Attractors for dynamics with geometrically constrained dissipation and critical exponents.

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Outline

- PDE model. -nonlinear wave eq. with **critical** sources, **nonlinear geometrically constrained** dissipation and **acoustic boundary conditions**.
- Main Results: **decay rates** to equilibria and existence of **smooth** attractors.
- Discussion of the **geometric hypotheses** necessary for **propagation of the damping**.
- The role of acoustic boundary conditions and "heavy" Wentzell potential energy. Why purely boundary dissipation can not work?
- Main tools used in the proofs:
  - **Observability** -inverse type- inequality attesting quasi-stability of dynamical system -control theory enters
  - **Carleman's estimates** : handling criticality of nonlinear potential energy
  - **Geometric construction of new flux multipliers** -handling the lack of Lopatinski condition: Neumann and Wentzell boundary conditions.
- Conclusions and Open Questions

The model

Let $\Omega \in R^3$, bounded domain with boundary Γ.

Equation:

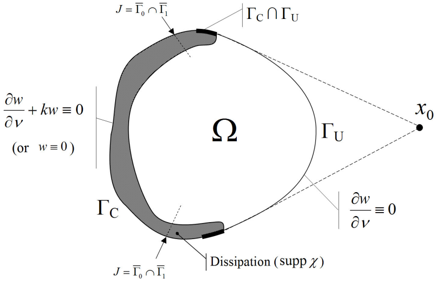
$$\Box u + a(x)g(u_t) + f(u) = 0 \tag{1}$$

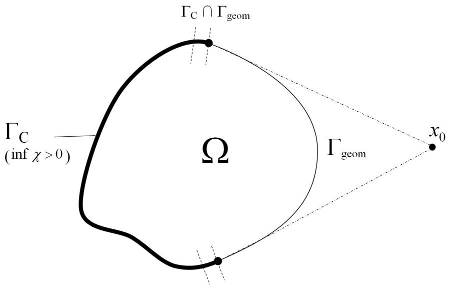
- **Nonlinear source** : $f(s)s \geq -\lambda|s|^2$, $|f'(s)| \leq M|s|^2, |s| \geq 1$.
- **nonlinear localized damping** : $g(s)$ monotone increasing,continuous . No growth conditions at the **origin**.

Wentzell Boundary conditions

$$\partial_\nu u + u = \alpha \Delta_T u, \quad on \quad \Gamma, \quad \alpha \geq 0$$

Initial Conditions: $u(0) = u_0(x) \in H^1(\Omega), u_t(0)(x) = u_1(x) \in L_2(\Omega)$


$J = \overline{\Gamma}_0 \cap \overline{\Gamma}_1$
$\Gamma_C \cap \Gamma_U$
$\frac{\partial w}{\partial \nu} + kw \equiv 0$
(or $w \equiv 0$)
Ω
Γ_U
x_0
Γ_C
$\frac{\partial w}{\partial \nu} \equiv 0$
Dissipation (supp χ)
$J = \overline{\Gamma}_0 \cap \overline{\Gamma}_1$


$\Gamma_C \cap \Gamma_{geom}$
Γ_C
$(\inf \chi > 0)$
Ω
Γ_{geom}
x_0

GOALS

Critical exponents + geometrically constrained and nonlinear damping + the lack of Lopatinski condition How much damping is needed in order to control long time behavior??

$$E(t) = ||\nabla u(t)||_\Omega^2 + |\nabla u(t)|_\Gamma^2 + ||u_t(t)||_\Omega^2$$

- Existence of compact attractors
- Attractors are smooth, finite-dimensional and exponential
- Decay of solutions to equilibria

with geometrically constrained damping

$$support \ a(x) \subset\subset \Omega$$

Previous work

- **Dirichlet boundary** conditions on undissipative part of the boundary and "star-shaped" constraints along the entire boundary.
  - **Stabilization** - $f(s)s \geq 0$: Chen, Russell, Littman, Lagnese, Triggiani, Komornik, Zuazua, Bardos, Lebeau, Rauch, Tataru...
  - **Existence of compact attractors** : Fereisel, Zuazua, Chueshov, Eller, IL,Kostin,
- **Neumann** conditions on **undissipative part** of the boundary and **no geometric constraints** on dissipative part of the boundary: **Lopatinski** condition does not hold :*stabilization* : Isakov, Tataru,Triggiani, I.L, X.Zhang, Yamamoto.
  - Microlocal analysis
  - Carleman's estimates

More recent work: Acoustic boundary conditons

- **Wentzell B.C.**

$$\partial_\nu u + u = -u_t + \Delta_T u, \quad on \ \ \Gamma$$

Heminna, Esaim 2000, *There is no uniform decay rates*. **NEGATIVE result**. It is easy to understand. Recovery of the finite energy requires observation of $u_t \in L_2(\Sigma)$ and $\partial_\nu u \in L_2(\Sigma)$. The dissipation gives the first, while

$$\partial_\nu u = L_2 + \Delta_T u \in L_2(H^{-1}(\Gamma))$$

- *supp* $a(x) = U(\partial\Gamma)$ -**full collar support**. Cavalcanti, Fukouka, Toundykov , JEE , 2010 -exponential decays.

Collar geometrically constrained

The interest is in

$$supp \;\; a(x) = U(\Gamma_1) \subset\subset U(\partial\Gamma)$$

$$E(t) = \int_\Omega [|\nabla u|^2 + |u_t|^2] dx + \int_\Gamma [|\nabla u|^2 + |u|^2] d\xi$$

The Difficulty: need to stabilize heavy part of the energy $H^1(\Gamma)$ by using dissipation from far away. Lopatinski does not hold

$$\int_{\Gamma_0} |\nabla u|^2, \quad from \quad \int_{\Gamma_1} a(x) g(u_t) dx$$

- far away from the source of the dissipation
- "heavy" -not bounded by standard potential energy of the wave.

Motivation: Acoustic boundary conditions.

Neumann uncontrolled boundary is BAD

$$\Box u = u_{tt} - \Delta u = f(u) \quad (2)$$

$$\partial_\nu u = 0, \quad \Gamma_0 \quad (3)$$

$$\partial_\nu u = -u_t, \ on \ \Gamma_1 \quad (4)$$

$E(t) = \int_\Omega |\nabla u|^2 + |u_t|^2$, **Goal** : $E(t) \leq Ce^{-\omega t}E(0)$
The difficulty : Multipliers (h) give rise to the boundary Lagrangian:

$$(h \cdot \nu)[|\nabla_T u|^2 - |u_t|^2] \ on \ \Gamma_0$$

Indefinite sign not controlled by the geometry. Dirichlet (Lopatinski case) has + Sign———

This difficulty for the Wave eq with Neumann uncontrolled B.C was handled in I.L., R. Triggiani, X. Zhang -Contemporary Mathematics, AMS, 2004.

Main result on stabilization: $f(s)s \leq 0$

Theorem

ASSUME the following geometric condition

- $(x - x_0)\nu \leq \Gamma_0$, *support* $a(x) \supset U(\Gamma_1)$
- Γ_0 *is* **convex**

ASSUME the following growth condition on g *continuous , monotone.*

$$m|s|^2 \leq g(s)s \leq M|s|^2, \quad |s| \geq 1$$

THEN: $E(t) \leq S(t) \to \infty,$

$$ODE: \quad S_t + h^{-1}(S) = 0, S(0) = E(0)$$

where $s^2 + g^2(s) \leq h(sg(s)), |s| \leq 1$

In the superlinear case $h^{-1}(t) = \sqrt{t}g(\sqrt{t})$,

In the sublinear case $h^{-1}(t) = \sqrt{t}g^{-1}(\sqrt{t})$

Decay Rates Driven by ODE

- Superlinear at the origin $g(s)$, $|g(s)| \sim |s|^p$

$$S_t + \sqrt{S} g(\sqrt{S}) = 0$$

$$E(t) \sim \frac{c}{t^{\frac{2}{p-1}}}$$

- Sublinear at the origin $g(s)$: $g(s) \sim s^{1/3}$.

$$S_t + \sqrt{S} g^{-1}(\sqrt{S}) = 0$$

Observability Inequality -Main Lemma

Gives a connection with controllability and Inverse problems.

Lemma

$$E(T) + \int_0^T E(t)dt$$

$$\leq C_T \int_0^T \int_\Omega a(x)g(u_t)[u_t + |u| + \nabla|u|] + a(x)u_t^2 dxdt$$

Energy reconstructed from the observation of the solution on the SUPPORT of $a(x)$ ONLY. **This leads to**

$$E(T) \leq \mathcal{H}(\int_0^T \int_\Omega a(x)g(u_t)u_t)$$

where $\mathcal{H}$ is concave and monotone determine from the dissipation g.

Interaction between Dissipation and the Energy

Energy

$$E_p = \int_\Omega |\nabla u|^2 + \int_\Gamma |\nabla u|^2, \quad E_k = \int_\Omega |u_t|^2$$

Dissipation

$$\int_{\Omega_1} |u_t|^2$$

TASKS:

- Propagate dissipation from Ω_1 into the full energy in Ω_1. (Partition of energy localised)
- Propagate reconstructed energy in Ω_1 into **the entire** Ω. Challenge!!!
- The main issue is that on Ω_0 **no information** on E_k.

How to handle- Multipliers identity

Let $h(x)$ be a smooth vector field.

- $w = \psi u$ with ψ supported near Γ_0 and zero near Γ_1 (no information)
- $v = \phi u$, ϕ supported in a collar near Γ_1 (information present).
- $\phi + \psi = 1$, in Ω
- **flux identity**

$$
\begin{aligned}
((\nabla w, [\nabla h - 1/2 \rho I] \nabla w))_\Omega + 1/2((\rho, w_t^2))_\Omega \\
= LOT(w, v) + \\
<< \partial_\nu w, 1/2(divh - \rho) w + h \nabla w >>_{\Gamma_0} \\
+1/2 << (h \cdot \nu), w_t^2 - |\nabla w|^2 >>_{\Gamma_0}
\end{aligned}
$$

- **equipartition**

$$((|\nabla v|^2 + v_t^2))_\Omega + << v^2 + |\nabla_T v|^2 >>_{\Gamma_1} = ((2, v_t^2))_\Omega + LOT(v, w) \quad (5$$

Comments

General strategy : Reconstruct the total energy in terms of the damping $((a(x), u_t^2))$ and $LOT(u, w)$.

- **reconstruction of** v **-equipartition** Since $v = \phi u$ supported in the area of the damping, Equipartition gives: $((v_t^2)) \leq c((a(x), v_t^2))$ and

$$\int_0^T E_v(t)dt \leq C((a(x), u_t^2)) + LOT(v)$$

- **reconstruction of** w **-flux idenity.** Requires the following three tasks:
  (1) reconstruction of potential interior energy $|\nabla w|^2$
  (2) reconstruction of boundary (Wentzell) energy $|\nabla_T w|^2$ on Γ_0
  (3) Getting rid of the unbounded and undetermined boundary terms.

Geometry enters

Reconstruction of the heavy Wentzell energy

Demands on the multiplier h :NEEDS

- $[\nabla h - 1/2\rho I] > c_0$ interior coercivity
- $< (h \cdot \nu), w_t^2 - |\nabla w|^2 >>_{\Gamma_0} = 0$
  $h \cdot \nu = 0$ on Γ_0 tangency on Γ_0
- Lift the energy from Wentzell part (coercivity on Γ_0) boundary coercivity

$$\int_{\Sigma_0} [\partial_\nu w 1/2(divh - \rho)w + h\nabla w]dx >> \geq c \int_{\Sigma_0} |\nabla_T w|^2$$

Then

$$\int_0^T E_w(t) \leq C((a(x), u_t^2)) + LOT(w, v)$$

When $\Gamma_0 = \emptyset$-dissipation in the full collar

- Heavy Wentzell boundary energy is encoded in v. No need to reconstruct on Γ_0
- Boundary terms with w are ZERO. (since ψ is supported away from $\Gamma_1 = \Gamma$)

$$<< \partial_\nu w, 1/2(divh-\rho)w + h\nabla w >>_\Gamma = << (h \cdot \nu), w_t^2 - |\nabla w|^2 >>_\Gamma \equiv 0$$

CONCLUSION: Standard radial $h(x) = x_0 - x$ will work. W. Strauss, C, Morawetz, Lagnese...

What if $\Gamma_0 \neq \emptyset$

Radial multipliers no longer work .
necessary condition : $h(x)$ **needs to be tangent to** Γ_0

Lemma

Let Γ_0, *defined by a level curve* $l(x)$, *be "convex".* $\Gamma_0 = \{x \in \Gamma, l(x) = 0\}$. *Assume the Hessian matrix* $H_l|_{\Gamma_0} \geq 0$. *Then, there exists* $\vec{h}(x)$ *such that*

1. $\vec{h} \cdot \nu = 0$ *on* Γ_0
2. *The Jacobian* $\nabla \vec{h} \geq \rho > 0$ *in* Ω.
3. $\int_{\Gamma_0}(-\partial_\nu w, 1/2(div\vec{h} - \rho)w + \vec{h} \cdot w)ds \geq \rho \int_{\Gamma_0}(|\nabla_T w|^2 + w^2)dx$

Construction -with D. Tataru

Vector field satisfying **1** and **2** was constructed in
I.L. R. Triggiani, X. Zhang, Contemporary Mathematics, AMS, 2002
NEEDS to secure the coercivity condition 3.

- Construct a parametric family $d_\lambda(x)$

$$d_\lambda(x) \equiv 1/2|x - x_0|^2 - (x - x_0)\nu \frac{l(x)}{|\nabla l(x)|} + \lambda l(x)^2$$

  where $\nu = \frac{\nabla l}{|\nabla l|}$.

$$d_\lambda(x) = 1/2|x - x_0|^2 - z(x) + \lambda l^2(x)$$

- Define $\vec{h}_\lambda(x) = \nabla d_\lambda(x)$

Properties of the vector field $\vec{h}_\lambda$.

- **Tangency condition** $\nabla d_\lambda \cdot \nu = 0, \ on\ \Gamma_0$.
  This relies on the cancellation with $z = pl, \ p = (1/2)(x - x_0)\nu \frac{1}{|\nabla l(x)|}$
  $\nabla z(x) \cdot \nu = \nabla (x - x_0)^2 \cdot \nu$
  and
  $\nabla l^2(x) \cdot \nu = 0, \ on\ \Gamma_0$
- $H_z = pH_l + \nabla l \oplus \nabla p + \nabla p \oplus \nabla l$
  $H_{l^2} = 2\nabla l \oplus \nabla l$
- This gives for large λ ($p \geq 0, H_l \geq 0, \ on\ \Gamma_0$) and
  $(H_z x, x) \geq -O(|\nabla|^2)|x|^2 - \epsilon |x|^2$ near Γ_0.
- **Interior coercivity condition** Hence $H_z + \lambda H_{l^2} \geq -\epsilon |x|^2$ for large λ.

Continue

Key calculations for **Coercivity on the boundary**.
Let D be the Riemanian (Levi-Civita) connection on the boundary manifold. Let $\vec{h}$ be tangential.

$$\nabla_T w \cdot \nabla_T(\vec{h} \cdot \nabla_T w) = (D_{\nabla_T w}\vec{h}) \cdot \nabla_T w + h \cdot D_{\nabla_T w} \nabla_T w$$

- $$\nabla_T w \cdot [\nabla_T \vec{h}] \nabla_T w \geq \rho |\nabla_T w|^2, \ \textit{in geodesic frame}$$
- $$\frac{d}{d\lambda} D_\nu(h_3(\lambda)) = 2|\nabla l|^2 > 0$$
- $div \vec{h} - div_T \vec{h} = D_\nu h_3 \geq \rho$
- $1/2(div \vec{h} - \rho)|_{\Gamma_0} \geq \rho > 0$
- $\{|\nabla_T h|^2 + (1/2)(div \vec{h} - div_T \vec{h} - \rho) I\} \nabla_T w \cdot \nabla_T w \geq \rho |\nabla_T w|^2$ on Γ_0.

Unique Continuation

The final step is absorbtion of lower order terms by compactness-uniqueness argument;

Lemma

$$Lot(v, w) \leq C \int_0^T \int_{\Omega_1} a(x) u_t^2 \, dx dt$$

This gives

$$E(t) \leq \mathcal{H}\left(\int_0^T \int_{\Omega_1} a(x) u_t^2 \, dx dt\right)$$

$\mathcal{H}^{-1}(t) = \sqrt{t} g(\sqrt{t})$.

$f(s)s \geq -\lambda|s|^2, |s| > 1$ and critical

Theorem

Attractors *Under the geometric condition of the previous theorem,*

1. *For all weak solutions there exists a global attractor* $\mathcal{A} \in H^1(\Omega) \times L_2(\Omega)$
2. *Each trajectory stabilizes to an equilibrium at the rate specified by the previous Theorem.*
3. *Assuming additionally that* $g'(0) > 0$ *, the said attractor is smooth ie* $\mathcal{A} \in H^2(\Omega) \times H^1(\Omega)$ *and* **finite dimensional**.
4. *If* $g;(0) > 0$*,* $f \in C^\infty, g \in C^\infty$*, then* $\mathcal{A} \in C^\infty(\Omega) \times C^\infty(\Omega)$.
5. *Under the above conditions, there exists an exponential attractor.*

Defined by the following **Observability Inequality**

$$\int_0^T \tilde{E}(u-v) \leq C_T \mathcal{H}(\int_0^T \int_\Omega |a(x)[g(u_t)-g(v_t)][u_t-v_t]dxdt)+LOT(u-v)$$

where $\mathcal{H}(s)$ is a concave, continuous increasing function determined from

$$s^2 \leq H(sg(s)), |s| \leq 1$$

.

$$LOT(u-v)$$

denote terms of lower order with respect to the energy.

Attractors

Theorem

Quasistable systems admit compact attractors. All weak trajectories stabilize to equilibria at the rate defined by ODE

$$S_t + \mathcal{H}^{-1}(S(t)) = 0,$$

$\mathcal{H}^{-1}(s) \sim \sqrt{s}g(\sqrt{s})$.

Theorem

If $LOT(u-v)$ is quadratic and $\mathcal{H}(s) = as$ (guaranteed by $g'(0) > m \geq 0$), then the attractor is finite dimensional and smooth.

I. Chueshov, I.L. *Memoires AMS 2008.* , D. Tataru, IL.

General theory: A. Babin, A.Haraux, J. Hale, O. Ladyzhenskaya, G. Raugel, G. Sell, R. Temam, A. Eden, Efendyev, A. Miranville, S. Zelik.P. Drabek and others. ,

Remarks

- Quasistability estimate motivated by **Control theory** (Inverse problems). The value of T must be sufficiently large (finite speed of propagation), but finite.
- Main PDE task is to prove **quasistability estimate**. Geometric condition allowing propagation will be used in the process.
- **Obstacles**
  - Critcality of the source and nonlinearity of the damping,
  - Support of the damping is constrained
  - Need for new flux multipliers-propagators
  - Need for Carleman's estimates with large parameter..

Criticality of the source -kinetic energy

Needs the folllowing estimate

$$\int_s^T (f(u) - f(v), u_t - v_t)dt \leq \epsilon \int_s^T E(u-v)dy + C_\epsilon LOT(u-v)$$

Due to criticality one can not rescale ϵ.
REMEDY: Backward regularity on the attractor for $T \to -\infty$ taking advantage of

- Trajectories are smooth when $t \to -\infty$
- Propagate forward
- $\mathcal{A} \subset H^2 \times H^1$ algebraically
- Topological bound due to the compactness

A. Kahmmendov, S. Zelik , I. Chueshov, D. Toundykov.

Critiality-potential energy recovery =Carleman

Needs the folllowing estimate

$$\int_s^T (f(u) - f(v), \vec{h}\nabla(u-v))_\Omega dt \leq \epsilon \int_s^T E(u-v)dt + C_\epsilon LOT(u-v)$$

Criticality of the first term.
Geometrically constrained damping forces the use of "critical" multipliers -$\vec{h}\nabla w$.
Carleman's estimates use weights $e^{\phi(t,x)\tau}$ for τ large with a suitable pseudoconvex function. This makes the energy level terms scaled like τ^{-1}. The ϵ term is obtained from τ^{-1}.

How to prove ?

Difficulty: D_Ω geometrically constrained damping.

$$\textit{support} \quad D_\Omega \subset\subset \Omega_0 \subset \Omega$$

Geometric propagation of the damping.

Tools:

- Observability (inverse) estimates from control theory.
- Geometric Multipliers $\vec{h} \cdot \nabla$.
- Carleman's estimates $\vec{h} \cdot \nabla e^{\tau\phi(x,t)}$
- Microlocal Analysis
- Unique Continuation

Role of Carleman's estimates

After some work with geometric multipliers $M(y)$ (of the energy level) .

$$\int_0^T E_y(t)dt \leq H(\int_0^T (D_\Omega(y_t), y_t)_{\mathcal{H}} dt) + \int_0^T (F(y), M(y))_{\mathcal{H}} dt + LOT(y)$$

$F(y(t))M(y(t)) \leq CE(t)$ is of the **energy level**. $y = u - v$???
Carleman's estimates enter the game: .
Let $P(u)$ be a differential operator of order m. Let ϕ be a psedoconvex function [L. Hormander] and let τ be a large parameter.
Classical Carleman's estimate

$$\tau |e^{\tau \phi} u|_{m-1,Q} \leq C |e^{\tau \phi} Pu|$$

for u **compactly supported** in $Q \in R^{n+1}$

$$e^{\tau \phi} Pu = P e^{\tau \phi} u + [P, e^{\tau \phi}] u$$

Psedoconvexity assures that the commutator generates the H^{m-1} norms.
L.Hormander - elliptic PDO with smooth coefficients

Carleman's estimates with boundary terms

D. Tataru -1993 developed this estimate for functions with support on the boundary.

$$\tau |ue^{\tau\phi}|_{m-1,Q} \leq C|e^{\tau\phi}Pu| + |BT(u)|_{m-1,\partial Q}$$

Became a **new tool** for the study of

- perturbation theory ,
- unique continuation in hyperbolic like PDE's,
- exact controllability,
- inverse problems,
- existence of attractors.

D. Tataru, Unique Continuation for solutions to PDE, between **Holmgren and Hormander**, - Comm PDE 1995.
Belishev, Uhlmann, Littman, Isakov, Yamamoto, Nakamura.

Role of Carleman's estimate-continue

- Unique Continuation and Inverse problems::

$$P(x,t,D_x,D_t)u = 0, \;\; u = 0 \; on \; \partial Q_0 \subset\subset \partial Q \Rightarrow u \equiv 0 \; in \; Q?$$

- Absorbtion of critical terms in the estimates. Taking advantage of the **"large parameter" τ.**
- Absorbtion of Lower Order Terms

$$Lot(y) \leq CH(\int_0^T (D_\Omega(y_t), y_t)_{\mathcal{H}})$$

via compactness-uniqueness argument .

Conclusions

- Dynamical system is quasistable
- Uniform decay rates for the energy not possible with boundary dissipation.
- Dissipation located in a small collar near the boundary requires the following geometric properties
  - star shaped on Γ_0 :
  - convexity of the level set function on Γ_0
- Key element of the proof:
  - NEW flux multiplier that is "flat" on Γ_0 -capturing the geometry.
  - Carleman's estimates

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