b|\|D(t)\|_{L^\infty}\|\rho(t)\|_{L^2}$$

- Smoluchowski Navier-Stokes systems

$$\begin{cases} \frac{\partial v}{\partial t} + v\cdot\nabla v - \nu\Delta v + \nabla p = \nabla\cdot\tau & \text{in} \quad \Omega\times(0,T) \\ \frac{\partial f}{\partial t} + v\nabla f + \nabla_g\cdot(Wf) - \Delta_g f = 0 & \text{in} \quad \Omega\times(0,T) \\ \nabla\cdot v = 0 & \text{in} \quad \Omega\times(0,T), \end{cases}$$

where
$\tau_{ij} = \int_M \gamma_{ij}^{(1)}(m)f(t,x,m)dm + \int_M\int_M \gamma_{ij}^{(2)}(m_1,m_2)f(t,x,m_1)f(t,x,m_2)dm$ and
$W = c_\alpha^{ij}\partial_j v_i$.

- The energy $E(t) = \frac{1}{2}\int_{\mathbb{R}^d}|u(t,x)|^2\,dx + \int_{\mathbb{R}^d}\int_M f\log f\,dx\,dm$ decreases in time.

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Regularity difficulties: the maximal derivatives and "the co-rotational parameter"

- Recall the system:

$$\begin{cases} (\partial_t + u\cdot\nabla)Q - \big(\xi D(u) + \Omega(u)\big)(Q + \frac{1}{3}Id) + (Q + \frac{1}{3}Id)\big(\xi D(u) - \Omega(u)\big) \\ -2\xi(Q + \frac{1}{3}Id)\mathrm{tr}(Q\nabla u) = \Gamma H \\ \\ \partial_t u + u\nabla u = \nu\Delta u_\alpha + \nabla p + \nabla\cdot\Big(QH - HQ\Big) \\ -\nabla\cdot\Big(\xi\left(Q + \frac{1}{3}Id\right)H + \xi H\left(Q + \frac{1}{3}Id\right)\Big) \\ +2\xi\nabla\cdot\Big((Q + \frac{1}{3})QH\Big) - L\nabla\cdot(\nabla Q\odot\nabla Q + \frac{1}{3}\mathrm{tr}(Q^2)) \\ \nabla\cdot u = 0 \end{cases}$$

with $H = L\Delta Q - aQ + b[Q^2 - \frac{\mathrm{tr}(Q^2)}{3}Id] - cQ\mathrm{tr}(Q^2)$.

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- Where's the difficulty?
- If $\xi = 0$ maximal derivatives only

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Energy dissipation revisited I

- Cancellations that appear in the energy dissipation destroy some high derivatives...
- The cancellation lemma:

Lemma

For any symmetric matrices $Q', Q \in \mathbb{R}^{d\times d}$ and $\Omega_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} - u_{\beta,\alpha}) \in \mathbb{R}^{d\times d}$ (decaying fast enough at infinity so that we can integrate by parts, in the formula below, without boundary terms) we have:

$$\int_{\mathbb{R}^d} tr((\Omega Q' - Q'\Omega)\Delta Q)\, dx - \int_{\mathbb{R}^d} \partial_\beta (Q'_{\alpha\gamma}\Delta Q_{\gamma\beta} - \Delta Q_{\alpha\gamma} Q'_{\gamma\beta}) u_\alpha \, dx = 0$$

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- Idea: differentiate the equations and to the highest derivatives the equations will keep the same structure (as the initial system) plus lower-order derivatives perturbation; thus we can use the high-derivatives cancellations available for the initial system to avoid the maximal derivatives
- Unlike in previous approaches (in complex fluids) we do not estimate the two equations separately but always together.

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Energy dissipation revisited II

- Littlewood-Paley language: take take $\chi \in \mathcal{D}(B(0,1))$ such that $\chi \equiv 1$ on $B(0,1/2)$ and let $\varphi(\xi) = \chi(\xi/2) - \chi(\xi)$. Define

$$\Delta_q u = \mathcal{F}^{-1}(\varphi(2^{-q}\xi)\mathcal{F}u), \; S_q u = \mathcal{F}^{-1}(\chi(2^{-q}\xi)\mathcal{F}u)$$

- Then we have in the sense of distributions $u = S_0 u + \sum_{q>0} \Delta_q u$.
- Bony's paraproduct decomposition:

$$\Delta_q(ab) = S_{q-1}a\Delta_q b + \Sigma_{|q'-q|\leq 5}[\Delta_q, S_{q'-1}a]\Delta_{q'}b$$
$$+\Sigma_{|q'-q|\leq 5}(S_{q'-1}a - S_{q-1}a)\Delta_q\Delta_{q'} + \Sigma_{q'>q-5}\Delta_q(S_{q'+2}b\Delta_{q'}a)$$

- In our case, the equation in Q becomes:

$$\partial_t \Delta_q u - \nu\Delta\Delta_q u = \Delta_q \nabla p + L\nabla\cdot(S_{q-1}Q\Delta_q\Delta Q - \Delta_q\Delta Q S_{q-1}Q)$$
$$-\Delta_q(u\nabla u) - L\nabla\cdot\Delta_q\left(\nabla Q\odot\nabla Q - \frac{1}{3}\mathrm{tr}(\nabla Q\odot\nabla Q)\right)$$
$$+ \text{ perturbative (lower derivatives) terms}$$

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The rate of increase of the high norms- the Brezis-Gallouet trick and beyond

- Let $y(t) = \|u(t)\|_{H^1}^2 + \|\nabla Q(t)\|_{H^1}$. An estimate of the form

$$y'(t) \leq f(t)y(t)\log(1+y(t))$$

  and $f(t) \leq Ce^t$ would give $y(t) \leq Ce^{e^{e^t}}$.

- Where does the logarithm come from? Brezis-Gallouet trick-logarithmic embedding!

$$\|u\|_{L^\infty(\mathbb{R}^2)} \leq \|u\|_{H^1}\left(1+\ln(e+\|u\|_{H^{1+\varepsilon}})\right) \sim f(t)\ln(1+y)$$

- This works in the co-rotational case $\xi = 0$ after "peeling out" the maximal derivatives.
- If $\xi \neq 0$ turns out that we can obtain an estimate of the form:

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The regularity result, in $2D$

Theorem

Let $s > 0$ and $(\bar{Q}, \bar{u}) \in H^{s+1}(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$. There exists a global a solution $(Q(t,x), u(t,x))$ of the coupled system, with restriction $c > 0$, subject to initial conditions

$$Q(0,x) = \bar{Q}(x),\ u(0,x) = \bar{u}(x)$$

and $Q \in L^2_{loc}(\mathbb{R}_+; H^{s+2}(\mathbb{R}^2)) \cap L^\infty_{loc}(\mathbb{R}_+; H^{s+1}(\mathbb{R}^2))$, $u \in L^2_{loc}(\mathbb{R}_+; H^{s+1}(\mathbb{R}^2) \cap L^\infty_{loc}(\mathbb{R}_+; H^s)$. Moreover, we have:

$$L\|\nabla Q(t,\cdot)\|^2_{H^s(\mathbb{R}^2)} + \|u(t,\cdot)\|^2_{H^s(\mathbb{R}^2)} \leq C\left(e + \|\bar{Q}\|_{H^{s+1}(\mathbb{R}^2)} + \|\bar{u}\|_{H^s(\mathbb{R}^2)}\right)^{e^{e^{e^{Ct}}}} \tag{4}$$

where the constant C depends only on $\bar{Q}, \bar{u}, a, b, c, \Gamma$ and L. If $\xi = 0$ the increase in time of the norms above can be made to be only doubly exponential.

The difference between $\xi = 0$ (co-rotational) and $\xi \neq 0$

- Recall the system:

$$\begin{cases} (\partial_t + u \cdot \nabla)Q - \big(\xi D(u) + \Omega(u)\big)(Q + \frac{1}{3}Id) + (Q + \frac{1}{3}Id)\big(\xi D(u) - \Omega(u)\big) \\ -2\xi(Q + \frac{1}{3}Id)\mathrm{tr}(Q\nabla u) = \Gamma H \\ \\ \partial_t u + u\nabla u = \nu\Delta u_\alpha + \nabla p + \nabla \cdot \Big(QH - HQ\Big) \\ -\nabla \cdot \Big(\xi\left(Q + \frac{1}{3}Id\right)H + \xi H\left(Q + \frac{1}{3}Id\right)\Big) \\ +2\xi\nabla \cdot \Big((Q + \frac{1}{3})QH\Big) - L\nabla \cdot \left(\nabla Q \odot \nabla Q + \frac{1}{3}\mathrm{tr}(Q^2)\right) \\ \nabla \cdot u = 0 \end{cases}$$

with $H = L\Delta Q - aQ + b[Q^2 - \frac{\mathrm{tr}(Q^2)}{3}Id] - cQ\mathrm{tr}(Q^2)$.

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A technical trick-how the "double logarithm" appears I

- We want to obtain $y' + \frac{\nu}{2}\|\nabla u\|^2_{H^s} + \frac{\Gamma L}{2}\|\Delta Q\|^2_{H^s} \le y\ln(e+y)\Big(1+\ln\big(e+\ln(e+y)\big)\Big)$ with $y = \|u\|^2_{H^s} + \|\nabla Q\|^2_{H^s}$.
- Attempt to estimate

$$\mathcal{I} \stackrel{\text{def}}{=} \sum_{|q'-q|\le 5} \|S_{q'-1}\nabla Q\Delta_{q'}u\|_{L^2}\|Q\|_{L^\infty}\|\Delta\Delta_q Q\|_{L^2}$$

($\mathcal{I} \sim \Delta_q($ worst term) so we want to estimate $2^{2qs}\mathcal{I} \sim y$)

- We have

$$|\mathcal{I}| \le \sum_{|q'-q|\le 5} \|S_{q'-1}\nabla Q\|_{L^{\frac{2}{\varepsilon}}}\|Q\|_{L^\infty}\|\Delta_{q'}u\|_{L^{\frac{2}{1-\varepsilon}}}\|\Delta\Delta_q Q\|_{L^2} \quad (5)$$

- Using the interpolation inequality

$$\|f\|_{L^{2p}} \le C\sqrt{p}\|f\|^{\frac{1}{p}}_{L^2}\|\nabla f\|^{1-\frac{1}{p}}_{L^2}$$

with $p = \frac{1}{1-\varepsilon} \in [1,2]$, we obtain:

$$|\mathcal{I}| \le C\sum_{|q'-q|\le 5} \|S_q\nabla Q\|_{L^{\frac{2}{\varepsilon}}}\|Q\|_{L^\infty}\|\Delta_q u\|^{1-\varepsilon}_{L^2}\|\Delta_q\nabla u\|^{\varepsilon}_{L^2}\|\Delta_q\Delta Q\|_{L^2},$$

where $C > 0$ is constant independent of $\varepsilon \in (0, \frac{1}{2})$.

- Using Young's inequality we obtain:

$$|\mathcal{I}| \le C\sum_{|q'-q|\le 5}\left(\left(\|S_q\nabla Q\|_{L^{\frac{2}{\varepsilon}}}\|Q\|_{L^\infty}\right)^{\frac{2}{1-\varepsilon}}\|\Delta_q u\|^2_{L^2} + \frac{\nu}{100}\|\Delta_q\nabla u\|^2_{L^2} + \frac{\Gamma L^2}{100}\|\Delta_q\Delta Q\|^2_{L^2}\right)$$

$$\le 2^{-2qs}\left(C\left(\|S_q\nabla Q\|_{L^{\frac{2}{\varepsilon}}}\|Q\|_{L^\infty}\right)^{\frac{2}{1-\varepsilon}}\|u\|^2_{H^s} + \frac{\nu}{100}\|\nabla u\|^2_{H^s} + \frac{\Gamma L^2}{100}\|\Delta Q\|_{H^s}\right)$$

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A technical trick-how the "double logarithm" appears I

- We want to obtain $y' + \frac{\nu}{2}\|\nabla u\|_{H^s}^2 + \frac{\Gamma L}{2}\|\Delta Q\|_{H^s}^2 \le y\ln(e+y)\Big(1+\ln\big(e+\ln(e+y)\big)\Big)$ with $y = \|u\|_{H^s}^2 + \|\nabla Q\|_{H^s}^2$.
- Attempt to estimate

$$\mathcal{I} \stackrel{\text{def}}{=} \sum_{|q'-q|\le 5} \|S_{q'-1}\nabla Q \Delta_{q'} u\|_{L^2} \|Q\|_{L^\infty} \|\Delta\Delta_q Q\|_{L^2}$$

($\mathcal{I} \sim \Delta_q(\text{ worst term})$ so we want to estimate $2^{2qs}\mathcal{I} \sim y$)

- We have

$$|\mathcal{I}| \le \sum_{|q'-q|\le 5} \|S_{q'-1}\nabla Q\|_{L^{\frac{2}{\varepsilon}}} \|Q\|_{L^\infty} \|\Delta_{q'} u\|_{L^{\frac{2}{1-\varepsilon}}} \|\Delta\Delta_q Q\|_{L^2} \tag{5}$$

- Using the interpolation inequality

$$\|f\|_{L^{2p}} \le C\sqrt{p}\|f\|_{L^2}^{\frac{1}{p}} \|\nabla f\|_{L^2}^{1-\frac{1}{p}}$$

with $p = \frac{1}{1-\varepsilon} \in [1,2]$, we obtain:

$$|\mathcal{I}| \le C \sum_{|q'-q|\le 5} \|S_q \nabla Q\|_{L^{\frac{2}{\varepsilon}}} \|Q\|_{L^\infty} \|\Delta_q u\|_{L^2}^{1-\varepsilon} \|\Delta_q \nabla u\|_{L^2}^{\varepsilon} \|\Delta_q \Delta Q\|_{L^2},$$

where $C > 0$ is constant independent of $\varepsilon \in (0, \frac{1}{2})$.

- Using Young's inequality we obtain:

$$|\mathcal{I}| \le C \sum_{|q'-q|\le 5} \left(\left(\|S_q \nabla Q\|_{L^{\frac{2}{\varepsilon}}} \|Q\|_{L^\infty} \right)^{\frac{2}{1-\varepsilon}} \|\Delta_q u\|_{L^2}^2 + \frac{\nu}{100}\|\Delta_q \nabla u\|_{L^2}^2 + \frac{\Gamma L^2}{100}\|\Delta_q \Delta Q\|_{L^2}^2 \right)$$

$$\le 2^{-2qs}\left(C\left(\|S_q \nabla Q\|_{L^{\frac{2}{\varepsilon}}} \|Q\|_{L^\infty} \right)^{\frac{2}{1-\varepsilon}} \|u\|_{H^s}^2 + \frac{\nu}{100}\|\nabla u\|_{H^s}^2 + \frac{\Gamma L^2}{100}\|\Delta Q\|_{H^s} \right)$$

A technical trick -how the "double logarithm" appears II

- We have obtained:

$$y' \leq \left(\|\nabla Q\|_{L^{\frac{2}{\varepsilon}}} \|Q\|_{L^\infty}\right)^{\frac{2}{1-\varepsilon}} y(t) \tag{6}$$

- On the other hand using again the interpolation inequality

$$\|g\|_{L^{2p}} \leq C\sqrt{p}\|g\|_{L^2}^{\frac{1}{p}} \|\nabla g\|_{L^2}^{1-\frac{1}{p}}$$

we get:

$$\|\nabla Q\|_{L^{\frac{2}{\varepsilon}}}^{\frac{2}{1-\varepsilon}} \leq \left(\frac{1}{\varepsilon}\right)^{\frac{1}{1-\varepsilon}} \|\nabla Q\|^{\frac{2\varepsilon}{1-\varepsilon}} \|\Delta Q\|_{L^2}^2 \leq \left(\frac{1}{\varepsilon}\right)^{\frac{1}{1-\varepsilon}} (1 + \underbrace{\|\nabla Q\|_{L^2}^2}_{\stackrel{\text{def}}{=} f(t)})\|\Delta Q\|_{L^2}^2$$

where for the last inequality we assumed $0 < \varepsilon < \frac{1}{2}$. Then (6) becomes:

$$y'(t) \leq C(1+f(t))\|\Delta Q\|_{L^2}^2 \left[(1+f(t))\ln(e+y(t))\right]^{\frac{1}{1-\varepsilon}} \left(\frac{1}{\varepsilon}\right)^{\frac{1}{1-\varepsilon}} y(t)$$

Observing that the constants in the interpolation inequality do not depend on the space L^p that we work with and denoting $N \stackrel{def}{=} \ln(e+y)$ we choose

$$\varepsilon \stackrel{def}{=} (1+\ln N)^{-1}$$

and observing that $[N(1+\ln N)]^{1+\frac{1}{\ln N}} \leq CN(1+\ln N)$ for some constant C independent of N, the last inequality becomes:

$$\varphi'(t) \leq C(1+f(t))^3 \|\Delta Q\|_{L^2}^2 \varphi(t) \ln\Big(e+\varphi(t)\Big)\Big(1+\ln(e+\ln(\varphi(t)+e))\Big)$$

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Current and future work

- Uniqueness of weak solutions
- Asymptotic behaviour
- Non-newtonian effects

Challenging open problems

- The optimal rate of increase of high norms: one, two, three exponentials? No exponential, polynomial growth?
- Are there regimes where the system is "better" than $3D$ Navier-Stokes?

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Statistical aspects of coarsening

- The quenching from the isotropic into the nematic occurs through the creation of nematically ordered islands into the ambient isotropic fluid.
- A scaling phenomenon: the pattern of domains at a later time looks statistically similar to that at an earlier time, up to a time-dependent change of scale.
-

$$C(r,t) = \frac{\langle Tr\,[Q(x+r,t)Q(x,t)]\rangle}{\langle Tr\,[Q(x,t)Q(x,t)]\rangle} \quad (7)$$

where the brackets $\langle , \rangle$ denote an average over $x \in \mathbb{R}^d$ and **over the initial conditions**.
- The statistical scaling hypothesis states that for late enough times the correlation function $C(r,t)$ will assume a scaling form:

$$C(r,t) \sim f\left(\frac{r}{L(t)}\right) \quad (8)$$

where $L(t)$ is the time-dependent length scale of the nematic domains.
- C.Denniston, E.Orlanding, J.M. Yeomans, Phys. Rev. E, 64 (2001)

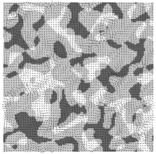

FIG. 1. Schlieren pattern (shading) and fluid velocity (vectors) associated with the ordering of a liquid crystal after a quench from the isotropic to the nematic phase. The director in the darkest regions is perpendicular to the director in the lightest regions (with the other shade in between). The shading on the Schlieren pattern has been rendered with only three shades of gray so as not to obscure the vector field. Disclinations of strength ±1/2 are located at the intersection of dark gray and white "brushes."

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Statistical solutions and the correlation function

- The simplest gradient flow of the energy $\int_{\mathbb{R}^d} \frac{L}{2}|\nabla Q|^2 + \frac{a}{2}\mathrm{tr}(Q^2) - \frac{b}{3}\mathrm{tr}(Q^3) + \frac{c}{4}\mathrm{tr}^2(Q^2)\, dx$

$$\partial_t Q_{ij} = \Delta Q_{ij} + a^2 Q_{ij} + b^2 \left(Q_{il}Q_{lj} - \frac{\delta_{ij}}{3}\mathrm{tr}(Q^2) \right) - c^2 Q_{ij}\mathrm{tr}(Q^2), i, j = 1, 2, 3$$

- Consider an averaging measure μ_0 on the infinite-dimensional functional space of initial datas, let us call it H. If Probability that $Q_0 \in A = \mu_0(A)$, for a Borel set $A \subset H$ then one can rigorously define the time-dependent family of measures $\mu_t(A) \stackrel{def}{=} \mu_0(\{S(t, Q_0) \in A\}) = \mu_0(S(t)^{-1}A)$(where $S(t, Q_0)$ is the solution with initial data Q_0 at time t) and study the evolution of these measures.
- We define then $C(r, t)$:

$$C(r,t) \stackrel{def}{=} \frac{\int_H \left(\int_{\mathbb{R}^3} \mathrm{Tr}\left[Q(x+r)Q(x)\right]\, dx\right)\, d\mu_t(Q)}{\int_H \left(\int_{\mathbb{R}^3} \mathrm{Tr}\left[Q(x)Q(x)\right]\, dx\right)\, d\mu_t(Q)}$$

The controversy about the behaviour of $L(t)$

- Heuristic arguments suggest that any length scale in the system should grow with a power law in time, more precisely $L(t) \sim t^{\frac{1}{2}}$ as $t \longrightarrow \infty$ (R. E. Blundell and A. J. Bray, Phys. Rev. E 49, 4925 (1994))
- An examination of the configuration of the order parameter in experiment or simulation clearly shows that the late stage ordering proceeds by defects moving to annihilate.
- Considerable controversy exists as to whether the ordering violates dynamical scaling (M. Zapotocky, P. M. Goldbart, and N. Goldenfeld, Phys. Rev. E 51, 1216 (1995).)
- A problem arose due to early calculations which showed that the friction coefficient diverges logarithmically with system size (H. Imura and K. Okano, Phys. Lett. 42A, 403 (1973)).
- Simulations of tensor models of liquid crystals reinforced this view when they failed to measure the $t^{\frac{1}{2}}$ behavior and, in fact, found exponents that appeared to be decreasing away from 1/2 at late times. (M. Zapotocky, P. M. Goldbart, and N. Goldenfeld, Phys. Rev. E 51, 1216 (1995); J.-I. Fukuda, Eur. Phys. J. B 1, 173 (1998).)

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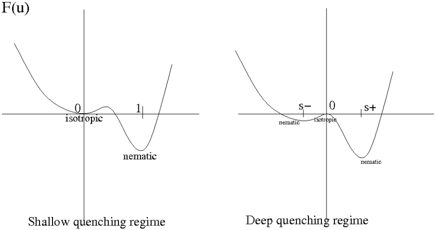
The evolutionary equation: just a bistable gradient system

- High dimensional (in the domain and target space) version of:

$$\partial_t u = u_{xx} - F'(u), \ \ u(t,x) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$$

- Choosing a suitable initial data, our system reduces to the scalar equation above.



- The equation is: $u_t = u_{xx} - au + bu^2 - c^2u^3$ with $a > 0$ in the shallow quenching and $a < 0$ in the deep quenching
- We continue referring just to the shallow quenching regime!
- (A. Zlatos, JAMS, 19 (2006)) For initial data $u(0,x) = \chi_{[0,L]}$ we have that there exists $L_0 > 0$ so that
  - If $L < L_0$ $u(t,x) \to 0$ uniformly on compacts
  - If $L = L_0$ $u(t,x) \to U$ uniformly on compacts, with U a stationary solution
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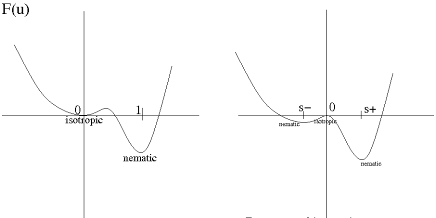

- Shallow quenching regime — Deep quenching regime

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Statistical solutions as averages of individual solutions and the individual behaviour I

- $$C_n(r,t) = \frac{\Sigma_{j=1}^{J(n)} \theta_j^{(n)} \int_H \left(\int_{\mathbb{R}^3} \mathrm{Tr}\left[Q(x+r)Q(x)\right]\, dx \right)\, d\delta_{Q_j^{(n)}}}{\int_H \left(\int_{\mathbb{R}^3} \mathrm{Tr}\left[Q(x)Q(x)\right]\, dx \right)\, d\sigma\, d\delta_{Q_j^{(n)}}} \to C(r,t) \quad (9)$$

  where $\Sigma_{j=1}^{J(n)} \theta_j^{(n)} = 1$.
- Understanding the behaviour of individual solution helps to understand the statistical solutions. But it might not be necessary...
- For small enough initial data we have a representation

  $$Q_{ij}(t,x) = A_{ij}(Q) \frac{e^{-\frac{|x|^2}{4(t+1)}}}{e^{a^2 t} (4\pi(t+1))^{3/2}} + w_{ij}(t,x)$$

  where w_{ij} decays faster than the first term.
- Let $C_\delta(r,t) \stackrel{def}{=} \frac{\int_H \left(\int_{\mathbb{R}^3} \mathrm{Tr}[Q(x+r)Q(x)]\, dx \right) d\delta_{Q(t)}(Q)}{\int_H \left(\int_{\mathbb{R}^3} \mathrm{Tr}[Q(x)Q(x)]\, dx \right) d\delta_{Q(t)}(Q)}$. Then $\| C_\delta(r,t) - e^{-\frac{|r|^2}{8(t+1)}} \|_{L^\infty(dr)} = o(1)$ as $t \to \infty$.
- Thus for small initial data, $L(t) \sim t^{\frac{1}{2}}$, but this scaling only captures the underlying brownian motion.

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Statistical solutions as averages of individual solutions and the individual behaviour II

- Let us denote $w_0 : \mathbb{R} \longrightarrow [0,1]$ the one dimensional travelling wave connecting 0 at $-\infty$ with 1 at $+\infty$. Then w_0 satisfies the ODE:
$$w_0'' + \bar{c}w_0' + f(w_0) = 0 \tag{9}$$
for some constant $\bar{c} > 0$ together with the conditions at infinity $\lim_{y\to\infty} w_0(y) = 0,\ \lim_{y\to-\infty} w_0(y) = 1$.
- If $u_0(x)$ is spherically symmetric with $\|u_0(x) - w_0(|x| - R)\|_X \leq \varepsilon$ then the solution $u(x,t)$ is spherically symmetric and
$$\|u(x,t) - w_0(|x| - \bar{c}t + \frac{2}{\bar{c}}\log t + L)\|_X \leq C\frac{\log t}{t}$$
hence $u(t,x) \sim \chi_{\bar{c}t - \frac{2}{\bar{c}}\log t}$.
- Then $C_\delta(r,t) \sim P(\frac{r}{t})$ as $t \longrightarrow \infty$ (where P is a third order polynomial).
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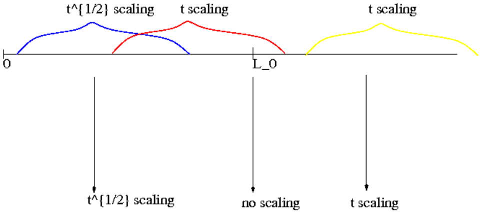

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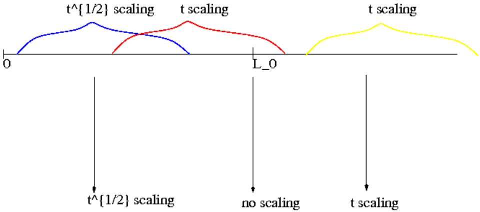

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