

# Generic properties of scalar parabolic equations

(Joint work with P. Brunovský and R. Joly)

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## Purpose - Motivations

Let  $\Omega$  be a (regular) bounded domain of  $\mathbf{R}^n$ ,  $n = 1, 2, 3$ .  
We consider the **scalar** parabolic equation

$$\begin{aligned}u_t &= \Delta u + f(x, u, \nabla u), t > 0, \\ \text{homogeneous Dirichlet (or Neumann) B.C.}, \\ u|_{t=0} &= u_0 \in Y \equiv X_p^{2\alpha},\end{aligned}\tag{1}$$

where  $f : (x, y, z) \in \Omega \times \mathbf{R} \times \mathbf{R}^n \mapsto \mathbf{R}$  is a function of class  $C^2$ . If  $\Omega = \mathbf{T}^n$ , we can also consider periodic conditions.

**Local existence and uniqueness of solutions:**

$X = L^p(\Omega)$ ,  $p > n$ ,  $2\alpha \in (1 + n/p, 2)$  s. t.

$X_p^{2\alpha} = D((-\Delta_D)^\alpha) \subset W^{2\alpha, p}(\Omega) \subset C^1(\overline{\Omega})$ .

In the case  $n = 1$ , choose  $Y \subset H^s(\Omega)$ , with  $3/2 < s < 2$ .

Equation (1) defines a **local nonlinear semi-flow**

$S(t) \equiv S_f(t) : u_0 \in X_p^{2\alpha} \mapsto S_f(t)u_0 = u(t) \in X_p^{2\alpha}$ .

Under additional conditions on  $f$  (growth and dissipation), e.g., (H1) there exist a function  $k(\cdot) \in C^0(\mathbf{R}^+, \mathbf{R}^+)$  and constants  $\varepsilon > 0$  and  $\kappa > 0$  such that

$$\forall r > 0, \forall \xi \in \mathbf{R}^n, \quad \sup_{(x,z) \in \Omega \times [-r,r]} |f(x, z, \xi)| \leq k(r)(1 + |\xi|^{2-\varepsilon}),$$
$$\forall |z| \geq \kappa, \forall x \in \Omega, \quad zf(x, z, 0) < 0,$$

the solutions of (1) are global.

### Theorem (Compact global attractor)

If (H1) holds,  $S_f(t)$  has a compact global attractor  $\mathcal{A}_f$ , i.e.

- $\mathcal{A}_f$  is a compact set in  $X_p^{2\alpha}$
- $\mathcal{A}_f$  is invariant (i.e.  $S_f(t)\mathcal{A}_f = \mathcal{A}_f$ , for any  $t \geq 0$ )
- $\mathcal{A}_f$  attracts every bounded set of  $X_p^{2\alpha}$ .

Even with Hypothesis (H1), the dynamics can be very complicated.

## Two types of questions

- I) **Stability of the dynamics under perturbations:**
  1. Local stability
  2. Global or structural stability
- II) **Generic dynamics (back to the sixties):**
  1. (generic) hyperbolicity of **critical elements** (i. e. **equilibria and periodic orbits**) ?
  2. (generic) **transversality** of stable and unstable manifolds of critical elements ?
  3. (generically) does the **non-wandering set** reduce to a **finite number of critical elements**?

Properties 1. + 2. = Kupka-Smale

Kupka-Smale + property 3. = Morse-Smale

(Morse-Smale  $\rightsquigarrow$  structural stability (Palis, Smale, Oliva))

(Kupka-Smale  $\rightsquigarrow$  local stability)

**Goal:** To generalize the Morse-Smale and Kupka-Smale properties to PDE's

## Basic definitions 1: critical elements

1. Let  $e$  be an equilibrium and  $L_e$  be the linearized operator

$$L_e v = \Delta v + D_u f(x, e, \nabla e) v + D_{\nabla u} f(x, e, \nabla e) \cdot \nabla v$$

- $e$  is *hyperbolic* if, for any eigenvalue  $\lambda$  of  $L_e$ ,  $\operatorname{Re} \lambda \neq 0$ .
- *Morse index*  $i(e) = \#$  eigenvalues  $\lambda$  of  $L_e$  with  $\operatorname{Re} \lambda > 0$ .

2. Let  $p(t)$  be a periodic solution of period  $T$  and  $\Pi(t, s)$  be the linear operator defined by the linearized equation:

$$v_t = \Delta v + D_u f(x, p, \nabla p) v + D_{\nabla u} f(x, p, \nabla p) \cdot \nabla v,$$

$$v(x, s) = v_{0,s} \in X_p^{2\alpha}.$$

- $\Gamma = \{p(t) | t \in [0, T]\}$  is *hyperbolic* if  $\sigma(\Pi(T, 0)) \cap \{|\mu| = 1\} = \{1\}$  and 1 is an isolated simple eigenvalue of the *period map*  $\Pi(T, 0)$ .
- *Morse index*  $i(\Gamma) = \#$  eigenvalues  $\mu$  s.t.  $|\mu| > 1$ .

The equilibrium points and periodic orbits are called *critical elements*.

## Basic definitions 2: stable and unstable sets

Let  $\mathcal{C}_0$  be a hyperbolic critical element and  $V_{\mathcal{C}_0}$  be a well-chosen neighborhood of  $\mathcal{C}_0$ ,

- $W^s(\mathcal{C}_0) = \{u_0 \in X_p^{2\alpha} \mid S_f(t)u_0 \rightarrow_{t \rightarrow +\infty} \mathcal{C}_0\}$  : **stable set**
- $W_{loc}^s(\mathcal{C}_0) = \{u_0 \in W^s(\mathcal{C}_0) \mid S_f(t)u_0 \in V_{\mathcal{C}_0}, \forall t \geq 0\}$ : **local stable set**
- $W^u(\mathcal{C}_0) = \{u_0 \in X_p^{2\alpha} \mid S_f(t)u_0 \text{ exists for } t \leq 0 \text{ and } S_f(t)u_0 \rightarrow_{t \rightarrow -\infty} \mathcal{C}_0\}$ : **unstable set**
- $W_{loc}^u(\mathcal{C}_0) = \{u_0 \in W^u(\mathcal{C}_0) \mid S_f(t)u_0 \in V_{\mathcal{C}_0}, \forall t \leq 0\}$  : **local unstable set**

$W_{loc}^s(\mathcal{C}_0)$  (resp.  $W^s(\mathcal{C}_0)$ ) is an (embedded) submanifold (resp. immersed manifold) of codimension  $i(\mathcal{C}_0)$ .

$W_{loc}^u(\mathcal{C}_0)$  is an (embedded) submanifold of dimension  $i(e_0)$  (resp.  $i(\Gamma_0) + 1$ ) if  $\mathcal{C}_0 = e_0$  (resp.  $\mathcal{C}_0 = \Gamma_0$ ).

$W^u(\mathcal{C}_0) = \cup_t S_f(t)W_{loc}^u(\mathcal{C}_0)$  is an immersed manifold of dimension  $i(e_0)$  or  $i(\Gamma_0) + 1$ .

## Basic definitions 3

1) A point  $u_0 \in Y$  is **non-wandering** if for any neighbourhood  $\mathcal{N}_{u_0}$  of  $u_0$  and any time  $t_0 > 0$ , there exists  $t \geq t_0$  such that  $S_f(t)\mathcal{N}_{u_0} \cap \mathcal{N}_{u_0} \neq \emptyset$  (non-wandering set  $\Omega(S_f)$ ).

2) For any  $u_0 \in Y$ , the  **$\omega$ -limit set**  $\omega(u_0)$  is defined by

$$\omega(u_0) = \{v \in Y \mid \exists t_n \rightarrow_{n \rightarrow +\infty} +\infty, \text{ s.t. } S_f(t_n)u_0 \rightarrow_{n \rightarrow +\infty} v\}.$$

3) The **Whitney topology** on  $C^r$  is generated by the neighbourhoods

$$\{g \in C^r(\bar{\Omega} \times \mathbf{R} \times \mathbf{R}^n, \mathbf{R}) \mid |D^i f(x, u, v) - D^i g(x, u, v)| \leq \delta(u, v), \\ \forall i, 1 \leq i \leq r, \forall (x, u, v) \in \bar{\Omega} \times \mathbf{R} \times \mathbf{R}^n \},$$

where  $f \in C^r(\bar{\Omega} \times \mathbf{R} \times \mathbf{R}^n, \mathbf{R})$  and  $\delta$  is any positive continuous function.

With this topology,  $C^r \equiv \mathcal{G}^r$  is a **Baire space**, i.e., **any countable intersection of open and dense sets** (i.e. **generic set**), is **dense** in  $\mathcal{G}^r$ .

## Part I: Back to the sixties (analogy with vector fields)

Let  $d \geq 1$ ,  $g \in C^1(\mathbf{R}^d, \mathbf{R}^d)$  be any vector field.  
We consider the ordinary differential equation

$$\begin{aligned}y_t(t) &= g(y(t)) , \quad t > 0 \\ y(0) &= y_0 \in \mathbf{R}^d .\end{aligned}\tag{2}$$

We assume that there exists  $M > 0$  large enough so that, for any  $y \in \mathbf{R}^d$  with  $\|y\| \geq M$ , we have  $\langle y | g(y) \rangle < 0$ .

Then (2) is a global dynamical system  $T_g(t)$  on  $\mathbf{R}^d$  and has a compact global attractor  $\mathcal{A}_g$ .

More generally, one could consider vector fields on a **compact manifold**  $\mathcal{M}^d$  of dimension  $d$ .

## Back to the sixties: Morse-Smale and Kupka-Smale

Andronov and Pontrjagin (1937)

Smale (1959, 1961)

Peixoto (1962) (genericity of M-S if  $\mathcal{M}^2$  compact, orientable)

Kupka, Smale (1963, genericity of K-S)

Smale (1965, Axiom A diffeomorphisms etc...)

Peixoto (1967)

Palis, Smale (1969,1970): Structural stability of M-S for any  $d$

Robbin (1979): Algebraic Kupka-Smale theory

### Summary:

1. If  $d = 1$ ,  $T_g(t)$  is a gradient system and is **Morse-Smale**, generically in  $g$  (automatic transversality of stable and unstable manifolds)
2. If  $d = 2$ ,  $T_g(t)$  has the **Poincaré-Bendixson** property and is **Morse-Smale**, generically in  $g$  (**Peixoto**).
3. If  $d > 2$ ,  $T_g(t)$  is **Kupka-Smale** generically in  $g$ . No density of Morse-Smale systems. **Particular case:**  $g = \nabla G$ .

## Earlier results for the PDE

- $n = 1, \Omega = (0, 1)$ :  $S_f(t)$  is a **gradient** system (Zelenyak, (1968))
- $n = 1, \Omega = (0, 1)$ :  $S_f(t)$  is **generically Morse-Smale** (D. Henry, (1985)).
- Particular case,  $n \geq 2, f(x, u, \nabla u) \equiv f(x, u)$ :  $S_f(t)$  is a **gradient** system and is **generically Morse-Smale** (Brunovský and Poláčik, (1997))  
Functional formulation of the transversality.
- Case  $n \geq 2, f(x, u, \nabla u) \equiv f(x, u)$ : the damped wave equation is **generically Morse-Smale** with respect to  $f(x, u)$ .  
(Brunovský and G.R. (2003), Joly ( $n = 1$ , local damping))
- **Structural stability** of Morse-Smale maps (in the book of Hale, Magalhães and Oliva, 1984).
- **"Realization of vectors fields"** in (1) (Poláčik, Dancer and Poláčik (1987-1989), etc.. )

## Part II: The case of $S^1$ - Morse-Smale Property

$$\begin{aligned} u_t(x, t) &= u_{xx}(x, t) + f(x, u, u_x), \forall (x, t) \in S^1 \times (0, +\infty), \\ u(x, 0) &= u_0 \in H^s(S^1) \end{aligned} \quad (3)$$

where  $s \in (3/2, 2)$ .

Theorem (Morse-Smale, (Joly-R, 2010) )

Assume that (H1) holds. There exists a **generic subset**  $\mathcal{O}_{MS}$  of  $\mathcal{G}^2$  such that, for any  $f$  in  $\mathcal{O}_{MS}$ ,  $S_f(t)$  is **Morse-Smale**, i.e.,

1) the non-wandering set is a **finite** set, which consists only of hyperbolic equilibria and hyperbolic periodic orbits.

1) if  $\mathcal{C}^-$  and  $\mathcal{C}^+$  are two (hyperbolic) critical elements, then  $W^u(\mathcal{C}^-) \cap W_{loc}^s(\mathcal{C}^+) = \emptyset$ .

### Non-trivial example of non-gradient Morse-Smale PDE

Particular case  $f(x, u, u_x) = f(u, u_x)$  (Fiedler, Rocha, Wolfrum)

Hypothesis (H1) is only used in Statement 1)

The proof consists in 4 steps

## The case of $S^1$ : Sturm properties

For  $\varphi \in C^1(S^1, \mathbf{R})$ , the **Sturm or zero number**  $z(\varphi)$  is the (even) number of strict sign changes of  $\varphi$ .

**Theorem (Sturm (1828); Angenent-Fiedler, Matano, Angenent, ...)**

1) Let  $T > 0$ ,  $a \in C^1(S^1 \times [0, T], \mathbf{R})$  and  $b \in C^0(S^1 \times [0, T], \mathbf{R})$ . Let  $v : S^1 \times (0, T) \rightarrow \mathbf{R}$  be a bounded non-trivial solution of

$$v_t = v_{xx} + a(x, t)v_x + b(x, t)v ,$$

*Then, the number of zeros of  $x \in S^1 \mapsto v(x, t)$  is finite, non-increasing in time  $t$  and strictly decreases at  $t = t_0$  if and only if  $x \mapsto v(x, t_0)$  has a multiple zero.*

2) *If  $u$  and  $w$  are two solutions of (3), then  $u_t$  and  $u - w$  satisfy the properties of Statement 1).*

**Consequence:** Special properties of eigenvalues of the period map  $\Pi(T_0, 0)$ .

## The case of $S^1$ : Morse-Smale property (Step 1)

In 2008, Czaja-Rocha have proved that, if  $\Gamma^\pm$  are two hyperbolic periodic orbits of (3), then  $W^u(\Gamma^-) \pitchfork W_{loc}^s(\Gamma^+)$ .

More generally, one proves:

### Proposition (Automatic results)

- 1) If  $e^-$  and  $e^+$  are two hyperbolic equilibria of (3) with *different Morse indices*, then  $W^u(e^-) \pitchfork W_{loc}^s(e^+)$ .
- 2) If  $\Gamma$  (resp.  $e$ ) is a hyperbolic periodic orbit (resp. equilibrium) of (3), then the unstable manifold of  $\Gamma$  (resp. of  $e$ ) *transversally intersects* the local stable manifold of  $e$  (resp. of  $\Gamma$ ).

### Ingredients of the proof:

- decay properties of the zero number
- a careful analysis of the asymptotics of the connecting orbits.

Notice that homoclinic orbits can exist (see Sanstede-Fiedler)

## The case of $S^1$ : Morse-Smale (steps 2 and 3)

### Proposition (Genericity of hyperbolicity)

There exists a **generic subset**  $\mathcal{O}_h$  of  $\mathcal{G}^2$  such that, for all  $f \in \mathcal{O}_h$ , all the equilibria and the periodic solutions of (3) are hyperbolic.

### Proposition (Generic non-existence of homoclinic orbits)

There exists a **generic subset**  $\mathcal{O}_{MS} \subset \mathcal{O}_h$  of  $\mathcal{G}^2$  such that, for any  $f \in \mathcal{O}_{MS}$ , there does not exist any connecting orbit  $u(t)$  of (3) between two equilibria  $e^\pm$  with same Morse index ( $\rightsquigarrow$  no homoclinic orbits).

### Ingredients of the proof of Proposition 3:

- Sard-Smale Theorem
- Functional characterization of transversality (as Brunovský-Polačik and Brunovský-R)
- zero-number of the difference of two solutions of (3)

The **previous 3 steps imply** that, for any  $f \in \mathcal{O}_{MS}$ ,  $S_f(t)$  is a **Kupka-Smale** system.

## The case of $S^1$ : Morse-Smale - back to Step 2

To prove the genericity of the hyperbolicity of the periodic orbits, one uses the **recursion argument** (on the period  $|T| \leq k$  and on the “norm of the periodic orbit  $\|p\|_{C^1(S^1)} \leq n$ ) of Peixoto (1966) (see also Mallet-Paret (1977)).

### Steps of the proof

1. Definition of the (Fredholm) operator  $\Phi(\cdot)$

$$\Phi(T, u_0, f) = S_f(T)u_0 - u_0.$$

If  $(T, u_0, f) \in \Phi^{-1}(0)$ , then  $p(t) = S_f(t)u_0$  is a periodic solution of (3) of period  $T$ .

2. The surjectivity of  $D_{t,u}\Phi(T, p(0), f)$  is equivalent to the simplicity (and hence to the hyperbolicity) of the periodic orbit.
3. To apply Sard-Smale theorem.
4.  $\rightsquigarrow$  To prove that, for any solution  $\varphi^* \neq 0$  of the adjoint equation  $(\Pi_{f,p}(T, 0))^* \varphi^* = \varphi^*$ , there exists  $g \in \mathcal{G}^2$  s. t.

$$\int_{S^1} \int_0^T g(x, p(x, s), p_x(x, s)) [(\Pi_{f,p}(T, s))^* \varphi^*](x) ds dx \neq 0 .$$

## The case of $S^1$ : Morse-Smale - back to Step 2

### Lemma

Let  $p(t)$  be a periodic solution of (3) of minimal period  $T > 0$ .  
The map

$$(x, t) \in S^1 \times [0, T) \mapsto (x, p(x, t), p_x(x, t))$$

is one to one.

Assume that there exist  $x_0, t_0 \in [0, T)$  and  $t_1 \in [0, T)$ ,  $t_0 \neq t_1$  such that

$$p(x_0, t_0) = p(x_0, t_1) \text{ and } p_x(x_0, t_0) = p_x(x_0, t_1).$$

The function  $v(x, t) = p(x, t + t_1 - t_0) - p(x, t)$  is a solution of

$$v_t(x, t) = v_{xx}(x, t) + b(x, t)v(x, t) + d(x, t)v_x(x, t),$$

Moreover,  $v(x_0, t_0) = 0$ ,  $v_x(x_0, t_0) = 0$  and  $v(x, t)$  does not vanish everywhere since  $|t_1 - t_0| < T$ .

$\rightsquigarrow$  the zero number  $z(v(t))$  drops strictly at  $t = t_0$ .

Since  $v(t)$  is periodic of period  $T_0$ , this leads to a contradiction.

## The case of $S^1$ : Morse-Smale (Last step)

Theorem (Poincaré-Bendixson, Fiedler-Mallet-Paret (1989))

*For any  $u_0 \in H^s(S^1)$ , the  $\omega$ -limit set  $\omega(u_0)$  of  $u_0$  satisfies exactly one of the following possibilities:*

- i) Either  $\omega(u_0)$  consists of precisely one periodic orbit,*
- ii) or the  $\alpha$ - and  $\omega$ -limit sets of any  $v \in \omega(u_0)$  consist only of equilibrium points.*

The proof uses the properties of the zero number

Proposition (non-wandering set)

*If  $f$  is a non-linearity such that  $S_f(t)$  satisfies the following properties:*

- there exists a compact global attractor  $\mathcal{A}_f$  for (3).*
  - All equilibria and periodic orbits are hyperbolic.*
  - There is no homoclinic orbit and all heteroclinic orbits are transversal,*
- then, the set of non-wandering elements consists in a finite number of equilibrium points and periodic orbits.*

Analogy with vector fields in  $\mathbf{R}^2$  (Use of Poincaré-Bendixson theorem)

## Part III: Case $n \geq 2$ , new aspects (compared to $S^1$ )

1. No Sturm-Liouville property - No “interesting” property of the zero number (negative result of Fusco and Lunel, 1997).
2. No Poincaré-Bendixson theorem.
3. Realisation of the ODE in the PDE. Persistent chaotic dynamics (PDE cannot be simpler than ODE)
4. Role of the “singular nodal set” and of the unique continuation theorems

Particular case of non-linearities  $f(x, u)$ :  $S_f(t)$  is gradient and generically Morse-Smale (Brunovský-Poláčik, 1997).

## Case $n \geq 2$ , genericity of Kupka-Smale

Theorem (Kupka-Smale (Brunovský-Joly-R, 2010))

There exists a *generic subset*  $\mathcal{O}_{KS}$  of  $\mathcal{G}^2$  such that, for any  $f$  in  $\mathcal{O}_{KS}$ ,  $S_f(t)$  is a *Kupka-Smale system*, that is,

- 1) all the critical elements (i.e. equilibria and periodic orbits) are *hyperbolic*,
- 2) if  $C^-$  and  $C^+$  are two (hyperbolic) critical elements, then

$$W^u(C^-) \cap W_{loc}^s(C^+) = \emptyset.$$

Generalization to Navier-Stokes equations on  $\mathbf{T}^n$

Ingredients of the proof:

1. Recursion argument of Peixoto (more difficult than for  $S^1$ )
2. Sard-Smale theorem
3. **Unique continuation** theorems and analysis of **singular nodal sets** (generalization of an argument of **Hardt-Simon (1989)**).
4. Generalization of the transversality arguments of Abraham-Robbin (1967)

## Non-vanishing condition

$$v_{tt} = \Delta v + D_u f(x, u, \nabla u)v + D_{\nabla u} f(x, u, \nabla u) \cdot \nabla v . \quad (4)$$

In the proof, we are led to the following type of problem:

- One obtains a finite-dimensional linear subspace  $\mathcal{L}^*$  of solutions of the adjoint equation of the linearized equation (4).
- For any  $\varphi^* \in \mathcal{L}^*$ , one needs to find  $g \in \mathcal{G}^2$  s.t.

$$\int_{t_0}^{t_1} \int_{\Omega} g(x, u(x, s), \nabla u(x, s)) \varphi^*(x, s) dx ds \neq 0 , \quad (5)$$

where either  $t_0 = 0$ ,  $t_1 = T$  or  $t_0 = -\infty$ ,  $t_1 = +\infty$ , and either,  $u(x, s) = p_0(x, s)$  is a periodic solution, or  $u(x, s)$  is a connecting orbit between critical elements.

## One-to-one property I

### Theorem (Case of periodic orbits)

Let  $f \in C^\infty(\Omega \times \mathbf{R} \times \mathbf{R}^n, \mathbf{R})$  and  $p_0(x, t)$  be a periodic solution of minimal period  $T_0$ . Then, there exists a *dense open set* of points  $(x_0, t_0) \in \Omega \times \mathbf{R}$  such that

$$(p_t(x_0, t_0), \nabla p_t(x_0, t_0)) \neq (0, 0) ,$$

$$(p(x_0, t_0), \nabla p(x_0, t_0)) \neq (p(x_0, t), \nabla p(x_0, t)) , \forall t \neq t_0 + mT_0.$$

- Argument of Hardt-Simon
- Unique continuation theorems, Escauriaza-Fernández (2003)...

## One-to-one property II

### Theorem (Case of connecting orbits)

Let  $f \in C^\infty(\Omega \times \mathbf{R} \times \mathbf{R}^n, \mathbf{R})$  and  $p^\pm(x, t)$  be periodic solutions of minimal periods  $T^\pm$ . Let  $u(t)$  be a global solution of (1) connecting  $p^-(t)$  to  $p^+(t)$ . Then there exists a **dense open set** of points  $(x_0, t_0) \in \Omega \times \mathbf{R}$  such that

$$(u_t(x_0, t_0), \nabla u_t(x_0, t_0)) \neq (0, 0)$$

$$(u(x_0, t_0), \nabla u(x_0, t_0)) \neq (u(x_0, t), \nabla u(x_0, t)) \quad \forall t \neq t_0$$

$$(u(x_0, t_0), \nabla u(x_0, t_0)) \neq (p^\pm(x_0, t), \nabla p^\pm(x_0, t)) \quad \forall t \in \mathbf{R}$$

- Argument of Hardt-Simon
- Unique continuation theorems, Escauriaza-Fernández (2003)...

## Singular nodal sets and unique continuation theorem

We study the linear parabolic equation with parameter in  $\Omega \subset \mathbf{R}^n$ ,

$$v_t = \Delta v + a(x, t, \tau)v + b(x, t, \tau) \cdot \nabla_x v, \quad (6)$$

### Theorem (Singular nodal set)

Let  $I$  and  $J$  be open intervals in  $\mathbf{R}$ . Let  $a \in C_b^\infty(\Omega \times I \times J, \mathbf{R})$  and  $b \in C_b^\infty(\Omega \times I \times J, \mathbf{R}^n)$  and  $v$  be a solution of (6). Assume that there no pair  $(t, \tau) \in I \times J$  such that  $v(\cdot, t, \tau) \equiv 0$ . Then,

1)  $M = \{(x, t, \tau) \in \Omega \times I \times J \mid v(x, t, \tau) = 0, \nabla_x v(x, t, \tau) = 0\}$  is contained in a countable union of manifolds of dimension  $n$ ,

- either parametrized by  $t, \tau$  and  $n - 2$  components of  $x$ ,
- or parametrized by  $\tau$  and  $n - 1$  components of  $x$ .

2) The set  $(TNS) = \{(x_0, t_0) \in \Omega \times I \mid \text{there does not exists } \tau \in J \text{ such that } (v(x_0, t_0, \tau), \nabla v(x_0, t_0, \tau)) = (0, 0)\}$  is **generic** in  $\Omega \times I$ .

- Argument of Hardt-Simon

- Unique continuation theorems, Escauriaza-Fernández (2003)...

## Sard-Smale Theorem

### Theorem (Sard-Smale)

Let  $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$  be three Banach manifolds,  $\Phi \in C^r(\mathcal{X} \times \mathcal{Y}; \mathcal{Z})$ ,  $r \geq 1$  and  $\xi \in \mathcal{Z}$ . Assume that:

1.  $\forall (x, y) \in \Phi^{-1}(\xi)$ ,  $D_x \Phi(x, y)$  is a Fredholm map of index  $< r$ ;
2.  $\forall (x, y) \in \Phi^{-1}(\xi)$ ,  $D\Phi(x, y) : T_x \mathcal{X} \times T_y \mathcal{Y} \rightarrow T_\xi \mathcal{Z}$  is surjective;
3. one of the following properties is satisfied:
  - (a)  $\mathcal{X}$  and  $\mathcal{Y}$  are separable metric spaces;
  - (b) the map  $(x, y) \in \Phi^{-1}(\xi) \mapsto y \in \mathcal{Y}$  is  $\sigma$ -proper, that is, there is a countable system of subsets  $V_n \subset \Phi^{-1}(\xi)$  such that  $\cup_n V_n = \Phi^{-1}(\xi)$  and for each  $n$  the map  $(x, y) \in V_n \mapsto y \in \mathcal{Y}$  is proper (ie any sequence  $(x_k, y_k) \in V_n$  such that  $y_k$  is convergent in  $\mathcal{Y}$ , has a convergent subsequence in  $V_n$ ).

Then, the set  $\{y \in \mathcal{Y} \mid \xi \text{ is a regular value of } \Phi(., y)\}$  is a countable intersection of open dense sets in  $\mathcal{Y}$ .