Computing Slowly Advancing Features in Fast-Slow Systems without Scale Separation - A Young Measure Approach

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Overview

1. Background
    - Classical Theory of Levinson–Tikhonov
    - Motivating Example for Limit Cycle

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    - Classical Theory of Levinson–Tikhonov
    - Motivating Example for Limit Cycle
2. The use of Young measures
    - Young measures and description of the limit of fast dynamics
    - Fast-slow systems without separation of state variables
    - Computing slow observables

Classical Fast-Slow System

We consider a classical singular perturbed system

$$\begin{aligned} \frac{dx}{dt} &= f(x,y) \\ \varepsilon\frac{dy}{dt} &= g(x,y) \end{aligned}$$

with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, and initial conditions

$$x(0) = x_0, \quad y(0) = y_0.$$

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- Moreover, what is the equation of motion that governs this limit behavior?
- Can we develop an efficient numerical algorithm for computing the above limit behavior?

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$$\begin{aligned} \frac{dx}{dt} &= f(x,y) \\ 0 &= g(x,y). \end{aligned}$$

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Thus, AFP stands for Asymptotically stable Fixed Point.

Motivating Examples

Consider the four dimensional system with planar slow and fast equations given by

$$\begin{aligned}
\frac{d\xi_1}{dt} &= \xi_2 \\
\frac{d\xi_2}{dt} &= -2\xi_1 - \xi_2 - \eta_1 + F(\eta_2) \\
\varepsilon\frac{d\eta_1}{dt} &= \eta_2 \\
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$$F_s(\eta) = -\eta - \eta^3 + \eta^5 - \eta^7$$

and

$$F_u(\eta) = \eta - \eta^3 + \eta^5 - \eta^7.$$

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When the slow variable $x = (\xi_1, \xi_2)$ is kept fixed, the fast solution $y(t) = (\eta_1(t), \eta_2(t))$ converges as $t \to \infty$ to the solution of the corresponding algebraic equations, namely, to the point $(-\xi_1 - \xi_2, 0)$.

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Inserting these values to the slow equation

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determines the limit, as $\varepsilon \to 0$, of the slow solutions.

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The stationary point determined by the algebraic equation is unstable with respect to the fast dynamics (hence the subscript u, standing for unstable, is placed); in particular, the solution to the full equation is not attracted to the mentioned manifold.

The Limit Cycle Case - continue

However, for a fixed (ξ_1, ξ_2) and with initial condition (η_1, η_2) different from the fixed point, the fast solution converges to a **limit cycle**, thus, in this case the following Assumption LC holds.

Assumption LC. In the region where the analysis is carried out, when x is held fixed the solutions of the fast equation, namely of $\frac{dy}{ds} = g(x,y)$, converge to a limit cycle which we denote by $\Gamma(x)$. Thus, LC stands for Limit Cycle.

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This kind of work is in Bogoliubov-Mitropolsky and in Pontryagin-Rodygin.

Direct Numerical Simulations

ε	F stable	F unstable
0.1	0.52 sec	3.9 sec
0.01	0.64 sec	43 sec
0.001	0.63 sec	433 sec
0.0001	0.65 sec	4339 sec

Z. Artstein, J. Linshiz and E.S. Titi, *SIAM, Multiscale Modeling and Simulation,* **6(4)** (2007), 1085–1097.

Description of the limit cycle

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One way in which the limit cycle can be represented is as a periodic function

$$\gamma_x(\cdot) : [0, T_x] \to \mathbb{R}^m \tag{2}$$

with a period T_x which depends on the fixed slow state x.

Description of the limit cycle by measure

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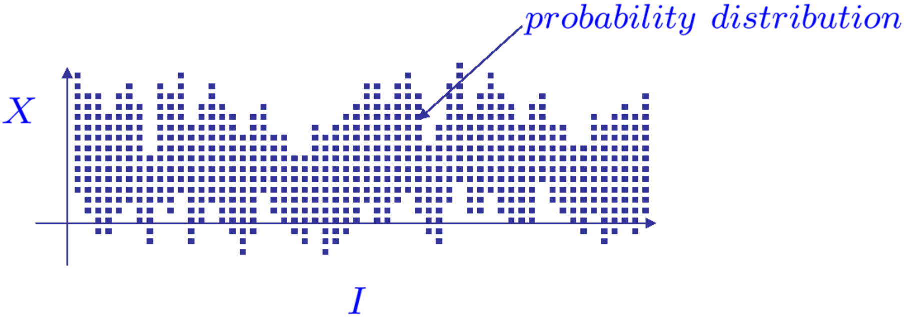
The measure μ_x is called in literature as Young Measure.

Young Measure



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Notice that the measure μ_x is an invariant measure of fast dynamics.

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Denote by $(x_\varepsilon(t), y_\varepsilon(t))$ the solution to our system, under Assumption LC. The goal is to describe the structure of the limit of $(x_\varepsilon(\cdot), y_\varepsilon(\cdot))$, as $\varepsilon \to 0$, on a fix time interval, say $[0, \tau_0]$.

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We also need a convergence notion for the probability measures μ_x. To this end we adopt the weak convergence of measures.

Equation of motion for the limit slow motion

Theorem *Under Assumption LC the slow parts* $x_\varepsilon(\cdot)$ *of the solutions converge as* $\varepsilon \to 0$, *uniformly on the time interval* $[0, \tau_0]$, *to a solution* $x_0(\cdot)$ *of*

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The present result follows from standard averaging techniques; e.g., Sanders and Verhulst.

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The topological limit is given as follows:

Proposition *Under Assumption LC, the couple* $(x_\varepsilon(t), y_\varepsilon(t))$ *of slow and fast trajectories approaches, as* $\varepsilon \to 0$, *the tube located in* $\mathbb{R}^n \times \mathbb{R}^m$, *having a circular* m*-cross section, and given by*

$$\{(x_0(t), \Gamma_{x_0(t)}) : \ t \in [0, \tau_0]\}. \tag{5}$$

where $x_0(t)$ *is the uniform limit of* $x_\varepsilon(t)$ *as* $\varepsilon \to 0$.

Quantitative description of the limit of the fast dynamics and Young measures

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Indeed, the family $\mu_{x_0(t)}$, parameterized by the time variable t over the interval of integration $[0, \tau_0]$, can be viewed as a probability measure-valued map $\mu_{x_0(\cdot)}$, called in the literature a Young measure.

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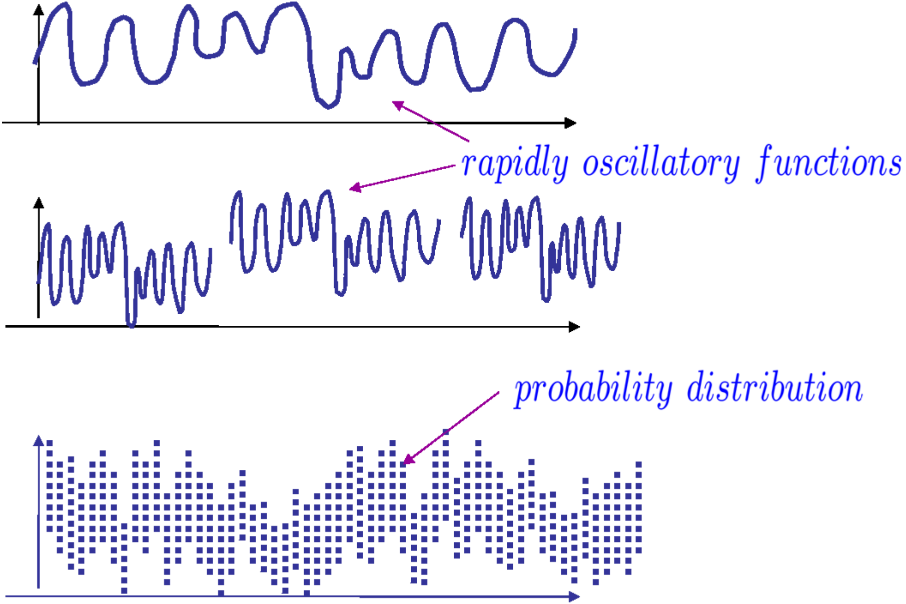
The mappings $y_\varepsilon(\cdot)$ can also be viewed as (degenerate) Young measures, namely, with values being Dirac measures. It can be represented as a measure

$$\delta_{y_\varepsilon(t)}(dy).$$

The Limit of the Graphs of Fast Oscillating Functions



rapidly oscillatory functions

probability distribution

Convergence of distributions

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Suppose that a small interval, say I, is given near, say, a point τ in the interval. Then for ε small enough the distribution of the fast dynamics $y_\varepsilon(\cdot)$ over the interval I is very similar to the distribution $\mu_{x_0(\tau)}$ on the corresponding limit cycle.

Numerical Computations using Young measures

The following computations are reported in:

Z. Artstein, J. Linshiz and E.S. Titi, *SIAM, Multiscale Modeling and Simulation,* **6(4)** (2007), 1085–1097.

System I

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Transient to Limit Cycle

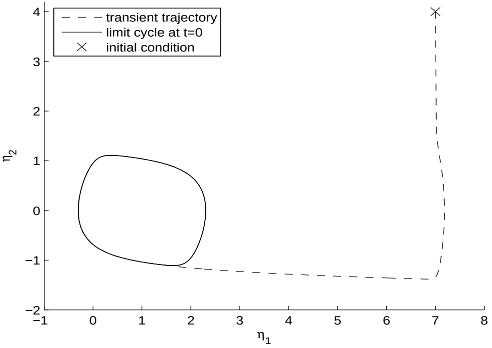


Figure 1

The drift of the slow dynamics

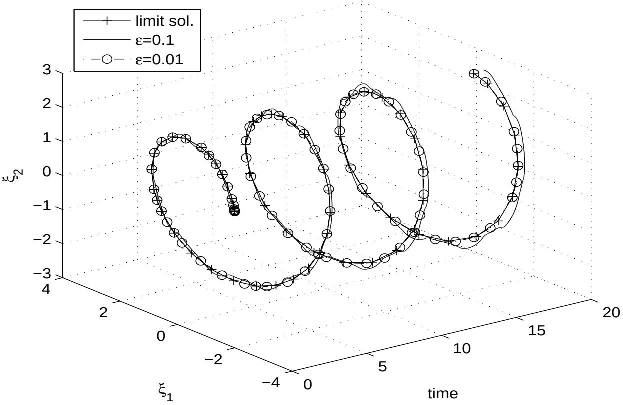


Figure 2

Limit Solution tube generated by the fast dynamics

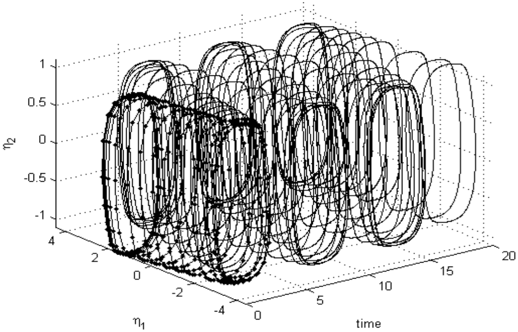


Figure 3

Computations System II

The second example is given by the following set of equations.

$$
\begin{aligned}
\frac{dx_1}{dt} &= x_2 - 1 \\
\frac{dx_2}{dt} &= 1 - \sqrt{y_1^2 + y_2^2} \\
\varepsilon \frac{dy_1}{dt} &= x_1 \frac{y_1}{\sqrt{y_1^2 + y_2^2}} - y_1 - y_2 x_2 \\
\varepsilon \frac{dy_2}{dt} &= x_1 \frac{y_2}{\sqrt{y_1^2 + y_2^2}} - y_2 + y_1 x_2 .
\end{aligned}
$$

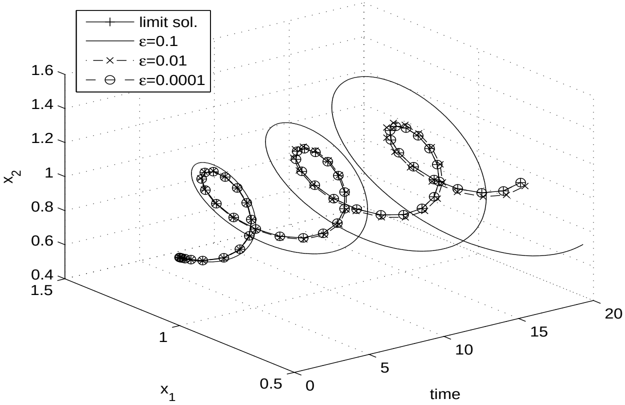
This system also has a limit solution which can be computed analytically. This can be seen when the variables in the latter two equations are written in polar coordinates as follows.

System II in Polar Coordinates

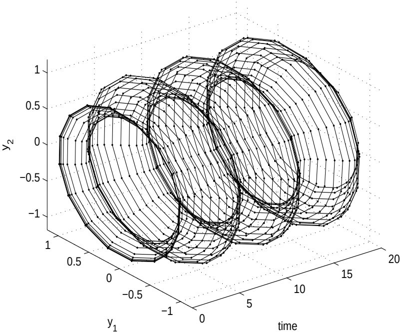
$$\begin{aligned}
\frac{dx_1}{dt} &= x_2 - 1 \\
\frac{dx_2}{dt} &= 1 - r \\
\varepsilon \frac{dr}{dt} &= x_1 - r \\
\varepsilon \frac{d\theta}{dt} &= x_2 .
\end{aligned}$$

In fact, it is easy to see that the fast dynamics converges toward a limit cycle parameterized by the slow dynamics, which, in turn, oscillates.

Numerics using our algorithm



Again, the numerics of the explicit expression for the slow dynamics cannot be distinguished from the solution obtained by our algorithm.



the limit tube of the fast dynamics (here the lines depict the limit cycles while the dots reflect the approximation to the corresponding invariant measures).

General fast dynamics with compact attractor

$$
\begin{aligned}
\frac{dx}{dt} &= f(x,y) \\
\varepsilon\frac{dy}{dt} &= g(x,y)
\end{aligned}
$$

with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, and initial conditions

$$x(0) = x_0, \quad y(0) = y_0.$$

Suppose the fast dynamics $\frac{dy}{ds} = g(x,y)$ for each fixed x in the domain of interest has a compact attractor with unique invariant measure μ_x.

Equation of motion for the limit slow dynamics

Theorem *Under assumption that for each fixed* x*, in the relevant domain of interest, the fast dynamics,* $\frac{dy}{ds} = g(x,y)$*, has a compact attractor with unique invariant measure* μ_x*, then the slow parts* $x_\varepsilon(\cdot)$ *of the solutions converge as* $\varepsilon \to 0$*, uniformly on the time interval* $[0, \tau_0]$*, to a solution* $x_0(\cdot)$ *of*

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$$\frac{dx}{dt} = \int_{\mathbb{R}^m} f(x,y)\mu_x(dy).$$

Moreover, the fast dynamics $y_\varepsilon(\cdot)$, *viewed as a delta Dirac measure, converges to* $\mu_{x_0(\cdot)}(dy)$ *weakly in the sense of Young measures.*

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Therefore, one might have to identify slow functionals/observables of the state variables and find an efficient algorithm to compute them.

Fast and slow dynamics, but not fast and slow state variables

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Consider a system of the form

$$\frac{dU}{d\tau} = \frac{1}{\varepsilon}F(U) + G(U),$$

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Can we develop efficient numerical algorithms to compute them?

Slow observables

The constants of motion of the fast dynamics, $\frac{dU}{ds} = F(U)$, to the above system are candidates for slow observables. And the idea is to find an equation of motion for they way they are drifted by the slow flow.

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Observe that whenever μ is an invariant measure for $\frac{dU}{ds} = F(U)$, and $v(U)$ is a constant of motion for it, then $v(U)$ is constant on the support of μ. We denote this value by $\hat{v}(\mu)$.

Equation of motion for the identified slow observables

Theorem *Let* $U_{\varepsilon_k}(\tau)$ *be as in the statement of last Theorem which converge, as* $k \to \infty$*, in the Young measures sense, to the Young measure* $\mu_0(\tau)$*, for* $\tau \in [0, T]$.

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where $G(U) = G(U_1, \ldots, U_N)$*, and the* ∇ *operator is with respect to the vector* U*. Furthermore, the sequence of measurements* $v(U_{\varepsilon_k}(\tau))$ *converge weakly to* $\hat{v}(\mu_0(\tau))$*, as* $k \to \infty$.

Application to discretized Burgers with small diffusion

$$\frac{dU_k}{d\tau} + \frac{U_k(U_{k+1} - U_{k-1})}{2h\varepsilon} = \frac{U_{k+1} - 2U_k + U_{k-1}}{h^2}.$$

Denote by U the vector $(U_1, \cdots, U_N)$; the above system can be rewritten as:

$$\frac{dU}{d\tau} = \frac{1}{\varepsilon} F(U) + G(U),$$

where

$$F(U) = \frac{-1}{2h} \begin{pmatrix} U_1(U_2 - U_N) \\ U_2(U_3 - U_1) \\ \cdot \\ \cdot \\ \cdot \\ U_N(U_1 - U_{N-1}) \end{pmatrix}, G(U) = \frac{1}{h^2} \begin{pmatrix} U_2 - 2U_1 + U_N \\ U_3 - 2U_2 + U_1 \\ \cdot \\ \cdot \\ \cdot \\ U_1 - 2U_N + U_{N-1} \end{pmatrix}.$$

Integrable systems and Toda Lattice

This system was investigated by Goodman and Lax on the whole line. Here we deal with the periodic case and find that the fast dynamics is integrable and has at least $N/2$ constants of motion.

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In joint work with Artstein, Gear, Kevrekidis , Slemrod and E.S.T. we investigate this computationally and find a great saving by using the Young measure approach

Invariant Torus for the fast dynamics

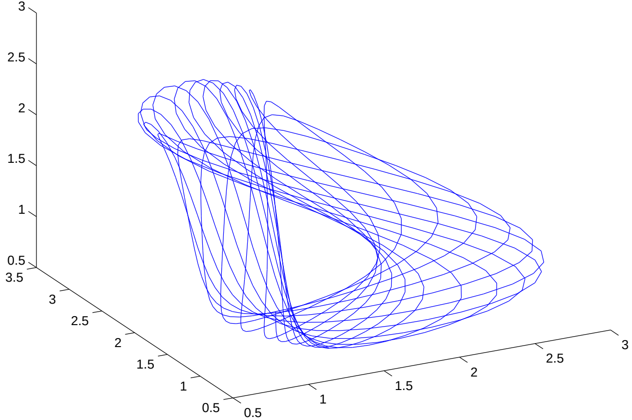

Figure: Torus for the case $N = 6$ of the fast system. Initial values were [1 1 1 3 2 1].

Evidence of fast dynamics

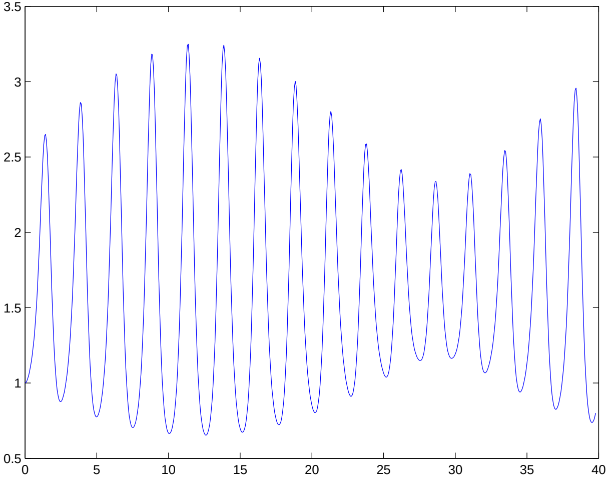

Figure: $U_1(\sigma)$ for the case in the first figure.

Fast evolution of the integrand

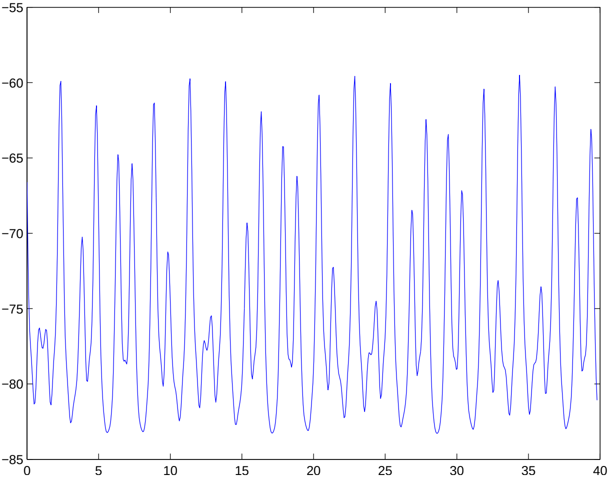

Figure: Evolution of $\nabla v_3(U(\sigma)) \cdot G(U(\sigma))$ along the trajectory $U(\sigma)$ of the first figure.

Averaged integrand

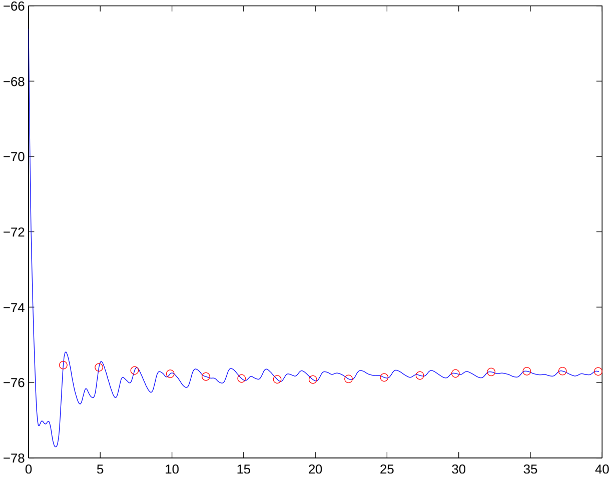

Figure: Averaged $\nabla v_3(U(\sigma)) \cdot G(U(\sigma))$ along the trajectory $U(\sigma)$ of the first figure.

The motion of slow observables

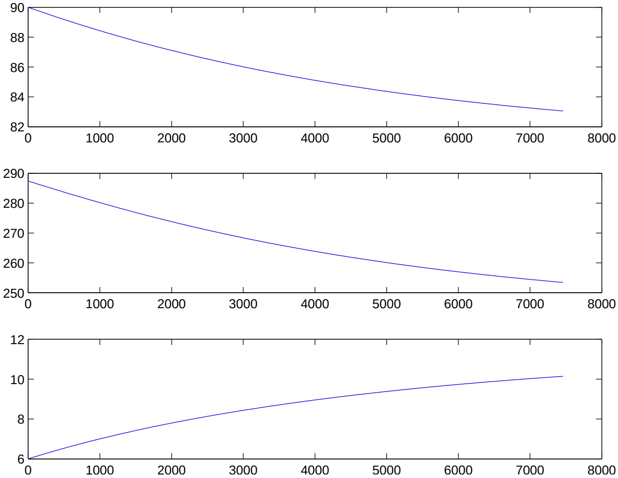

Figure: Behavior of the slow observables v_2, v_3 and v_4 as they are drifted by the slow diffusion.

evolution of tori

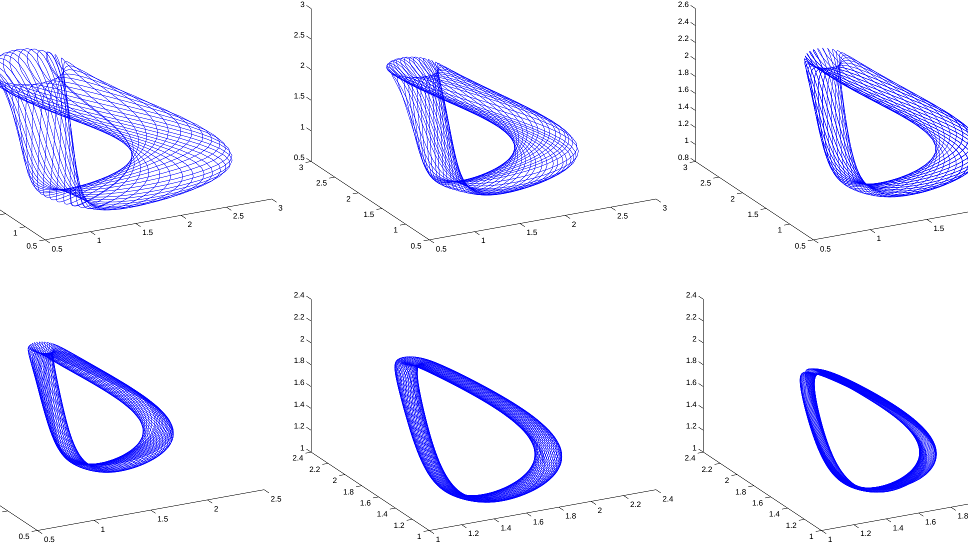

Figure: Evolution of Tori of system (**22**) for initial condition [1, 1, 1, 1, 4