A New Theoretical Scheme for the Analysis of Equations with Memory

Vittorino Pata
Politecnico di Milano

Coauthors

V.V. Chepyzhov (RAS - Moscow)

M. Conti (Milano - Politecnico)

M. Fabrizio (Bologna)

S. Gatti (Modena)

C. Giorgi (Brescia)

M. Grasselli (Milano - Politecnico)

E.M. Marchini (Milano - Politecnico)

A. Miranville (Poitiers)

J.E. Muñoz Rivera (LNCC - Rio de Janeiro)

S. Zelik (Surrey)

- This talk will mainly refer to the recent papers:

- M. Fabrizio, C. Giorgi, VP:
  A new approach to equations with memory
  ARMA 2010

- M. Conti, E.M. Marchini, VP:
  Semilinear wave equations of viscoelasticity
  in the minimal state framework
  DCDS 2010

Equations with memory

Roughly speaking, an abstract evolution equation with memory has the following structure:

$$\partial_t w(t) = \mathcal{F}(w(t), w^t(\cdot)) \qquad t > 0 \qquad (\boldsymbol{A})$$

where

$$w^t(s) = w(t-s) \qquad s > 0$$

and $\mathcal{F}$ is some operator acting on $w(t)$ as well as on the *past values* of w up to the actual time t

The function w is supposed to be *known* for all $t \leq 0$, where need not solve the equation. Accordingly, the initial datum reads

$$w(t)_{|t\leq 0} = w_0(t)$$

where w_0 is a given function defined on the time interval $(-\infty, 0]$

The history approach

A way to circumvent the intrinsic difficulties of the problem (mainly due to the nonlocal character of the equation) is to rephrase ($\boldsymbol{A}$) as an ODE in some abstract space by introducing an auxiliary variable carrying the information on the past history of w, in order to exploit the powerful machinery of the theory of dynamical systems

This line was traced by C.M. Dafermos (1970) in the context of linear viscoelasticity, who proposed to view w^t as an additional variable ruled by its own equation, so translating ($\boldsymbol{A}$) into a differential system acting on an extended space accounting for the memory component

Within Dafermos history approach, nowadays quite popular in the literature, significant progresses in the analysis of equations with memory have been made in the last years

The minimal state approach

On the other hand, when dealing with $(\boldsymbol{A})$, what one can actually measure is the function $w(t)$ for $t \geq 0$. In particular, it might happen that two *different* initial past histories w_0 lead to the *same* $w(t)$

From the viewpoint of the dynamics, those two different initial past histories are by all means indistinguishable

This observation suggests that, rather than the past history, a different variable should be employed to describe the initial state of the system, satisfying a natural minimality property:

different initial states entail different evolutions $w(t)$

In which case, the knowledge of $w(t)$ for all $t \geq 0$ determines in a unique way the initial state of the problem, the only object that really influences the future dynamics

Main task

- Determine, if possible, what is the minimal state associated to $(\boldsymbol{A})$

Unfortunately, a universal strategy is not available, and the correct choice depends on the particular concrete realization of $(\boldsymbol{A})$

Nonetheless, for a large class of equations with memory where the memory contribution enters in the form of convolution integral a general scheme seems to be applicable

A concrete model arising from viscoelasticity

Let $\alpha > 0$, let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, and let $A = -\Delta$ be the Laplace-Dirichlet operator acting on $L^2(\Omega)$ with $\mathrm{dom}(A) = H^2(\Omega) \cap H^1_0(\Omega)$

For $t > 0$ we consider the equation in $u = u(\boldsymbol{x}, t) : \Omega \times \mathbb{R} \to \mathbb{R}$

$$\partial_{tt} u + A\Big[\alpha u - \int_0^\infty \mu(s) u(t-s) ds\Big] + g(u) = 0 \qquad (\boldsymbol{E})$$

Equation $(\boldsymbol{E})$ is supplemented with the initial conditions

$$u(0) = u_0 \qquad \partial_t u(0) = v_0 \qquad u(-s)_{|s>0} = \phi_0(s)$$

where u_0, v_0 and the function $\phi_0(s)$ are prescribed data

General assumptions

• The *memory kernel* $\mu : \mathbb{R}^+ = (0,\infty) \to [0,\infty)$ is AC, nonincreasing and summable with total mass in $(0,\alpha)$. For simplicity we take

$$\alpha > 1 \qquad \alpha - \int_0^\infty \mu(s)ds = 1$$

We also define

$$\tau_\infty = \sup\{s \in \mathbb{R}^+ : \mu(s) > 0\}$$

We speak of *infinite delay* if $\tau_\infty = \infty$ and *finite delay* if $\tau_\infty < \infty$

• The nonlinearity $g \in \mathcal{C}^2(\mathbb{R})$ fulfills the standard growth and dissipativity assumptions

$$|g''(u)| \leq c(1+|u|) \qquad \liminf_{|u|\to\infty} g(u)/u > -\lambda$$

where $\lambda > 0$ is the first eigenvalue of A (e.g. $g(u) = u^3 - u$)

The problem of initial conditions

The classical (history) approach to equations with memory requires the knowledge of the past history of u up to time $t = 0$ (i.e. the function ϕ_0) playing the role of an initial datum. In that case, calling

$$F_0(t) = \int_0^{\infty} \mu(t+s)\phi_0(s)ds$$

the problem reads

$$\partial_{tt}u + A\Big[\alpha u - \int_0^t \mu(s)u(t-s)ds - F_0\Big] + g(u) = 0$$

It is then clear that what really determines the dynamics (besides u_0 and v_0) is the function F_0 rather than ϕ_0

• *A trivial example*

Consider the memory kernel

$$\mu(s) = \mathrm{e}^{-s}$$

Then

$$F_0(t) = \mathrm{e}^{-t} \int_0^\infty \mathrm{e}^{-s} \phi_0(s) ds$$

i.e. there are infinitely many functions ϕ_0 yielding the same $F_0(t)$

$\Rightarrow$ Construct a new theory where F_0, and not ϕ_0, is the correct initial datum of the problem, for it contains all the information needed to capture the future dynamics

The state formulation (heuristic derivation)

For $t \geq 0$ and $\tau \geq 0$ introduce the *minimal state* variable

$$\xi^t(\tau) = \int_0^\infty \mu'(\tau + s)\Big[u(t-s) - u(t)\Big]ds$$

which (formally) fulfills the equation

$$\partial_t \xi^t(\tau) = \partial_\tau \xi^t(\tau) + \mu(\tau)\partial_t u(t)$$

with initial datum at $t = 0$

$$\xi^0(\tau) = \int_0^\infty \mu'(\tau + s)\Big[\phi_0(s) - u_0\Big]ds$$

Accordingly, equation ($\boldsymbol{E}$) turns into

$$\partial_{tt} u + A\Big[u + \int_0^\infty \xi(\tau)d\tau\Big] + g(u) = 0$$

The equation in the state framework

Defining the new memory kernel

$$\nu(\tau) = \begin{cases} 1/\mu(\tau) & \text{if } \tau < \tau_\infty \\ 0 & \text{if } \tau \geq \tau_\infty \end{cases}$$

we introduce the *state space*

$$\mathcal{S} = L^2_\nu(\mathbb{R}^+; H^1_0(\Omega))$$

along with the *extended state space*

$$\mathbb{S} = H^1_0(\Omega) \times L^2(\Omega) \times \mathcal{S}$$

We also consider the infinitesimal generator L of the semigroup of left translations on $\mathcal{S}$, that is

$$L\xi = \xi' \qquad \text{dom}(L) = \{\xi \in \mathcal{S} : \xi' \in \mathcal{S}\}$$

where the *prime* stands for distributional derivative

Then we view the original equation ($\boldsymbol{E}$) as the equation in $\mathbb{S}$ in the unknown variables $u = u(t)$ and $\xi = \xi^t(\tau)$

$$\begin{cases} \partial_{tt} u + A\Big[u + \int_0^\infty \xi(\tau)d\tau\Big] + g(u) = 0 \\ \partial_t \xi = L\xi + \mu \partial_t u \end{cases} \qquad (\boldsymbol{ES})$$

Thm 1. ($\boldsymbol{ES}$) generates a strongly continuous semigroup

$$S(t) : \mathbb{S} \to \mathbb{S}$$

Remark. The state variable ξ^t is *minimal* in the following sense: if $(u(t), \partial_t u(t), \xi^t)$ is a solution to ($\boldsymbol{ES}$) the knowledge of $u(t)$ for all $t \geq 0$ uniquely determines ξ^t

The equation in the memory framework

With an analogous heuristic derivation, the equation is translated into the equation in the unknown variables $u = u(t)$ and $\eta = \eta^t(s)$

$$\begin{cases} \partial_{tt} u + A\Big[u + \int_0^\infty \mu(s)\eta(s)ds\Big] + g(u) = 0 \\ \partial_t \eta = R\eta + \partial_t u \end{cases} \qquad (\boldsymbol{EM})$$

R is the infinitesimal generator of the semigroup of right translations on the *memory space*

$$\mathcal{M} = L^2_\mu(\mathbb{R}^+; H^1_0(\Omega))$$

Defining the *extended memory space*

$$\mathbb{M} = H^1_0(\Omega) \times L^2(\Omega) \times \mathcal{M}$$

we have

Thm 2. $(\boldsymbol{EM})$ generates a strongly continuous semigroup

$$M(t) : \mathbb{M} \to \mathbb{M}$$

Comparison between the two formulations

• *The map* Λ. Define the linear map $\Lambda : \mathcal{M} \to \mathcal{S}$ by the formula

$$(\Lambda\eta)(\tau) = -\int_0^\infty \mu'(\tau + s)\eta(s)ds$$

Accordingly, define $\mathbf{\Lambda} : \mathbb{M} \to \mathbb{S}$ as

$$\mathbf{\Lambda}(u, v, \eta) = (u, v, \Lambda\eta)$$

Lemma. $\mathbf{\Lambda}$ is continuous from $\mathbb{M}$ into $\mathbb{S}$ with unitary operator norm

The link between the history and the state formulations is detailed in the next result, which tells that the state approach is more general

Thm 3. For every $w \in \mathbb{M}$ the following equality holds:

$$S(t)\mathbf{\Lambda}w = \mathbf{\Lambda}M(t)w$$

Global attractors

Further hypotheses on μ are needed. The typical assumption is

$$\mu'(s) + \delta\mu(s) \leq 0$$

Remark. We are in presence of a *very weak* dissipation mechanism entirely contributed by the memory (further *instantaneous* dissipation terms such as $\partial_t u$ or $A\partial_t u$ would dramatically simplify the analysis)

Then we can prove:

Thm 4. The semigroup $S(t)$ on $\mathbb{S}$ has the global attractor $\mathfrak{A}$

Thm 5. The semigroup $M(t)$ on $\mathbb{M}$ has the global attractor $\mathfrak{A}'$

Asymptotic comparison

Thm 6. $\mathfrak{A} = \mathbf{\Lambda}\mathfrak{A}'$

Remark. In fact the proof of the more recent Thm 4 (2010) is obtained by *defining* $\mathfrak{A}$ as $\mathbf{\Lambda}\mathfrak{A}'$ and then showing that $\mathfrak{A}$ is the global attractor of $S(t)$ (using Thm 5 and the properties of the map $\mathbf{\Lambda}$)

To some extent Thm 6 says that the two theories entail the *same* asymptotic dynamics. However $M(t)$ satisfies a further property:

Thm 7. The restriction of $M(t)$ on the global attractor $\mathfrak{A}'$ is a bijection which can be extended to a strongly continuous group of operators on $\mathfrak{A}'$ through the standard formula

$$M(-t) = M^{-1}(t)$$

• The analogous result for $S(t)$ cannot be inferred from Thm 7 since the map $\mathbf{\Lambda}$ is not in general injective on $\mathbb{M}$

On the injectivity of $\mathbf{\Lambda}$

Lemma. $\mathbf{\Lambda}$ is injective on $\mathbb{M}$ if and only if whenever $\eta \in \mathcal{M}$ the following implication holds:

$$\int_0^\infty \mu(\tau + s)\eta(s)ds = 0 \quad \forall \tau \geq 0 \qquad \Rightarrow \qquad \eta = 0$$

- For the relevant case $\mu(s) = \mathrm{e}^{-s}$ this is false

Exploiting the *Titchmarsh Convolution Theorem* we can prove the injectivity in the finite delay case

Thm 8. If $\tau_\infty < \infty$ then $\mathbf{\Lambda}$ is injective on $\mathbb{M}$

Still, in the infinite delay case $\tau = \tau_\infty$ there are several examples of kernels for which $\mathbf{\Lambda}$ is not injective

The injectivity of $S(t)$ on $\mathfrak{A}$

$\Rightarrow$ The main question is then whether or not $\boldsymbol{\Lambda}$ is injective on $\mathfrak{A}'$

Remark. From the philosophical point of view, the injectivity of $S(t)$ on $\mathfrak{A}$ means that, although the state framework does not require the knowledge of a *specific* initial history, any solution appearing in the longterm is in fact generated by a *unique* past history

• For some kernels the answer is trivially positive, since $\boldsymbol{\Lambda}$ is injective on the *whole* space (e.g. in the finite delay case)

A quite immediate example when $\tau_\infty = \infty$ is given by the kernel

$$\mu(s) = \mathrm{e}^{-s^2}$$

Another (not so trivial) example is

$$\mu(s) = \frac{\mathrm{e}^{-s}}{s^\vartheta} \qquad \vartheta \in (0,1)$$

Even if we will not answer the question in full generality, we establish the backward uniqueness property on the attractor for a relatively large class of (non-compactly supported) memory kernels

Thm 9. Let there exists $\varepsilon > 0$ such that μ is the restriction of a holomorphic function in the complex domain

$$\{z \in \mathbb{C} : \ \Re z > -\varepsilon, \ |\Im z| < \varepsilon\}$$

Then $\mathbf{\Lambda}$ is injective on $\mathfrak{A}'$

• Perhaps the most relevant example is the finite sum of exponentials

$$\mu(s) = \sum_{\jmath=1}^{n} a_\jmath \mathrm{e}^{-b_\jmath s} \qquad a_\jmath, b_\jmath > 0$$

whose corresponding map $\mathbf{\Lambda}$ can be shown not to be injective on $\mathbb{M}$

• When $\tau_\infty = \infty$ the full answer is an open problem

THANK YOU