Non-dissipative regularization of first order balance models

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Plan

1. Motivation: inviscid regularizations
2. Lagrangian fluid dynamics
3. The generalized LSG equations
4. Global well-posedness

1. Inviscid Regularization I: Leray's velocity smoothing

$$\partial_t u + (\phi_\varepsilon * u) \cdot \nabla u + \nabla p = \nu \, \Delta u$$
$$\nabla \cdot u = 0$$

where ϕ_ε is a standard mollifier.

Leray (1934)

- The regularized equation has a unique global smooth solution
- As $\varepsilon \to 0$, a subsequence converges weakly to a weak solution of Navier–Stokes. (No uniqueness...)

Observations

- No advected vorticity
- No material frame invariance

Can we restore these properties?

2. Second grade fluids

Constitutive equations should be

- Observer objective
- Material frame invariant

(Noll, Truesdell, Rivlin, Erikson, ...)

Necessary and sufficient condition

Stress tensor depends only on deformation tensor

$$\operatorname{Def} u = \tfrac{1}{2}\left(\nabla u + \nabla u^T\right)$$

and its Jaumann derivative

$$\overset{\circ}{\operatorname{Def}}\, u = (\partial_t + u \cdot \nabla) \operatorname{Def} u + [\operatorname{Def} u, \operatorname{Rot} u] \qquad \text{where} \qquad \operatorname{Rot} u = \tfrac{1}{2}\left(\nabla u - \nabla u^T\right)$$

Equivalent viewpoints

- Build *differential fluids* using these ingredients
- Make Leray-filtered equations frame invariant (Guermond *et al.*, 2003)

3. Inviscid Regularization II: Vortex Methods

Ansatz in 2D

$$\omega(x,t) = \sum_{i=1}^{N} \gamma_i \, \delta(x - x_i(t)) \qquad u = K * \omega$$

Problems with unsmoothed equations

- Vector field is singular: does not generate a flow $\eta(x,t)$
- Can have collapse and singularities in finite time
- System of ODEs can become stiff: Not a good numerical scheme
- Sense of convergence is weak

Regularization (Chorin, 1973, …)

$$K^\alpha \equiv K * \phi^\alpha$$

with some scaled “blob function”

$$\phi^\alpha(x) = \frac{1}{\alpha^2} \, \phi\Big(\frac{x}{\alpha}\Big)$$

Note: The resulting scheme can be interpreted again as a PDE or PΨDE (Cottet, 1988)

4. Inviscid Regularization III: Euler-α

Arnold (1966), Ebin & Marsden (1970)

> *The motion of a perfect incompressible fluid is geodesic flow on the group of volume preserving diffeomorphisms with respect to the right-invariant L^2 metric.*

Regularization by changing the metric

(Holm, Marsden, Ratiu, 1997)

$$L = \frac{1}{2} \int_\Omega \left(|u|^2 + \alpha^2 \, |\nabla u|^2 \right) \mathrm{d}x$$

gives an advected vorticity of the form

$$\omega = (1 - \alpha^2 \, \Delta) \nabla \times u$$

Justification via "Lagrangian averaging"

Holm (1999), Marsden & Shkoller (2001)

5. What now?

Facts and Conjectures

- Regularization helps numerically
- Can be “structure preserving” if done right
- May constitute a physical model as well (turbulent fluctuation, modified material properties)

There is also the belief that geometric regularization is somehow ubiquitous or generic.

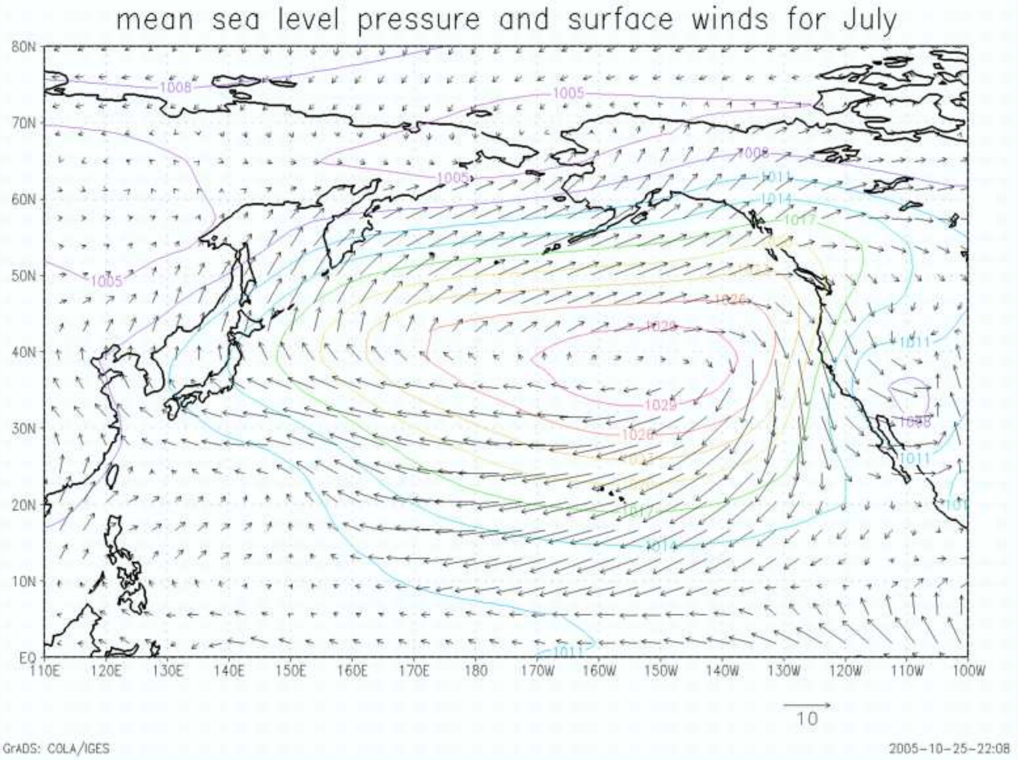
But:

- Few clear examples exist
- It’s difficult to get sign-definite energies
- Even vortex methods are not engineering mainstream

Program

- There is at least one “natural” example from shallow water theory
- There is progress in the analytical understanding
- Useful for backward error analysis for variational integrators?

6. Nearly geostrophic flow



(From: `geography.berkeley.edu/ProgramCourses/CoursePagesFA2005/Geog142/geog142_9-2.pdf\verb`)

7. Early history

- Salmon (1985): Variational asymptotics for nearly geostrophic shallow water
  Large-scale semigeostrophic (LSG) equations
- Salmon (1996): Same for stratified flow
- Shepherd and Ford (2000): Single thermally active layer
  depth invariant temperature (DIT) equations:

$$\partial_t \omega + J(\psi, \omega) = -J(\theta, \Delta\theta)$$
$$\partial_t \theta + J(\psi, \theta) = 0$$
$$\omega = \Delta\psi$$

where $J(\psi, \omega) = \nabla^{\perp}\psi \cdot \nabla\omega$.

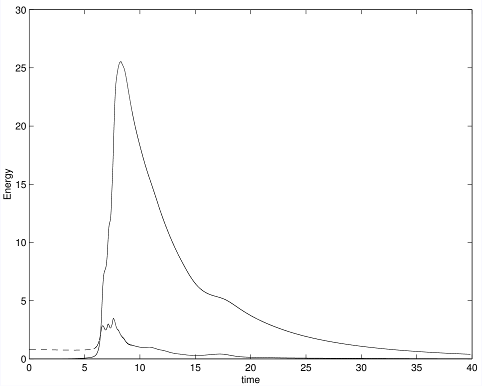
Observations:

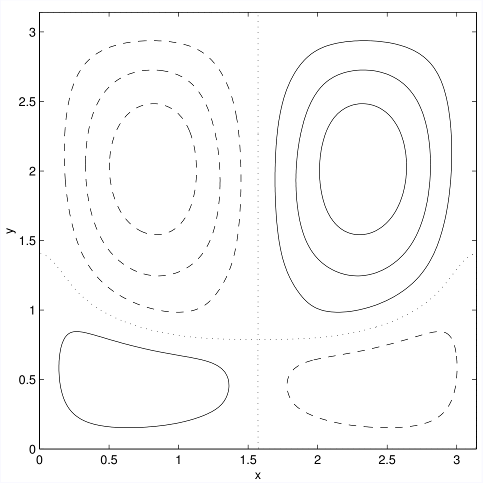
- Hamiltonian with conserved energy

$$E = E_{\mathrm{bt}} - E_{\mathrm{bc}} \equiv \frac{1}{2}\int |\nabla\psi|^2 \,\mathrm{d}x - \frac{1}{2}\int |\nabla\theta|^2 \,\mathrm{d}x$$

- Ill-posed

8. Simulation with hyperviscosity





Left:
"barotropic kinetic energy" (solid line) and
"baroclinic kinetic energy" (dashed line)

Right:
Vorticity (q at $t = 30$)

9. A new approach to variational asymptotics

DIT-α by higher order asymptotics?

- Does the "right thing" for shallow water (Ford, Malham, O., 2002):

$$H = \frac{1}{2} \int \left[h^2 - \varepsilon \, h \, |\nabla h|^2 + \varepsilon^2 \, h^2 \, |\text{Hess} \, h|^2 \right] \mathrm{d}x \, .$$

- *Still ill-posed*, definiteness is not the cure!

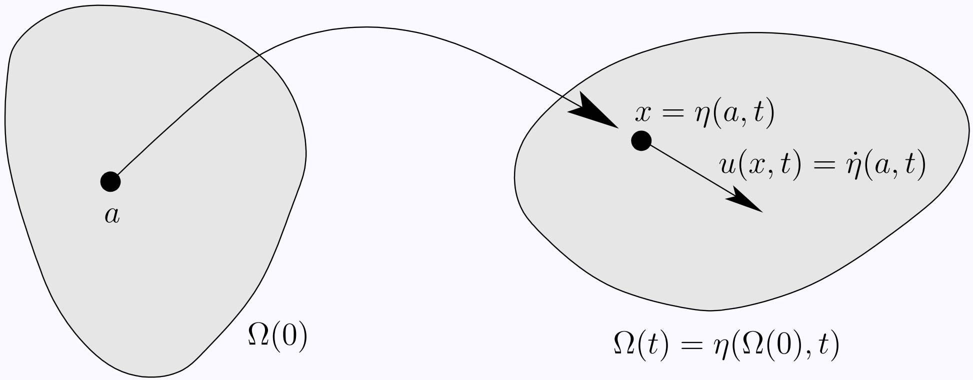
What *is* the Salmon approach?

- All reduced models arise from an *affine* Lagrangian
- They are related via asymptotically consistent changes of variables
- Observation: asymptotically consistent changes of variables can yield affine Lagrangians (when truncated to finite order)

Working hypothesis

Target structures should be *affine* (for semigeostropy) or *incompressible* (for quasigeostrophy).

10. Lagrangian fluid mechanics I: Notation



Ingredients

- Configuration space is group of diffeomorphisms ("flow maps") η
- Lagrangian $L = \frac{1}{2}\int_{\Omega(t)} |\dot{\eta}(a,t)|^2 \, \mathrm{d}a + E_{\text{pot}}$
- Incompressible flow: η is volume preserving; Shallow water: $h \circ \eta = \frac{1}{\det \nabla \eta}$
- Seek stationary points of Action $S = \int_{t_1}^{t_2} L \, \mathrm{d}t$ under variations of the flow map η

11. Lagrangian fluid mechanics II: Conservation laws

Writing the Lagrangian in the form

$$L = \int (F(h) + u) \circ \eta \cdot \dot{\eta} \, \mathrm{d}a - H \qquad \text{with} \qquad H = \frac{1}{2} \int (|\dot{\eta}|^2 + g(h) \circ \eta) \, \mathrm{d}a$$

we obtain

- Euler–Lagrange equations
- Conservation of energy: $\dot{H} = 0$
- Potential vorticity (PV) advection: $(\partial_t + u \cdot \nabla) q = 0$ where $q = \dfrac{\nabla^\perp \cdot F + \nabla^\perp \cdot u}{h}$

Affine Lagrangians

- Red terms disappear
- Euler–Lagrange equations become kinematic

12. Two approaches to constrained dynamics

Salmon, 1985

- Constrain the Hamilton principle
- Obtain affine Lagrangian
- Choose approximate convenient (canonical) coordinates

New approach

- Change into new coordinate system and expand L
- Fix transformation at each order to simplify system
- Obtain constraint

13. Degenerate variational asymptotics – the procedure

Step 1: Near identity transformation

Set up change of coordinates

$$\eta_\varepsilon = \eta \circ \xi_\varepsilon$$

Regard ξ_ε as flow in ε,

$$\xi_\varepsilon' = v_\varepsilon \circ \xi_\varepsilon \qquad \text{with} \qquad \xi_0 = \mathrm{id}$$

Plug into Lagrangian and expand:

$$L_\varepsilon = L_0 + \varepsilon\, L_1 + \tfrac{1}{2}\, \varepsilon^2\, L_2 + \dots$$

Step 2: degeneracy condition

Choose v, v', etc. such that $L_1, \dots, L_n$ become affine:

$$L_i = \int h\, (F_i(h) \cdot u + g_i(h))\, \mathrm{d}x$$

Now truncate at order n.

14. Model problem: semigeostrophic shallow water

$$\varepsilon\,(\partial_t u + u\cdot\nabla u) + u^\perp + \nabla h = 0$$
$$\partial_t h + \nabla\cdot(hu) = 0$$

Shallow water Lagrangian

$$L = \int \left[R\circ\eta\cdot\dot\eta + \tfrac12\,\varepsilon\,|\dot\eta|^2 - \tfrac12\,h\circ\eta \right] \mathrm{d}a$$

where

$$\nabla^\perp\cdot R = f \equiv 1 \qquad \dot\eta = u\circ\eta \qquad h\circ\eta = \frac{1}{\det\nabla\eta} \qquad \delta\eta = w\circ\eta$$

Variational derivation of the shallow water equations

$$\delta\iint R\circ\eta\cdot\dot\eta\,\mathrm{d}a\,\mathrm{d}t = -\iint \dot\eta^\perp\cdot\delta\eta\,\mathrm{d}a\,\mathrm{d}t = -\iint u^\perp\cdot w\,h\,\mathrm{d}x\,\mathrm{d}t$$
$$\tfrac12\,\varepsilon\,\delta\iint |\dot\eta|^2\,\mathrm{d}a\,\mathrm{d}t = -\varepsilon\iint \ddot\eta\cdot\delta\eta\,\mathrm{d}a\,\mathrm{d}t = -\iint \varepsilon\,(\dot u + u\cdot\nabla u)\cdot w\,h\,\mathrm{d}x\,\mathrm{d}t$$
$$-\tfrac12\,\delta\iint h\circ\eta\,\mathrm{d}a\,\mathrm{d}t = -\iint \nabla h\circ\eta\cdot\delta\eta\,\mathrm{d}a\,\mathrm{d}t = -\iint \nabla h\cdot w\,h\,\mathrm{d}x\,\mathrm{d}t$$

15. Semigeostrophic expansion of the Lagrangian

Write

$$L_\varepsilon = L_0 + \varepsilon\, L_1 + \tfrac{1}{2}\,\varepsilon^2\, L_2 + O(\varepsilon^3)$$

where

$$\begin{aligned}
L_0 &= \int \bigl[R \circ \eta \cdot \dot\eta - \tfrac{1}{2}\, h \circ \eta \bigr]\, \mathrm{d}a \\
L_1 &= \int \bigl[v^\perp \cdot u + \tfrac{1}{2}\, |u|^2 + \tfrac{1}{2}\, h\, \nabla \cdot v \bigr] \circ \eta\, \mathrm{d}a \\
L_2 &= \int \bigl[u \cdot (v' + \nabla v\, v)^\perp + (v^\perp + 2u) \cdot (\dot v + \nabla v\, u) \\
&\qquad + \tfrac{1}{2}\, h\, (\nabla \cdot v' + v \cdot \nabla\nabla \cdot v - (\nabla \cdot v)^2) \bigr] \circ \eta\, \mathrm{d}a
\end{aligned}$$

Goal: Make $L_1, L_2, \ldots$ affine
At first order, this leads to the choice

$$v = \tfrac{1}{2}\, u^\perp + \lambda\, \nabla h$$

16. Generalized L_1 dynamics

Affine L_1 Lagrangian:

$$L_1 = (\lambda + \tfrac{1}{2}) \int h \, \nabla^\perp h \cdot u \, \mathrm{d}x - \lambda \int h \, |\nabla h|^2 \, \mathrm{d}x$$

EL equations up to $O(\varepsilon)$:

$$u - \varepsilon \, (\lambda + \tfrac{1}{2}) \, (h \, \Delta u + 2 \, \nabla h \cdot \nabla u) = \nabla^\perp \big[h - \varepsilon \, \lambda \, (2 \, h \, \Delta h + |\nabla h|^2) \big]$$

Potential Vorticity:

$$q = \frac{1 + \varepsilon \, (\lambda + \tfrac{1}{2}) \, \Delta h}{h}$$

Special cases:

- $\lambda = \frac{1}{2}$: Salmon's L_1 dynamics
- $\lambda = -\frac{1}{2}$: Salmon's LSG, *ill-posed*
- $\lambda = 0$: Optimal regularity!

17. PV inversion when $\lambda = 0$

Potential Vorticity with $\lambda = 0$:

$$q = \frac{1 + \frac{\varepsilon}{2}\,\Delta h}{h}$$

Get elliptic equation for h:

$$\bigl(q - \tfrac{\varepsilon}{2}\,\Delta\bigr)h = 1$$

Solution procedure:

- Given q compute h: gain 2 derivatives
- From h use the EL equation to compute u: gain 1 derivative
- Use u to advect q

Claim:

Functional setting as for Lagrangian averaged Euler!

18. Generalized LSG equations

We consider the generalized LSG system on $\mathbb{T}^2$, for convenience $\sigma = \varepsilon(\lambda + \frac{1}{2})$ and $\varepsilon \equiv 1$:

$$\partial_t q + u \cdot \nabla q = 0$$
$$L_q h \equiv (q - \sigma\,\Delta)h = 1$$
$$\Lambda_h u \equiv (1 - \sigma\,(h\Delta + 2\nabla h \cdot \nabla))u = \nabla^{\perp}\big[h - \lambda\,(2\,h\,\Delta h + |\nabla h|^2)\big]$$

Hamiltonian structure and local classical solutions (O., Vasylkevych, 2010)

- The generalized LSG equations are Hamiltonian in an open neighborhood of the H^s class diffeomorphism group about the volume preserving H^s diffeomorphisms
- The Hamiltonian formulation holds true even if the Coriolis parameter is not the curl of a vector potential
- For $s > 2$ and every initial datum $q(0) = 1 + \tilde{q}(0)$ with $\tilde{q}(0)$ sufficiently small in H^s, there exists a unique local classical solution of class

$$q \in \bigcap_{k=0}^{m} C^k([0,T); H^{s-k}(D))$$

19. The road to global classical solutions

q is materially preserved

$$\min q(t) = \min q(0) \qquad \text{and} \qquad \max q(t) = \max q(0)$$

Note: $q(0) > 0$ is a physical restriction

Elliptic maximum principle

$$\min \frac{1}{q} \leq h \leq \max \frac{1}{q}$$

Weak formulation of Λ_h

Let

$$B(u,v) = \langle u, \Lambda_h v \rangle$$

Then

$$B(u,u) = \int \big(u \cdot u + \sigma\, h\, \nabla u : \nabla u - \tfrac{1}{2}\, \sigma \nabla h \cdot \nabla |u|^2 \big)\, \mathrm{d}x = \int \big(\tfrac{1}{2}\, (1 + hq)\, |u|^2 + \sigma\, h\, |\nabla u|^2 \big)$$

Conclusion: Coercivity provided $q(0) > 0$ uniformly in t.

20. Recall classical solutions for 2D Euler

Abstract vorticity formulation

$$\partial_t q + u \cdot \nabla q = 0$$
$$u = Kq$$

where K is a linear "Biot–Savart" operator.

H^s estimate

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|q\|_{H^s}^2 &\leq \left| \int \nabla \cdot u \, |\mathrm{D}^s q|^2 \, \mathrm{d}x \right| + \text{lots of other terms} \\ &\leq \|\nabla u\|_{L^\infty} \|q\|_{H^s}^2 + \text{lots of other terms} \end{aligned}$$

Observation: need

$$\|Kq\|_{W^{1,\infty}} \leq c \, \|q\|_{L^\infty} \left(1 + \ln_+ \|q\|_{H^s}\right)$$

This holds in 2D and 3D with $s = 2$, implying global classical solutions for 2D Euler and the Beale, Kato, and Majda (1984) criterion for 3D Euler.

21. The Yudovich trick

- Direct verification of the $W^{1,\infty}$ bound is hard when the integral kernel of K is not explicitly known.
- If K is of the general structure $u = \mathrm{D}\Delta^{-1} q$, elliptic L^p theory tells us that

$$\|u\|_{W^{1,p}} \leq c \, \frac{p}{p-1} \, \|q\|_{L^p}$$

  for $p \in (1, \infty)$.
- Use Gagliardo–Nirenberg interpolation

$$\|v\|_{L^\infty} \leq c(p) \, \|v\|_{H^{s-1}}^{\theta} \, \|v\|_{L^p}^{1-\theta}$$

  where $c(p) \to 1$ as $p \to \infty$ and

$$\theta = \frac{1}{p \left(\frac{s-1}{d} - \frac{1}{2} \right) + 1}$$

- Now optimize in p to obtain the desired estimate

$$\|u\|_{W^{1,\infty}} \leq c \, \|q\|_{L^\infty} \left(1 + \ln_+ \|q\|_{H^s} \right)$$

- *Problem:* For generalized LSG, $K(q)$ is nonlinear

22. q-h inversion

Write $q = 1 + \tilde{q}$, then $(q - \sigma\Delta)h = f$ can be written

$$h = (1 - \sigma\Delta)^{-1}(f - \tilde{q}h)$$

so that

$$\|h\|_{L^p} \leq \frac{1}{1 - \|\tilde{q}|_{L^\infty}} \|f\|_{L^p}$$

uniformly for $p \in [1, \infty]$. Moreover,

$$\|\nabla h\|_{L^p} \leq \|\nabla f\|_{L^p} + \|\nabla\tilde{q}\|_{L^p} \|h\|_{L^\infty} + \|\tilde{q}\|_{L^\infty} \|\nabla h\|_{L^p}$$

so that

$$\|\nabla h\|_{L^p} \leq c \left(\frac{1}{1 - \|\tilde{q}|_{L^\infty}}\right)^2 \left(\|\nabla f\|_{L^p} + \|\nabla\tilde{q}\|_{L^p}\right)$$

In general:

$$\|h\|_{W^{m,p}} \leq c \left(\frac{1}{1 - \|\tilde{q}|_{L^\infty}}\right)^{m+1} \left(\|f\|_{W^{1,p}} + \|\tilde{q}\|_{W^{1,p}}\right)$$

$$\|h\|_{W^{m+2,p}} \leq c \frac{p}{p-1} \left(\frac{1}{1 - \|\tilde{q}|_{L^\infty}}\right)^{m+1} \left(\|f\|_{W^{1,p}} + \|\tilde{q}\|_{W^{1,p}}\right)$$

23. h-u inversion

Write $h^{-1} \equiv b \equiv 1 + \tilde{b}$. Then

$$\Lambda_h u \equiv (1 - \sigma\,(h\Delta + 2\nabla h \cdot \nabla))u = g$$

can be written as

$$(b - \sigma\Delta)u = g + 2\sigma\,\nabla \ln h \cdot \nabla u$$

or

$$u = L_b^{-1}\,(g + 2\sigma\,\nabla \ln h \cdot \nabla u)$$

"Yudovich estimates"

$$\|u\|_{W^{1,p}} \leq \frac{C(q)}{\sigma}\,\frac{p}{p-1}\,\|f\|_{W^{-1,p}} \qquad \text{where} \qquad \|f\|_{W^{-1,p}} = \sup_{\phi \in W^{1,p'}} \frac{\langle \phi, f\rangle_{L^2}}{\|\phi\|_{W^{1,p'}}} \text{ with } \frac{1}{p} + \frac{1}{p'} = 1$$

and

$$\|u\|_{W^{m+1,p}} \leq C(q)\,c(p)\,\|q\|_{W^{m,p}}$$

where $C(q)$ may only depend on $\dfrac{1}{1 - \|\tilde{q}\|_{L^\infty}}$

24. Results

- For any $\lambda > -\frac{1}{2}$ and under the additional assumption that q initially satisfies the solvability condition, we obtain a global classical solution theorem as for $2D$ Euler.
- Note, in particular, that the solvability condition is a constant of the motion.
- Weak L^∞ solutions follow from the *a priori* estimates we already have.
- When $\lambda = 0$, it is likely that we obtain weak solutions in the space of strictly positive Radon measures. This is technically more challenging.
- Numerically, the contraction map formulation of the PV inversion provides a stable and rapidly converging formulation.
- Systematic numerical investigation of convergence rates vs. the full shallow water model are underway (joint work with GAG and DGD). Excellent results so far for the L_1 case ($\lambda = \frac{1}{2}$).