

# Dimensions, embeddings, and attractor

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Edinburgh, September 21st, 2010

# Finite-dimensional asymptotic dynamics?

Fundamental question (Eden, Foias, Nicolaenko, & Temam, 1994)

Suppose that  $\dot{u} = F(u)$  ( $u \in H$ ) has a finite-dimensional global attractor  $\mathcal{A}$ .

Is there a finite system of ODEs with 'the same attractor'?

- $\dot{u} = F(u)$  gives rise to a semigroup  $\{S(t)\}_{t \geq 0}$  on  $H$ ;
- the global attractor is compact, invariant, and attracts all bounded subsets of  $H$ ;
- it has 'finite dimension' in some appropriate sense;
- we want a map  $L : H \rightarrow \mathbb{R}^k$  for some  $k$  and an ODE  $\dot{x} = f(x)$  on  $\mathbb{R}^k$  generating a group  $T(t)$  such that  $T(t)$  has an attractor  $X = L\mathcal{A}$  and

$$T(t)|_X = L \circ S(t) \circ L^{-1}.$$

## What kind of set can be an attractor?

A set  $C \subset \mathbb{R}^m$  is an *m-cell* if there exists a homeomorphism from  $\overline{B}(0,1)$  onto  $C$ .

A set  $X \subset \mathbb{R}^m$  is *cellular* if given any neighbourhood  $U$  of  $X$  there exists an *m-cell*  $C \subseteq U$  that is a neighbourhood of  $X$ ; equivalently if there exists a decreasing sequence  $C_i$  of *m-cells* such that

$$X = \bigcap_{i=1}^{\infty} C_i.$$

### Theorem

A (compact) set  $X \subset \mathbb{R}^m$  is the global attractor of a homeomorphism  $h : \mathbb{R}^m \rightarrow \mathbb{R}^m$  if and only if it is cellular.

## Attractors of homeomorphisms are cellular

Since  $X$  is compact, it is contained in  $\overline{B}(0, R)$  for some  $R > 0$ .  
Since  $X$  attracts  $\overline{B}(0, 2R)$ , there exists an  $n \in \mathbb{N}$  such that

$$h^n(\overline{B}(0, 2R)) \subset \overline{B}(0, R)$$

and

$$\bigcap_{k=1}^{\infty} h^{nk}(\overline{B}(0, 2R)) \subseteq X. \quad (*)$$

But  $X \subset \overline{B}(0, 2R)$  and  $X$  is invariant,  $X \subset h^{nk}\overline{B}(0, 2R)$  for all  $k$ ,  
so we have equality in (\*).

Since  $h^{nk}$  is a homeomorphism for each  $k \in \mathbb{N}$ ,  $h^{nk}\overline{B}(0, 2R)$  is an  $m$ -cell. So  $X$  is cellular.

# Cellular sets are 'pointlike' 1

Let  $X \subset Q := \overline{B}(0, R)$ , with

$$X = \bigcup_{k=1}^{\infty} Q_k \quad \text{where} \quad Q_{k+1} \subset \text{int } Q_k.$$

Following Brown (1960) we construct a map  $f : Q \rightarrow Q$  such that  $f|_{\partial Q} = \text{Id}$ ,  $f(X) = \{0\}$ , and  $f : Q \setminus X \rightarrow Q \setminus \{0\}$  is a homeomorphism, i.e.  $X$  is 'pointlike'.

Let  $h_1 : Q \rightarrow Q$  be a homeomorphism such that  $h_1|_{\partial Q} = \text{Id}$  and  $h_1(Q_1) = \overline{B}(0, 1)$ .

Now, given  $h_{k-1}$ , let  $h_k : Q \rightarrow Q$  be a homeomorphism such that

$$h_k(x) = h_{k-1}(x) \quad x \in Q \setminus Q_{k-1} \quad \text{and} \quad h_k(Q_k) = \overline{B}(0, 1/k).$$

## Cellular sets are 'pointlike' 2

Set

$$f(x) = \lim_{k \rightarrow \infty} h_k(x);$$

then  $f : Q \rightarrow Q$  has  $f|_{\partial Q} = \text{Id}$ , and  $f(X) = \{0\}$  by construction.

Suppose that  $x, y \in Q$  with  $x \notin X$ , and hence  $x \notin Q_j$  for some  $j$ .

(i) If  $y \in X$  then  $f(y) = 0$  and  $f(x) \neq 0$  since  $h_j$  is a homeomorphism and  $h_j(Q_j) = B(0, 1/j) \ni \{0\}$ .

(ii) If  $y \notin X$  then  $x, y \notin Q_i$  for some  $i$ ,  $f(z) = h_i(z)$  for  $z \notin Q_i$ , and  $h_i : Q \rightarrow Q$  is a homeomorphism.

So  $X$  is pointlike.

Now define a homeomorphism  $s : Q \setminus X \rightarrow S^{m-1} \times (0, 1]$  by setting

$$s(x) = (p(x), r(x)) = \left( \frac{f(x)}{|f(x)|}, \frac{|f(x)|}{R} \right).$$

Note that  $r(x) \rightarrow 0$  as  $x \rightarrow X$ .

# Pointlike sets are attractors of homeomorphisms

Finally (after Garay, 1991), choose another homeomorphism  $\varphi : [0, R] \rightarrow [0, R]$  such that  $\varphi(0) = 0$  and  $\varphi^n(r) \rightarrow 0$  as  $n \rightarrow \infty$  for every  $r \in [0, R]$ , and in such a way that the map

$$h(x) := s^{-1}(p(x), \varphi(r(x))) \quad x \notin X, \quad h(x) = x, \quad x \in X$$

is continuous. The function  $\varphi$  'slows down the dynamics' sufficiently that one recovers a homeomorphism about  $X$ ...

$X$  is an attractor for the homeomorphism  $h : Q \rightarrow Q$ . Note, however, that the dynamics on  $X$  are trivial, i.e.  $h(x) = x$  for all  $x \in X$ .

## An aside: 'almost attractors' of homeomorphisms are cellular

Suppose that  $X \subset \mathbb{R}^m$  is such that for any  $\epsilon > 0$  there exists a homeomorphism  $h : \mathbb{R}^m \rightarrow \mathbb{R}^m$  that has an attractor  $X'$  with

$$X \subseteq X' \subseteq N(X, \epsilon).$$

Then  $X$  still must be cellular: given any  $\epsilon > 0$ , find a homeomorphism  $h$  that has an attractor  $X' \subset N(X, \epsilon)$ .

By the above,  $X'$  must be cellular. So there exists a cellular neighbourhood of  $X'$  contained in  $N(X, \epsilon)$ , i.e. a cellular neighbourhood of  $X$  contained in  $N(X, \epsilon)$ .

So the requirement of cellularity of  $X$  is not only an artefact of our proof.

## Attractors for flows and semiflows are cellular

### Theorem (Garay, 1991)

*A set  $X$  is the attractor of a flow in  $\mathbb{R}^m$  if and only if it is cellular.*

A similar construction to the above.

### Theorem (Günther, 1993; PRS)

*A set  $X$  is the attractor of an ODE in  $\mathbb{R}^m$  if and only if it is cellular.*

Set  $V(x) = r(x)^{-1}$ , “smooth”, and consider  $\dot{x} = -\nabla V(x)$ .

### Theorem (Garay, 1991)

*A set  $X$  is the attractor of a semigroup in an infinite-dimensional space if and only if it is cellular.*

Uses infinite-dimensional topology; not resolved in  $\mathbb{R}^m$ .

# Embedding sets with finite box-counting dimension

Theorem (Hunt & Kaloshin, 1999)

If  $d_{\text{box}}(\mathcal{A}) = d$ ,  $\tau(\mathcal{A}) = \tau$ ,  $k > 2d$ , and

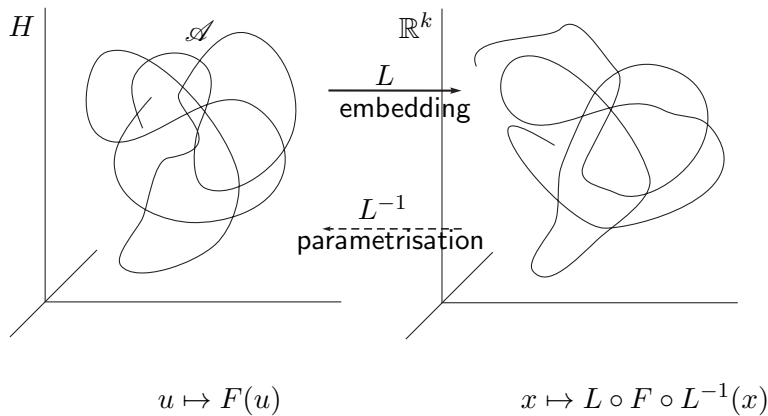
$$0 < \theta < \frac{k - 2d}{k(1 + \tau/2)},$$

then 'a.e.' linear map  $L : H \rightarrow \mathbb{R}^k$  is injective on  $\mathcal{A}$  and

$$\|x - y\| \leq C_L |x - y|^\theta \quad \text{for all } x, y \in \mathcal{A} \quad \text{for some } C_L > 0.$$

- $\tau(\mathcal{A})$  is the 'thickness exponent' of  $\mathcal{A}$ ,  $\tau(\mathcal{A}) \leq d_{\text{box}}(\mathcal{A})$ ;
- $\tau(\mathcal{A}) = 0$  if  $\mathcal{A}$  consists of 'smooth functions';
- $\tau(\mathcal{A})$  can be replaced by  $\text{dev}(\mathcal{A})$ , which is zero for 'many' equations (Pinto de Moura & JCR, 2010);
- BUT Hölder continuity here cannot be improved in general (Hunt & Kaloshin, 1999; PdeM & JCR, 2010).

# Embedding $\mathcal{A}$ into $\mathbb{R}^k$



# Making $X$ cellular

## Problem

Cellularity is not a topological property.

## Solution

Borsuk's 'shape' is a topological property.

$\mathcal{A}$  has the same shape as  $H$ .  $H$  has the same shape as a point. Shape is a topological property, so  $X = L\mathcal{A}$  has the shape of a point. If  $X \subset \mathbb{R}^k$  has the shape of point,  $X \times \{0\}$  is a cellular subset of  $\mathbb{R}^{k+1}$  (Daverman).

# Attractors of homeomorphisms

## Theorem

Let  $\mathcal{A}$  be the global attractor of a map  $S : H \rightarrow H$ , such that  $S|_{\mathcal{A}}$  is a homeomorphism and  $d_B(\mathcal{A}) = d < \infty$ . Then for any  $k > 2d + 2$  and any  $\epsilon > 0$  there exist homeomorphisms  $L : H \rightarrow \mathbb{R}^k$  and  $f : \mathbb{R}^k \rightarrow \mathbb{R}^k$  such that

- (i)  $f|_{L\mathcal{A}}$  reproduces the dynamics on  $\mathcal{A}$ , i.e.  
 $f|_{L\mathcal{A}} = L \circ S \circ L^{-1}$ , and
- (ii) the dynamical systems on  $\mathbb{R}^k$  generated by  $f$  has an attractor  $\mathcal{X}$  with

$$L\mathcal{A} \subseteq \mathcal{X} \subset N(L\mathcal{A}, \epsilon).$$

Choose any  $m > 2d + 1$  and a linear embedding  $L'$  of  $\mathcal{A}$  into  $\mathbb{R}^m$ . Then set  $L = (L', 0)$ , and put  $k = m + 1$ , giving a linear embedding  $L : H \rightarrow \mathbb{R}^k$  such that  $X = L\mathcal{A}$  is cellular.

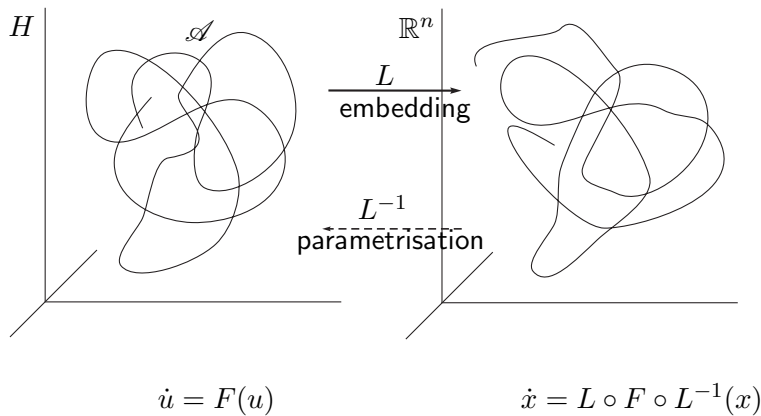
Define  $h : X \rightarrow X$  by  $h'(x) = LS(L^{-1}x)$ ; clearly  $|h'(x) - x| \leq \delta := \text{diam}(X)$  for every  $x \in X$ .

Since  $X$  is cellular, one can use an extension theorem due to Geoghegan & Summerhill (1973) to extend  $h'$  to a homeomorphism  $h : \mathbb{R}^k \rightarrow \mathbb{R}^k$  such that  $|h(x) - x| \leq \delta$  for every  $x \in \mathbb{R}^k$  and  $h(x) = x$  for all  $x \notin N(X, \delta)$ . Note that  $N(X, 2\delta)$  is invariant under  $h$ .

Now consider the homeomorphism  $\varphi$  that has  $X$  as an attractor (consisting of fixed points). There exists an  $n \in \mathbb{N}$  such that  $h^n(N(X, 2\delta)) \subset N(X, \epsilon)$ .

The homeomorphism  $f = \varphi \circ h$  has all the required properties.

# Embedding $\mathcal{A}$ into $\mathbb{R}^k$



## 'Embedded ODE' (cf. Eden et al., 1994)

The equation on  $X = L\mathcal{A}$  should be

$$\dot{x} = f(x) := L \circ F \circ L^{-1}(x).$$

Does this have unique solutions?

Osgood's criterion:  $\dot{x} = g(x)$  has unique solutions if

$$\int_0^1 \frac{1}{\omega(r)} dr = \infty, \quad (*)$$

where

$$|g(x) - g(y)| \leq \omega(|x - y|).$$

The condition (\*) is satisfied if  $g$  is  $\alpha$ -log Lipschitz with  $\alpha \leq 1$ ,

$$\omega(r) = r |\log r|^\alpha.$$

## Ensuring that $f(x) = L \circ F \circ L^{-1}$ is $\alpha$ -log Lipschitz

- As before, we will use an  $L$  that is linear and bounded, so Lipschitz.
- One can show that  $F$  is 1-log Lipschitz *when restricted to  $\mathcal{A}$*  for a wide class of equations (PdeM & JCR, 2010, using an argument of Kukavica, 2007); this means that  $L^{-1}$  would have to be Lipschitz to ensure uniqueness (which is too strong).
- So we will have to *assume* that  $F$  is  $\beta$ -log Lipschitz on  $\mathcal{A}$  for some  $\beta < 1/2$ . This is no restriction given a system of ODEs on  $\mathbb{R}^N$ , with  $N$  'large'.

Under this assumption on  $F$ , we need  $L^{-1}$  to be (essentially)  $1/2$ -log Lipschitz to guarantee uniqueness.

*We cannot obtain this only assuming that the box-counting dimension of  $\mathcal{A}$  is finite.*

# Assouad dimension

$X$  is  $(M, s)$ -homogeneous if

$$N(X \cap B(x, r), \rho) \leq M(r/\rho)^s.$$

[For box-counting dimension,  $N(X, \rho) \leq \rho^{-d}$  for  $d > d_{\text{box}}(X)$ .]

- Any subset of  $\mathbb{R}^k$  is homogeneous.
- Homogeneity is invariant under bi-Lipschitz maps.

Any set that can be bi-Lipschitz embedded into  $\mathbb{R}^k$  must be homogeneous. (But it is known that this is not sufficient.)

$$d_A(X) = \inf\{s : X \text{ is } (M, s)\text{-homogeneous}\}.$$

[Also  $d_A(X) < \infty$  does not imply  $d_A(X - X) < \infty$  (Olson, 2002).]

### Theorem (Robinson, 2010; after Olson, 2002; EJO & JCR, 2010)

Let  $X$  be a compact subset of a Hilbert space  $H$  such that  $d_A(X - X) < s < N$ . If

$$\gamma > \frac{(N/2) + 1}{N - s},$$

then 'a.e.' linear map  $L : H \rightarrow \mathbb{R}^N$  is injective on  $X$  and  $\gamma$ -almost bi-Lipschitz on  $X$ : for some constant  $c_L > 0$  and  $\rho_L > 0$

$$\|x - y\| \leq c_L |Lx - Ly| (-\log |Lx - Ly|)^\gamma$$

for all  $x, y \in X$  with  $\|x - y\| < \rho_L$ .

Uses the following beautiful result:

### Theorem (Ball, 1986)

Every section of the unit cube in  $\mathbb{R}^n$  by an  $(n - 1)$ -dimensional subspace has volume at most  $\sqrt{2}$ .

## Theorem (EPdeM, JCR, & JJS-G, 2010)

Suppose that  $d_A(\mathcal{A} - \mathcal{A}) = d$ , and an  $F$  that is  $\beta$ -log Lipschitz on  $\mathcal{A}$ ,  $\beta < \frac{1}{2}$ . For any  $m > [2(1 + d(1 - \beta))/(1 - 2\beta)] + 1$  and any  $\epsilon > 0$  there exists a system of ODEs

$$\dot{x} = \mathcal{F}(x) \quad (\dagger)$$

on  $\mathbb{R}^m$  and a bounded linear map  $L : H \rightarrow \mathbb{R}^m$  such that

1. the ODE  $(\dagger)$  has unique solutions;
2. the restriction  $L|_{\mathcal{A}}$  is an embedding whose image  $L\mathcal{A}$  is invariant under the dynamics of  $(\dagger)$ ;
3. for every solution of  $\dot{u} = F(u)$  on  $\mathcal{A}$  there exists a unique solution of  $(\dagger)$  such that  $u(t) = L^{-1}(x(t))$ ; and
4. the ODE  $(\dagger)$  has a global attractor  $\mathcal{X}$  such that

$$L\mathcal{A} \subseteq \mathcal{X} \subseteq N(L\mathcal{A}, \epsilon).$$

# Open problems

- A simpler problem: let  $X \subset \mathbb{R}^m$  be a cellular set and  $h : X \rightarrow X$  a homeomorphism. Can one extend  $h$  to a homeomorphism on  $\mathbb{R}^m$  for which  $X$  is the attractor?
- There is no known method to bound  $d_A(\mathcal{A})$  or (what we need)  $d_A(\mathcal{A} - \mathcal{A})$ ... but Eden, Foias, Nicolaenko, & Temam (1988) show that when the 'squeezing property' holds then

$$N(S(1)[\mathcal{A} \cap B(x, r)], \rho) \leq M(r/\rho)^s$$

for some appropriate  $M$ ,  $s$ , which is very suggestive...