Exponential Mixing of stochastic Navier-Stokes equations driven by mildly degenerate noises

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Outline

This is the joint work with S. Albeverio, A. Debussche, and M. Romito.

- A sketchy review for ergodicity of 3D stochastic Navier-Stokes equations (SNS) driven by Wiener noises.
- Galerkin approximations approach for the above ergodicity (Malliavin calculus+A control problem).
- A control problem (which includes the idea of using Malliavin calculus).
- Proof of exponential mixing by coupling method.

Contents

Navier-Stokes equations driven by noises

Let us consider 3D NS on $\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$

$$\begin{cases} \partial_t u = \nu \Delta u - (u \cdot \nabla) u + \nabla p + \dot{\eta} \\ \nabla \cdot u = 0 \\ u(0, x) = u_0(x) \end{cases} \tag{1}$$

where

- $u : \mathbb{R}_+ \times \mathbb{T}^3 \to \mathbb{R}^3$, $u_0 : \mathbb{T}^3 \to \mathbb{R}^3$, and $p : \mathbb{R}_+ \times \mathbb{T}^3 \to \mathbb{R}$.
- η is some (Wiener) noises.

Stochastic Navier Stokes equations (SNS)

Let $H = [L^2(\mathbb{T}^3, \mathbb{R})]^3 + Diverdence\ free$ with an orthonormal basis $(e_k^1, e_k^2)_{k}$

Then The previous Navier-Stokes equation driven by Wiener noises rigorously reads as

$$du = [\Delta u - (u \cdot \nabla)u + \nabla p]dt + QdW_t, \tag{2}$$

where $W_t = \sum_{k \in \mathbb{Z}^3; i=1,2} w_k^i(t) e_k^i$ is the cylindrical Brown motion on H, Q is some operator with some *smoothing effect*.

One motivation to study SNS is to see as $\epsilon \to 0+$ what happens for

$$du = [\Delta u - (u \cdot \nabla)u + \nabla p]dt + \epsilon QdW_t. \tag{3}$$

Projection

Let $\mathcal{P} : [L^2(\mathbb{T}^3, \mathbb{R})]^3 \to H$ be the projection, then the previous equation can be simplified as

$$dX(t) = [\Delta X(t) - B(X(t))]dt + QdW_t \tag{4}$$

where $B(X) = \mathcal{P}[(X \cdot \nabla)X]$.

About Q

To solve SNS, Q needs to have some smoothing effect. To take a little oversimplified case, assume that

$$Q = diag\{q_k\}_{k \in \mathbb{Z}^3}$$

where each q_k is a 2×2 matrix as the following

$$q_k = c_k k^{-\beta} I_2$$

with $\beta > 3/2$ being some fixed constant and $c_k = 0, 1$.

- Non-degenerate: $c_k = 1$ for all k.

$\exists \ N_0 > 0$

- Mildly degenerate: $c_k = 1$ for $|k| > N_0$ and $c_k = 0$ for $|k| \leq N_0$.
- Highly degenerate: $c_k = 0$ for $|k| > N_0$ and $c_k = 1$ for $|k| \leq N_0$.

Martingale solution

- **Global martingale solution** (Flandolli & Gatarek): There exists at lease a global martingale solution (weak solution in probability sense) for (4).
- Martingale solutions: Look for P on some space Ω so that

$$M_t^{\phi} = \langle u(t) - u(0), \phi \rangle - \int_0^t \langle u(s), \Delta \phi \rangle ds - \int_0^t \langle (u(s) \cdot \nabla)\phi, u(s) \rangle ds$$

  is a martingale with quadratic variance $t|Q\phi|^2$.
- Sketch proof: Galerkin approximation+Prokhorov theorem+Skorohod Theorem. The limit process (i.e. solutions) is possibly not Markov!

Markov Selection

Well-posedness (PDE): Existence, Uniqueness, Continuity (w.r.t.u_0). Any Stochastic version? Heuristically,

1. Existence.
2. Uniqueness → Markov property.
3. Continuity → Strong Feller property

Can we select one among martingale solutions with Markov and Strong Feller property? $\Longrightarrow$ Markov Selection!

The candidate set (i.e. Ω) for martingale solutions is too large to find a Markov solution → Shrinking the candidate set by adding some restrictions: Kolmogorov equation approach V.S. Krylov's approach

Krylov's approach for Markov selection (Flandoli & Romito 2008)

- Krylov's approach: Add the energy inequality restriction to shrink the candidate set Ω (Stroock-Varadhan's book 'Multidimensional Diffusion').
- This approach works for degenerate noises case, even for deterministic case.
- In FR's paper, they only proved the ergodicity in non-degenerate noises case.
- Romito and X (2009) proved the ergodicity in mildly degenerate noises case (Malliavin calculus).

Kolmogorov equation approach for Markov selection (Da Prato & Debussche 2003)

- Recall the limit of Galerkin approximation (i.e. solutions) can not preserve Markov property ⟶ To keep Markov property, the semigroups of Galerkin approximation need to have (uniformly) strong Feller property ⟹ Kolmogorov (Backward) Equation to study semigroups
- In DD's paper, this approach only works for non-degenerate noises case.
- Albeverio, Debussche and X (2009) proved the Markov selection and ergodicity in mildly degenerate noises case (Malliavin calculus).

A brief review on proving ergodic theorem

$(X_t, \mathcal{F}, \mathcal{F}_t, P)$ is ergodic if it has a unique invariant measure.

- General method:Existence and Uniqueness of invariant measure.
  - Existence (Krylov-Bogoliubov): Proving the tightness of $\frac{1}{T}\int_0^T p(t;x,dy)dt$
  - Uniqueness: Irreducibility+Strong Feller (Doob), Other methods.
- The uniqueness step (Doob's method):
  - Gradient Bound i.e. $|\nabla P_t f(x)| \le C(t,|x|)||f||_\infty \Rightarrow$ Strong Feller;
  - Control problem $\Rightarrow$ Irreducibility.
- The difficult part is to prove the strong Feller property (Malliavin calculus).

Hypoelliptic Operator and Hormander Theorem

- Hypoelliptic operator: A partial differential operator P defined on an open subset $U \subset \mathbb{R}^n$ is called hypoelliptic if for every distribution u defined on an open subset $V \subset U$ such that if $Pu \in C^\infty$, then $u \in C^\infty$.
- Hormander Theorem: Let $L = \sum_{i=1}^m X_i^2 + Y$ with X_i, Y all being smooth vector fields in some domain $U \subset \mathbb{R}^n$. If the Lie bracket system

$$H = \{X_i, [Y, X_i], [[Y, X_i], X_j]\ \cdots\}$$

spans $\mathbb{R}^n$ at every point x, then the transition probability of the Markov semigroup 'e^{tL}' has smooth density.

Malliavin Calculus

- Malliavin's proof for Hormander Theorem(Malliavin Calculus): Study L by the underlying SDE: $dx_t = Y(x_t)dt + \sum_{i=1}^m X_i(x_t) \circ dW_t^i$. Define the Malliavin matrix by

  $$\mathcal{M}_t = \int_0^t (J_{s,t}Q)(J_{s,t}Q)^* ds,$$

  where $Q = [X_1, \cdots, X_m]$ and $J_{s,t}$ is the Jacobi matrix between $x(t)$ and $x(s)$.
- The key point for Malliavin approach is to prove that $\mathcal{M}_t$ is invertible and $\mathcal{M}_t^{-1} \in L^p(\mathbb{P})$ for all $p > 1$.

Integration by parts formula

- Malliavin derivative: take $X(t) = X(t, W)$ and define $\mathcal{D}_v X(t) = \lim_{\epsilon \to 0} \frac{X(t, W+\epsilon v) - X(t, W)}{\epsilon}$.
- Integration by parts formula: $E[\mathcal{D}_v f(X(t))] = E[f(X(t)) \int_0^t v(s) dW_s]$

  (Heuristically proving by Girsanov Theorem)

Strong Feller property

In the case of Q being non-degenerate, one can apply Bismut-Elworthy-Li (BEL) formula

$$D_h \mathbb{E}[f(X(t,x))] = \frac{1}{t}\mathbb{E}[f(X(t,x)) \int_0^t \langle Q^{-1} D_h X(s,x), dW_s \rangle] \tag{5}$$

Since our Q is not invertible, (5) is not available.

According to the degeneracy of Q, we divide (4) into two parts:

$$dX^L = [\Delta X^L - B^L(X,X)]dt \tag{6}$$

$$dX^H = [\Delta X^H - B^H(X,X)]dt + Q^H dW_t^H \tag{7}$$

Heuristically, apply Malliavin calculus to (6) and BEL formula to (7).

Strong Feller property: Cutoff v.s. Negative potential

To use Malliavin calculus, one needs to use the stochastic calculus. But (6) & (7) only has martingale solution and thus cannot apply Ito formula. The trouble is from the nonlinearity $B(X, X)$, so

1. Cut off Nonlinearity $B(X, X)$ to obtain regular solution (Flandoli $\&$ Romito).
2. Multiply X by a large negative potential $e^{-\int_0^t \cdots}$ to suppress the blow up due to $B(X, X)$ (Da Prato & Debussche).

We can extend the above two approaches to the mildly degenerate case (Romito and X. $\&$ $\&$ $\&$ Albeverio, Debussche and X.).

More fundamental reason for applying Malliavin calculus

We know that if Lie brackets span the space R^n, then one has Malliavin matrix invertible and then many nice results. However, what is the more fundamental mechanism for these Lie brackets?

We explain it by SNS driven by degenerate noises: We do not like the nonlinearity $B(X, X)$ because it makes many troubles such as (possibly) blow up and obstruction for transferring strong Feller. On the other hand, $B(X, X)$ can transfer the noise to make Hörmander condition satisfied.

A remarkable result by this reason

- Hairer and Mattingly proved the ergodicity of 2D SNS driven by highly degenerate noises by Malliavin calculus.

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1. The sketch of ergodicity of 3D SNS
2. Kolmogorov equation approach (at Galerkin approximation level)
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Negative potential (together with Galerkin approximation)

Before going into the details, let us first give a picture of the method.

- Galerkin approximation $\longrightarrow$ to make strong solution X_m and Ito formula available $\longrightarrow$ obtain nice semigroups $P_t^m f = \mathbb{E}[f(X_m(t))]$.
- Can P_t^m converge to a Markov semigroup P_t? Multiply $X_m(t)$ by a negative potential $e^{-\int_0^t \cdots}$ $\longrightarrow$ An auxiliary Kolmogorov equations with solution $S_t^m f$, moreover

$$P_t^m f = S_t^m f + \int_0^t S_{t-s}^m [\cdots P_s^m] ds$$

$\longrightarrow$ Ascoli-Arzela's Theorem (together with some strong Feller) to select subsequence $\{m_k\}$ s.t. $P_t^{m_k}$ converges to a Markov semigroup.

Galerkin approximation for 3D SNS

$$dX_m = -[AX_m + B_m(X_m, X_m)]dt + Q_m dW_t \tag{8}$$

with $X_m(0) = x_m$, whose Kolmogorov equation is

$$\partial_t u_m = \mathcal{L}_m u_m, \quad \mathcal{L}_m := \frac{1}{2} Tr[Q_m Q_m^* D^2] - \langle Ax_m + B_m(x), D \rangle \tag{9}$$

with $u_m(0) = f$. Clearly, $P_t^m f(x) = u_m(t) = \mathbb{E}[f(X_m(t))]$. Our auxiliary Kolmogorov equation with a negative potential is

$$\partial_t v_m = \mathcal{L}_m v_m - K|Ax_m|^2 v_m \tag{10}$$

with solution $S_t^m = v_m(t) = \mathbb{E}[f(X_m(t)) e^{-K\int_0^t |AX_m(s)|^2 ds}]$. Moreover,

$$P_t^m f = S_t^m f + K \int_0^t S_{t-s}^m [|Ax_m|^2 P_s^m f] ds \tag{11}$$

Uniformly strong Feller property of S_t^m

We shall briefly prove the following key theorem

Theorem: Let $\gamma \in (0,1]$ and $||g||_2 := \sup_{x \in D(A)} \frac{|g(x)|_E}{(1+|Ax|)^2}$ for any $g : D(A) \to E$ (where E is some Banach space), then

$$||A^{-\gamma} D S_t^m f||_2 \leq C(\gamma) t^{-r} ||f||_2. \tag{12}$$

Uniformly strong Feller property of P_t^m

With (11) and (12), we can easily show that

$$||A^{-\gamma}P_t^m f||_{\color{red}{2}} \leq C(\gamma)t^{-r}||f||_{\color{red}{0}}$$

Sketch proof of Theorem

We drop the index m and write $\mathcal{E}_K(t) = e^{-K\int_0^t |AX(s)|^2 ds}$, given any $h \in H$, let us consider

$$D_h S_t f(x) = \mathbb{E}[D_h f(X(t))\mathcal{E}_K(t)] + \mathbb{E}[f(X(t))D_h\mathcal{E}_K(t)] \qquad (13)$$

Let us estimate $\mathbb{E}[D_h f(X(t))\mathcal{E}_K(t)]$ (omitting $\mathcal{E}_K(t)$ for simplicity):

$$\begin{aligned}
\mathbb{E}[D_h f(X(t))] &= \mathbb{E}[D^H f D_h X^H(t)] + \mathbb{E}[D^L f D_h X^L(t)] \\
&= \mathbb{E}[D^H f D_{h^H} X^H(t)] + \mathbb{E}[D^H f D_{h^L} X^H(t)] \\
&\quad + \mathbb{E}[D^L f D_{h^H} X^L(t)] + \mathbb{E}[D^L f D_{h^L} X^L(t)] \\
&= I_{HH} + I_{LH} + I_{HL} + I_{LL}
\end{aligned}$$

Roughly, I_{HH} by BEL formula, I_{LL} by Malliavin calculus, I_{LH} and I_{HL} by large $K > 0$.

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A control problem

To study the irreducibility, we need to solve control problem: Given any $T > 0$, $x, y \in D(A)$ and $\epsilon > 0$, there exist $\rho_0 = \rho_0(|Ax|, |Ay|, T)$, u and w such that

- $w \in L^2([0,T]; H)$ and $u \in C([0,T]; D(A))$,
- $u(0) = x$ and $|Au(T) - Ay| \leq \epsilon$,
- $\sup_{t \in [0,T]} |Au(t)| \leq \rho_0$,

and u, w solve the following problem,

$$\partial_t u + Au + B(u,u) = Qw, \tag{14}$$

where Q is mildly degenerate.

Solving the control problem

Due to the degeneracy of Q, we consider

$$\partial_t u^L + Au^L + B^L(u,u) = 0, \tag{15}$$

$$\partial_t u^H + Au^H + B^H(u,u) = Q^H w. \tag{16}$$

Step1 : Leading high frequency part to zero on $[0, T_1]$: Define $u^H(t) = x^H \phi(t)$ with $\phi(0) = 1$ and $\phi(T_1) = 0$, hence, $u^H(T_1) = 0$.

- plug this $u^H(t)$ into (15), we can solve $u^L(t)$ as $T_1 > 0$ sufficiently small.
- $w = (Q^H)^{-1}[\partial_t u^H + Au^H + B^H(u,u)]$.

Solving the control problem (continued)

Step2 : Leading low frequency part to y^L on $[T_1, T_2]$: Let

$$u^L(t) = \frac{t - T_1}{T_2 - T_1} y^L + \frac{T_2 - t}{T_2 - T_1} u^L(T_1),$$

hence, $u^L(T_2) = y^L$. Write low frequency part (15) as

$$\dot{u}_k + |k|^2 u_k + B_k(u, u) = 0, \quad |k| \le N_0.$$

We have $B_k(u,u) = B_k(u^L, u^L) + B_k(u^L, u^H) + B_k(u^H, u^L) + B_k(u^H, u^H)$,

- $B_k(u^L, u^L)$ is known.
- Choose some u^H depending on u^L s.t. $B_k(u^H, u^L) = B_k(u^L, u^H) = 0$.
- $w = (Q^H)^{-1}[\partial_t u^H + B^H(u,u) + Au^H]$.

Solving the control problem (continued)

Step3 : Leading the high modes to y in a short time $[T_2, T]$: Let

$$u^H(t) = \frac{t - T_2}{T - T_2} y^H + \frac{T - t}{T - T_2} u^H(T_2),$$

Hence, $u^H(T) = y^H$.

Since $T - T_2$ is sufficiently small and $u^H(t)$ is uniformly bounded, $u^L(T)$ is not far from y^L. Take

$$w = (Q^H)^{-1}[\partial_t u^H + B^H(u, u) + Au^H].$$

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Coupling of Markov chain

Let $\{X(k)\}_{k\in\mathbb{N}}$ be the Markov chain with transition probability $\{p_{ij}\}_{i,j}$. Let $X_1(k)$ and $X_2(k)$ be the two chains starting from i_1 and i_2 respectively (i.e. $X_1(0)=i_1$, $X_2(0)=i_2$). We construct the coupling Markov chain $(\tilde X_1(k),\tilde X_2(k))$:

$$(\tilde X_1(1),\tilde X_2(1))=\begin{cases}\tilde X_1(1)=\tilde X_2(1), & \text{if } \ i_1=i_2;\\ (Z_1,Z_2), & \text{otherwise.}\end{cases} \tag{17}$$

where (Z_1,Z_2) is the maximal coupling of $p_{i_1,\cdot}$ and $p_{i_2,\cdot}$. By maximal coupling,

$$||p_{i_1,\cdot}-p_{i_2,\cdot}||_{TV}=\mathbb{P}\{Z_1\neq Z_2|i_1,i_2\}.$$

Coupling of Markov chains (continued)

If

$$||p_{i_1,\cdot} - p_{i_2,\cdot}||_{TV} \leq p \quad \forall \; i_1 \neq i_2, \tag{18}$$

with $p < 1$. Hence,

$$\mathbb{P}\{Z_1 = Z_2 | i_1, i_2\} \geq 1 - p > 0,$$

and

$$\begin{aligned} \mathbb{P}\{\tilde{X}_1(m) = \tilde{X}_2(m)\} &= 1 - P\{\tilde{X}_1(m) \neq \tilde{X}_2(m)\} \\ &= 1 - P\{\tilde{X}_1(k) \neq \tilde{X}_2(k); k = 1, \cdots, m\} \\ &\geq 1 - p^m \\ &\to 1. \end{aligned} \tag{19}$$

Coupling of SNS

For two independent Wiener processes W and $\tilde{W}$, denote X and $\tilde{X}$ the solutions of SNS driven by W and $\tilde{W}$ respectively. For any fixed $0 < T \leq 1$, given initial data $x_1, x_2 \in D(A)$, construct the coupling of $(P_T)^*\delta_{x_1}$ and $(P_T)^*\delta_{x_2}$:

$$(V_1, V_2) = \begin{cases} (X(T, x_0), X(T, x_0)) & \text{if } x_1 = x_2 = x_0, \\ (Z_1(x_1, x_2), Z_2(x_1, x_2)) & \text{if } x_1, x_2 \in B_{D(A)}(0, \delta) \ \textit{with} \ x_1 \neq x_2, \\ (X(T, x_1), \tilde{X}(T, x_2)) & \text{otherwise,} \end{cases}$$

Coupling of SNS (continued)

set $X_i(0) = x_i$ $(i = 1, 2)$ and define

$$X_i((n+1)T) = V_i(X_1(nT), X_2(nT)) \quad i = 1, 2.$$

The key point for this coupling is to show a similar estimate as (18). This means that as long as $\tilde{X}_1$ and $\tilde{X}_2$ enter a small ball together, they will collide with a positive probability.

To show the exponential mixing, define some stopping time τ to describe the moment that $\tilde{X}_1$ and $\tilde{X}_2$ enter a small ball together.

Thanks A Lot!