Analysis of a model for amorphous surface growth

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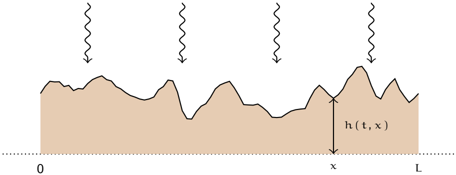
Summary

The mathematical model

$$\partial_t h + \Delta^2 h + \Delta|\nabla h|^2 + \Delta h - |\nabla h|^2 = \eta,$$



- $h(t,x)$ height profile in moving frame, i. e. $\int h\,dx = 0$,
- $\Delta^2 h \quad \rightsquigarrow \quad$ surface diffusion,
- $+\Delta h \quad \rightsquigarrow \quad$ linear instability,
- $\Delta|\nabla h|^2 - |\nabla h|^2 \quad \rightsquigarrow \quad$ coarsening
- η is space–time white noise (fluctuations in incoming particles)

[2D growth of amorphous surfaces: Raible, Mayr, et al. (Phys. Rev. E 2000, Europ. Lett. 2000)]
[Surface erosion for ion–beam sputtering: Frisch and Verga (PRL2006), Munoz–Garcia et al. (PRL2010)]
[1D terraces in the epitaxy of silicon: Cuerno et al. (PRL2005)]

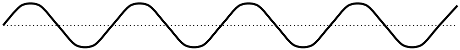
Typical behaviour

Consider

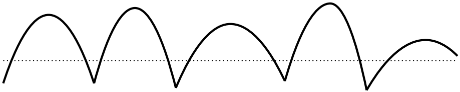
$$\dot{h} + h_{xxxx} + h_{xx} + (h_x^2)_{xx} = \eta$$

with $L >> 1$ and $h_0 \approx 0$,

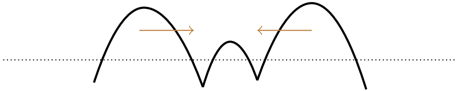
- Linear instability,



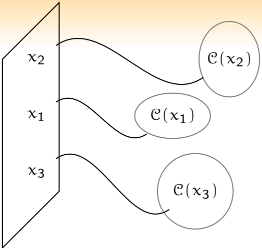
- Formation of parabola–shaped hills,



- Coarsening,



The Markovian framework



To bypass the non global well–posedness of the problem we consider a special class of solutions, which constitute a Markov process.

- Consider the set $\mathscr{C}(x)$ of all solutions starting at each i. c. x,
- prove that each $\mathscr{C}(x)$ is compact convex non–empty,
- prove a "set" version of the Markov property,
- short time coupling with a smooth problem,
- hence strong Feller,
- ergodicity and strong mixing of each Markov process.

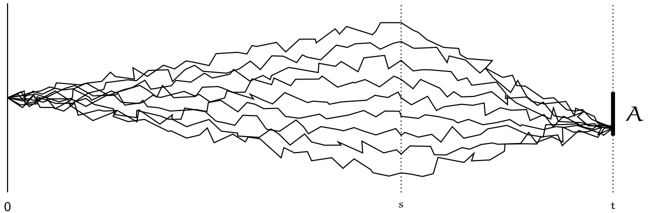
[krylov, stroock–varadhan, flandoli–mr]

The Markov property

A family $(\mathbb{P}_x)_{x\in L^2}$ of probability measures on $\Omega = C([0,\infty; H^{-2})$ is a Markov family if

$$P(t,x,A) = \int P(t-s,y,A)\,P(s,x,dy),$$

where $P(\cdot,x,A) = \mathbb{P}_x[h(t)\in A]$.



Weak martingale solutions

A probability measure $\mathbb{P}$ is a **weak martingale solution** if

- $\mathbb{P}[L^2_{loc}([0,\infty); H^1)] = 1$,
- the marginal at time 0 of P is μ_0
- for every test function ϕ, the process

$$M_t^\phi = \langle h(t) - h(0), \phi\rangle + \int_0^t \langle h(s), \phi_{xxxx} + \phi_{xx}\rangle + \int_0^t \langle (h_x)^2, \phi_{xx}\rangle$$

is a Brownian motion with diffusion coefficient $\|\phi\|^2_{L^2}$,

The Wiener and OU processes

Given a weak martingale solution $\mathbb{P}_x$, one can reconstruct the driving Wiener process (hence a function of the solution!). Given an orthonormal basis $(e_k)_{k\in\mathbf{N}}$ of $L^2(0,L)$,

$$(M_t^{e_k})_{k\in\mathbf{N}} \quad \text{are independent standard Bm}$$

and so

$$W(t) = \sum M_t^{e_k} e_k$$

is a cylindrical Wiener process. Likewise, one gets the solution Z to the corresponding linear problem,

$$\begin{cases} dZ + Z_{xxxx}\,dt = dW, \\ Z(0) = 0. \end{cases}$$

Energy solutions

A probability measure $\mathbb{P}$ on the path space Ω is an **energy martingale solution** if

- $\mathbb{P}$ is a weak martingale solution,
- $\mathbb{P}[V \in L^\infty_{loc}([0,\infty); L^2) \cap L^2_{loc}([0,\infty); H^2)] = 1$,
- there is a set $T_\mathbb{P} \subset (0,\infty)$ of null Lebesgue measure such that for all $s \notin T_\mathbb{P}$ and all $t \geqslant s$,

$$\mathbb{P}[\mathcal{E}_t(V,Z) \leqslant \mathcal{E}_s(V,Z)] = 1,$$

where $V(t,\omega) = h(t,\omega) - Z(t,\omega)$, for $t \geqslant 0$, and

$$\mathcal{E}_t(v,z) = \frac{1}{2}|v(t)|^2_{L^2} + \int_0^t \left(|v_{xx}|^2_{L^2} - |v_x|^2_{L^2}\right) ds + \\ - \int_0^t \left(\langle v_x, z_x\rangle_{L^2} + \langle 2v_x z_x + (z_x)^2, v_{xx}\rangle_{L^2}\right) ds.$$

Control of the energy: exceptional times

The energy inequality says,

$$\mathbb{P}[\mathcal{E}_t(V,Z) \leqslant \mathcal{E}_s(V,Z)] = 1,$$

for all $s \notin T_{\mathbb{P}}$ and all $t \geqslant s$, which seems hard. Indeed, it follows from a stability result that

Theorem

The following properties are equivalent,

- $\mathbb{P}\Big[\mathcal{E}_t(V,Z) \leqslant \mathcal{E}_s(V,Z) \text{ for all } t \text{ and a. e. } s\Big] = 1,$
- $\mathbb{P}\Big[\mathcal{E}_t(V,Z) \leqslant \mathcal{E}_s(V,Z)\Big] = 1 \quad \text{for all } t \text{ and a. e. } s.$

Remark: The inequality is robust enough to allow for modifications such as:

$$dZ + Z_{xxxx}\,dt + \alpha Z = dW.$$

Existence of Markov solutions

The first part of the strategy is complete.

Theorem

There exists at least one family $(\mathbb{P}_x)_{x\in L^2}$ *of probability measures on* Ω *such that for each* $x \in L^2$, $\mathbb{P}_x$ *is an energy martingale solution with initial distribution concentrated on* x*. Moreover the (**almost sure**) Markov property holds,*

$$\mathbb{E}^{\mathbb{P}_x}[\phi(h'_t)|\mathscr{B}_s] = \mathbb{E}^{\mathbb{P}_{h_s}}[\phi(h'_{t-s})],$$

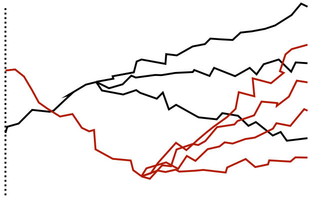
for all $s \notin T_{\mathbb{P}_x}$, $t \geqslant s$, *bounded measurable* $\phi : L^2 \to \mathbf{R}$.

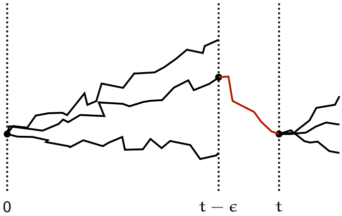
[blömker–flandoli–mr]

Short time coupling with a smooth process

We use now the existence of local "strong" solutions.

For smooth initial conditions there is a small (random) time τ such that up to τ all solutions coincide.
Essentially, any two solutions have the same distribution on the event $\{\tau > t\}$.





The real picture is that the "uniqueness of strong solutions" argument is applied at the very last moment only, thanks to the Markov property.

Continuity w.r.t. initial conditions

Distance from the local solution:

$$
\begin{aligned}
|P_\epsilon\phi(x) - \widetilde{P}_\epsilon\phi(x)| &\leqslant |\mathbb{E}^{\mathbb{P}_x}[\phi(h_\epsilon)\mathbb{1}_{\{\widetilde{\tau}_x\leqslant\epsilon\}}] - \mathbb{E}^{\widetilde{\mathbb{P}}_x}[\phi(h_\epsilon)\mathbb{1}_{\{\widetilde{\tau}_x\leqslant\epsilon\}}]| \\
&\leqslant \mathbb{P}_x[\widetilde{\tau}_x \leqslant \epsilon] + \widetilde{\mathbb{P}}_x[\widetilde{\tau}_x \leqslant \epsilon], \\
&\approx \text{short time tail of } \widetilde{\tau},
\end{aligned}
$$

(exponentially small) hence,

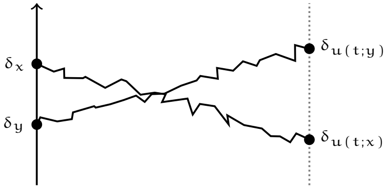
$$
\begin{aligned}
P_t\phi(y) - P_t\phi(x) &= P_\epsilon(P_{t-\epsilon}\phi)(y) - P_\epsilon(P_{t-\epsilon}\phi)(x) \\
&= o(\epsilon) + o(\|x-y\|) \\
&= \text{Error(non–uniqueness)} + \widetilde{\text{Error}}(x-y)
\end{aligned}
$$

Theorem

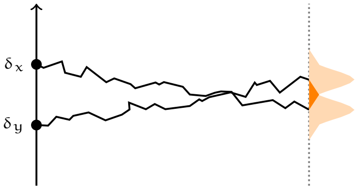
For any Markov family $x \mapsto \mathbb{P}_x$, the map $x \mapsto P(t,x,\cdot)$, for $x \in H^1$, is continuous in total variation.

It cannot work without noise...

Deterministic...



Stochastic...



In the stochastic case the two distributions have a common mass for two reasons:

- there is a tiny probability that $\widetilde{\tau}$ is large enough,
- the diffusive effect of Brownian motion.

Invariant measures

The noise is rough enough, so it is sufficient to have

- Existence of at least one invariant measure,
- regularity of transition densities,
- irreducibility.

The "hard" part is existence: the invariant measure must see the dynamics!

$$\mathbb{P}[\|h\|_{H^{1+\epsilon}} \geqslant R] \leqslant \mathbb{P}[\|V_{xx}\|_{L^2} \geqslant cR] + \mathbb{P}[\|Z\|_{H^{1+\epsilon}} \geqslant cR].$$

The energy inequality provides

$$\|V(t)\|_{L^2}^2 + \int_0^t \|V_{xx}\|_{L^2}^2 \leqslant c \int_0^t \|Z_x\|_{L^4}^{\frac{16}{3}} \|V\|_{L^2} + \text{lower order terms}\ldots$$

...not enough.

[blömker–hairer]

Strong mixing

We need to use the energy inequality corresponding to

$$dZ + Z_{xxxx}\,dt + \alpha Z = dW.$$

in order to control the term

$$\|V(t)\|_{L^2}^2 + \int_0^t \|V_{xx}\|_{L^2}^2 \leqslant c \int_0^t \|Z_x\|_{L^4}^{\frac{16}{3}} \|V\|_{L^2} + \text{lower order terms}\ldots$$

[es sahir–stannat]

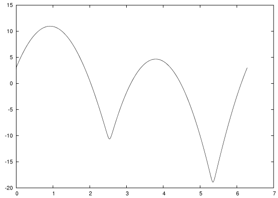
A smart choice of α and the ergodic theorem conclude the proof.

Theorem

Let $(\mathbb{P}_x)_{x\in L^2}$ be any a. s. Markov family of energy martingale solutions. Then the transition semigroup associated to $(\mathbb{P}_x)_{x\in L^2}$ has a unique invariant measure, fully supported on H^1.

Blow–up?

- coarsening suggests negative blow–up,
- numerical approximations (convergence rate strong and N–independent),
- no self–similar solutions (numerical evidence),
- blow–up criteria (NS–like).



[blömker–gugg–raible, blömker–mr]

A toy model

We consider the formulation in Fourier variables of the surface growth problem,

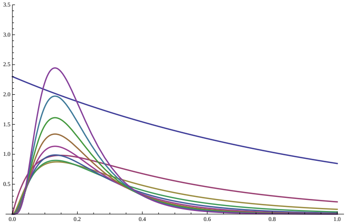
$$\dot{h}_k + k^4 h_k + k^2 \sum_{m=1}^{k-1} m(k-m) h_m h_{k-m} = 0, \qquad k \geqslant 1.$$

on the (invariant) subspace

$$\{h_k = 0 \text{ for } k \leqslant 0 \text{ and } h_k \geqslant 0 \text{ for } k \geqslant 1\}$$

We have

- global solutions for "small" initial data,
- blow–up if the initial data is large in a finite patch.



[mr–blömker]

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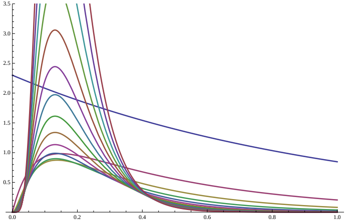

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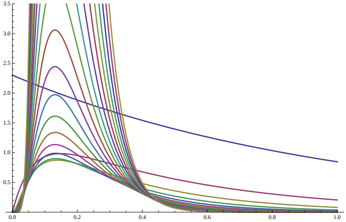


[mr–blömker]

Local existence and uniqueness in $\mathbb{R}^d$

We look for mild solutions

$$h(t) = e^{-t\Delta^2} h_0 - \int_0^t e^{-(t-s)\Delta^2} (\Delta|\nabla h|^2)\, ds \left[+ \int_0^t e^{-(t-s)\Delta^2}\, dW_s \right]$$

with initial conditions in the space that satisfies the minimal requirements to define the solution.
For two reasons:

- allow for rough initial data (essentially $\log \|x\|$)
- minimal requirement on the covariance for the corresponding SPDE.

[koch–tataru, koch–lamm]

The basic idea: minimal requirements

A weak solution on $\mathbf{R}^d$ is any space–time distribution h such that for each test function ϕ,

$$\int_0^\infty \int_{\mathbf{R}^d} h \frac{\partial \phi}{\partial t} - \int_0^\infty \int_{\mathbf{R}^d} h \Delta^2 \phi - \int_0^\infty \int_{\mathbf{R}^d} |\nabla h|^2 \Delta \phi = - \int_{\mathbf{R}^d} h_0 \phi(0)$$

hence

- at least $\nabla h \in L^2_{\text{loc}}([0,\infty) \times \mathbf{R}^d)$ (hence $h \in L^2_{\text{loc}}$),
- the equation is translation invariant,
- the equation is invariant for the scaling: $h(t,x) \rightsquigarrow h(\lambda^4 t, \lambda x)$,

so a translation– and scale–invariant version is

$$\|h\|^2_{\mathcal{X}^0} := \sup_{x \in \mathbf{R}^d, R>0} \left(\frac{1}{R^{d+2}} \int_0^{R^4} \int_{B_R(x)} |\nabla h|^2 \, dy \, dt \right).$$

The basic idea: the initial condition

The initial condition h_0 is chosen so that $\| e^{-t\Delta^2} h_0\|_{\mathcal{X}^0} < \infty$. It turns out that

$$\| e^{-t\Delta^2} h_0\|_{\mathcal{X}^0} \quad \approx \quad \sup_{t>0} t^{\frac{1}{4}} \|\nabla\big(e^{-t\Delta^2} h_0\big)\|_\infty$$

that is $h_0 \in B^{0,\infty}_{\infty}(\mathbf{R}^d)$.

Theorem

Global existence and uniqueness for small initial conditions in $B^{0,\infty}_{\infty}$. Local existence and uniqueness for initial condition with vanishing local norm.

In particular, $h_0(x) = \alpha_d \log\|x\|$ is a stationary solution for a suitable α_d.

Theorem

If $d = 4$, non–uniqueness of mild solutions for α_d sufficiently small.

The main characteristics

What does create non–uniqueness or blow–up?

We are essentially interested in the general problem

$$\dot{u} + \nu A u + B(u,u) = \text{forcing},$$

- viscous linear part,
- quadratic nonlinearity
- purely rotational nonlinearity (balance of energy),
- global weak solutions,
- local unique smooth solutions for regular initial conditions,
- existence of an invariant state.

The dyadic model

The system of differential equations

$$\dot{x}_n = -\nu\lambda_n^2 x_n + \lambda_{n-1}^{\beta} x_{n-1}^2 - \lambda_n^{\beta} x_n x_{n+1}$$

where $x_0 \equiv 0$ and $\lambda_n = 2^n$ has the following characteristics:

- formal balance of energy (whatever it is!),
- existence of weak solutions,
- smoothness for short times.

In fact,

$$\frac{d}{dt} x_n^2 + 2\nu\lambda_n^2 x_n^2 = \lambda_{n-1} x_{n-1}^2 x_n - \lambda_n^{\beta} x_n^2 x_{n+1}$$

and

$$\frac{d}{dt}\Big(\frac{1}{2}\sum_{n=1}^{N} x_n^2\Big) + \nu\sum_{n=1}^{N}\lambda_n^2 x_n^2 = -\lambda_N^{\beta} x_N^2 x_{N+1}$$

[cheskidov–friedlander–katz–pavlovic]

The dyadic model: known facts

$$\dot{x}_n = -\nu\lambda_n^2 x_n + \lambda_{n-1}^{\beta} x_{n-1}^2 - \lambda_n^{\beta} x_n x_{n+1}$$

It is also known that:

- positive initial conditions give positive solutions.
- if $\beta \leqslant 2$, there is well–posedness (2DNS–regime)
- if $\beta > 3$ there is blow–up (for large enough positive initial conditions).

By similarity (scaling properties), the three dimensional case corresponds to $\beta \approx \frac{5}{2}$.

[cheskidov]

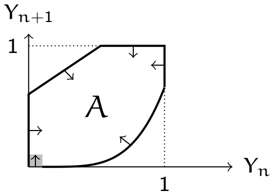
Smoothness and non–uniqueness

$$\dot{x}_n = -\nu\lambda_n^2 x_n + \lambda_{n-1}^{\beta} x_{n-1}^2 - \lambda_n^{\beta} x_n x_{n+1}$$

The range between 2 and 3 is the difficult one. From the scaling point of view neither the linear nor the nonlinear term are dominant in magnitude.

Theorem

Well–posedness for positive solutions if $\beta \in (2, \frac{5}{2}]$



Moreover there exists a (negative) solution, which is stationary,

$$\lambda_{n-1}^{\beta}\gamma_{n-1}^2 - \lambda_n^{\beta}\gamma_n\gamma_{n+1} = \nu\lambda_n^2\gamma_n, \qquad n \geqslant 1,$$

and non smooth: $\gamma_n \approx \lambda_n^{\beta-2}$.

[barbato–morandin–mr]

Playing with noise

Let $\sigma_n \in \mathbf{R}$,

$$\dot{x}_n = -\nu\lambda_n^2 x_n + \lambda_{n-1}^{\beta} x_{n-1}^2 - \lambda_n^{\beta} x_n x_{n+1} + \sigma_n \dot{w}_n$$

It is known that

- if $\beta \leqslant 2$ trivial well-posedness,
- if $\beta > 3$ and $\{\sigma_n \neq 0\}$ is finite, then blow–up,
- if $\beta \leqslant 1$ well posedness with a **special** multiplicative noise and $\nu \equiv 0$.

The following are **open problems**:

- well–posedness if $\beta \in (2, 3)$???
- well–posedness if $\beta \in (2, 3)$ and multiplicative noise which preserves positivity: $\sigma_n x_n \, dw_n$???

[flandoli–barbato–morandin, mr]

An example of blow–up

Consider

$$\dot{x}_n = -\nu\lambda_n^2 x_n + \lambda_{n-1}^{\beta} x_{n-1}^2 - \lambda_n^{\beta} x_n x_{n+1} + \sigma_n \dot{w}_n$$

with $\sigma_n \neq 0$ for all n. Assume $\beta > 3$.

Theorem

The probability that the blow–up time is finite is positive for every initial condition.

Three ideas:

- there is a part of the state space where the solution blows up when forced by "small noise",
- the noise is "small" with positive probability on any time interval,
- the system is irreducible.

The trick is to switch between ℓ^2–like and ℓ^∞–like topologies.

[de bouard–debussche, mr]

Everything fine?

There are a few reason that make the counterexample not completely satisfactory:

1. It is a counterexample to **smoothness**, not to **uniqueness**!
2. Does the example cover all the required properties?
   - [OK] viscous linear part,
   - [OK] quadratic nonlinearity,
   - [OK] purely rotational nonlinearity (balance of energy),
   - [OK] global weak solutions,
   - [OK] local unique smooth solutions for regular initial conditions,
   - [NO!] existence of an invariant state.

The crucial assumption $\beta > 3$ makes the linear part too weak. The process lives in the "critical" space with decay (at least)

$$\lambda_n^{\beta-2}|x_n| \approx O(1)$$

and $\beta - 2 > 1$.