Incompressible flows that promote the creation of hot spots

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Drift independent L^∞ bounds.

- Let $\Omega \subset \mathbb{R}^d$ be a domain, and $\varphi \in C^2(\Omega) \cap C(\bar{\Omega})$ be a solution of

$$\begin{cases} -\triangle\varphi + u \cdot \nabla\varphi = f & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \end{cases}$$

- ABP Maximum principle: $\|\varphi\|_{L^\infty(\Omega)} \leqslant c\|f^+\|_{L^d(\Omega)}$, where

$$c = c(d, \Omega, u) = \operatorname{diam}(\Omega) \exp\left(\frac{2^{d-2}}{\omega_d d^d}\left(\|u\|_{L^d(\Gamma^+)} + 1\right) - 1\right)$$

- If $\nabla \cdot u = 0$, there exists $c = c(p, d, \Omega)$, *independent of* u such that

$$\|\varphi\|_{L^\infty(\Omega)} \leqslant c\|f\|_{L^p(\Omega)}$$

  if $f \in L^p(\Omega)$, $p > \frac{d}{2}$. (Berestycki, Kiselev, Ryzhik, Novikov)

- What is the best constant c?
- More importantly, for a given f, is there an incompressible drift u for which the above constant is attained?
- If yes, such flows u should promote the creation of hot spots.
- Existence of such *incompressible* flows is surprising.
- We can prove very little.

Incompressible flows usually ‘help’ mixing.

- If $\nabla \cdot u = 0$, principal eigenvalue of $L^u = (u \cdot \nabla) - \triangle$ (with Dirichlet B.C.) is larger than that of $L^0 = -\triangle$.

  Follows immediately by minimising the Raleigh quotient: If $\phi \in H^1_0(\Omega)$ and $L^u \phi = \lambda^u \phi$, with $\int_\Omega \phi^2 = 1$ then

  $$\lambda^u \int_\Omega \phi^2 = \int_\Omega \phi L^u \phi = \int_\Omega |\nabla \phi|^2 \geqslant \lambda^0$$

- Consequently, solutions to $\partial_t \theta + u \cdot \nabla \theta - \triangle \theta = 0$ approach the equilibrium state faster than solutions to $\partial_t \theta - \triangle \theta = 0$ with same initial data.

- In the periodic setting, the *effective diffusivity* in the presence of an incompressible drift is larger than the diffusivity without. (Fannjiang, Papanicolaou ’94).

The explosion problem.

- Explosion problem: (E) $$\begin{cases} -\triangle\phi = \lambda e^{\phi} & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$
- (E) is the steady state of (H) $$\begin{cases} \partial_t\varphi - \triangle\varphi = \lambda e^{\varphi} & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$
- 'Hot spots' may prevent global existence to (H).
- (H) has global (positive) solutions $\iff$ (E) has a minimal positive solution.
- There exists an *explosion threshold* λ_*:
  - For all $\lambda < \lambda_*$, eqn (E) has a minimal positive solution.
  - For all $\lambda > \lambda_*$, eqn (E) has no positive solutions.
  - Joseph, Lundgreen '72/73; Keener, H. Keller '74; Crandall, Rabinowitz '75.

The explosion problem with an incompressible flow.

- For a given, divergence free u, the explosion problem becomes

$$\begin{aligned} -\triangle\phi + (u\cdot\nabla)\phi &= \lambda e^{\phi} && \text{in } \Omega,\\ \phi &= 0 && \text{on } \partial\Omega. \end{aligned}$$

- Berestycki, Kiselev, Novikov, Ryzhik '09: There exists an explosion threshold $\lambda_* = \lambda_*(u)$.
- Further, $\inf\limits_{u} \lambda_*(u) = \lambda_0 > 0$.
- *(Open)* What is λ_0? Is it $\lambda_*(u)$ for some u? What is this u?
  - Natural guess: $\lambda_0 = \lambda_*(0)$.
  - Intuition: Incompressible stirring 'should' help mixing, and cool hot spots, and so *increase* the explosion threshold.
  - This is known to be *false*. Numerical example in a long rectangle. (Berestycki, Kagan, Joulin, Sivashinsky '97)

The exit time problem.

- Can address a few, similar in spirit, questions for the exit time problem.
- Let $\nabla \cdot u = 0$ and consider $\begin{cases} -\triangle \tau^u + u \cdot \nabla \tau^u = 1 & \text{in } \Omega, \\ \tau^u = 0 & \text{on } \partial\Omega \end{cases}$
- τ^u is the expected exit time of the diffusion
$$dX_t = u(X_t)\,dt + \sqrt{2}\,dW_t$$
from the domain Ω.

Main problem: Under the constraints

$$\nabla \cdot u = 0 \quad \text{and} \quad u \cdot \hat{n} = 0 \text{ on } \partial\Omega,$$

what drift maximizes τ^u in some sense.

- Such flows ‘promote hot spots’, or ‘mix badly’.

A few remarks

- Without the divergence free constraint, can make τ^u arbitrarily large by a strong inward stirring.
- Using fast incompressible cellular flows, we can always make τ^u arbitrarily small.
- If incompressible stirring only 'helps' mixing, then $u \equiv 0$ should produce the largest τ^u.
- Surprisingly(?) this is false.

Theorem. *Let* $\Omega \subset \mathbb{R}^2$ *be nice*[1] *Then* $u \equiv 0$ *maximises* $\|\tau^u\|_{L^\infty}$ *if and only if* Ω *is a disk.*

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[1] Nice = Bounded, simply connected and Lipschitz

Exit times in a disk.

In a disk, no incompressible stirring can ever increase the expected exit time. More generally, we have:

Proposition. *Let $\Omega \subset \mathbb{R}^d$ be nice, and v be any divergence free vector field which is tangential on $\partial\Omega$. Then*

$$\|\tau^v\|_{L^p(\Omega)} \leqslant \left\|\tau^{0,B}\right\|_{L^p(B)}$$

where $B \subset \mathbb{R}^n$ is a ball with $|B| = |\Omega|$, and $\tau^{0,B}$ is the expected exit time from B with 0 drift.

Proof

- Given any $\tau = \tau^v$, consider the symmetric rearrangement τ^*:
  - B is a ball with $|B| = |\Omega|$, and $\tau^* : B \to \mathbb{R}^+$ is radial.
  - For all h, $|\{\tau > h\}| = |\{\tau^* > h\}|$.
- Let $\Omega_h = \{\tau > h\}$, $\Omega_h^* = \{\tau^* > h\}$. Then
$$\int_{\partial\Omega_h^*} |\nabla\tau^*|\,d\sigma \int_{\partial\Omega_h^*} \frac{1}{|\nabla\tau^*|}\,d\sigma = |\partial\Omega_h^*|^2 \leqslant |\partial\Omega_h|^2 \leqslant \int_{\partial\Omega_h} |\nabla\tau|\,d\sigma \int_{\partial\Omega_h} \frac{1}{|\nabla\tau|}\,d\sigma.$$
- Co-area implies $\int_{\partial\Omega_h} \frac{1}{|\nabla\tau|}\,d\sigma = -\frac{d}{dh}|\Omega_h| = -\frac{d}{dh}|\Omega_h^*| = \int_{\partial\Omega_h^*} \frac{1}{|\nabla\tau^*|}\,d\sigma$
- $\implies \displaystyle\int_{\partial\Omega_h^*} |\nabla\tau^*|\,d\sigma \leqslant \int_{\partial\Omega_h} |\nabla\tau|\,d\sigma = |\Omega_h^*|$.
- Since τ^* is radial $\implies \frac{d\tau^*}{dr} \leqslant \frac{|\Omega_r^*|}{|\partial\Omega_r^*|} \implies$ QED.

Increasing the exit times for non-circular domains.

Consider 'infinite amplitude' flows:

- For $A \in \mathbb{R}$, $\nabla \cdot u = 0$, let τ^{Au} solve

$$\begin{aligned} -\triangle \tau^{Au} + Au \cdot \nabla \tau^{Au} &= 1 && \text{in } \Omega, \\ \tau^{Au} &= 0 && \text{on } \partial\Omega \end{aligned}$$

- Let $\bar{\tau}^u \overset{\text{def}}{=} \lim_{A \to \infty} \tau^{Au}$ (convergence is uniform in Ω).
- The limit $\bar{\tau}^u$ satisfies the *Freidlin problem.*
- If $u = \nabla^\perp \psi \overset{\text{def}}{=} \begin{pmatrix} -\partial_2 \psi \\ \partial_1 \psi \end{pmatrix}$, and ψ has 'one hill', then $\bar{\tau}^u$ is given explicitly by

$$\bar{\tau}^u(y) \overset{\text{def}}{=} \lim_{A \to \infty} \tau^{Au}(y) = -\int_0^{\psi(y)} \frac{|\Omega_{\psi,h}|}{\int_{\Omega_{\psi,h}} \triangle \psi \, dx} \, dh$$

  where $\Omega_{\psi,h} = \{x \mid \psi(x) > h\}$.

- If D is not a disk, we will show that there is some ψ such that for $u = \nabla^\perp \psi$, $\|\bar{\tau}^u\|_{L^\infty} > \|\tau^0\|_{L^\infty}$.
    - Will of course imply that for large A, $\|\tau^{Au}\|_{L^\infty} > \|\tau^0\|_{L^\infty}$.

Main idea

- Let $I(\psi) = \|\bar{\tau}^{\nabla^\perp \psi}\|_{L^\infty}$.
- Set up a variational principle for 'one hill' stream functions using the explicit solution of the Freidlin problem.
- Show τ^0 is not a critical point.
- If τ^0 is not a critical point of the said variational principle, then for some u, large A,

$$\left\|\tau^{Au}\right\|_{L^\infty} > \|\bar{\tau}^u\|_{L^\infty} - \varepsilon > \left\|\bar{\tau}^{\nabla^\perp \tau^0}\right\|_{L^\infty} = \left\|\tau^0\right\|_{L^\infty}.$$

Localization

If τ^0 doesn't have 'one hill', can reduce to the one hill case as follows.

Lemma 1. *Assume that for all C^3 incompressible vector fields u tangential to $\partial\Omega$, we have $\|\tau^0\|_{L^\infty(\Omega)} \geqslant \|\bar\tau^u_\Omega\|_{L^\infty(\Omega)}$. Then for any C^3 incompressible vector field w tangential to $\partial\Omega_{h_0}$, we have*

$$\left\|\tau^0_{\Omega_{h_0}}\right\|_{L^\infty(\Omega_{h_0})} = \|\tau^0 - h_0\|_{L^\infty(\Omega_{h_0})} \geqslant \left\|\bar\tau^w_{\Omega_{h_0}}\right\|_{L^\infty(\Omega_{h_0})}.$$

Lemma 2. *The set of maxima of τ^0 are discrete.*

Thus, choosing h_0 large enough, $\tau^0\big|_{\Omega_{h_0}}$ will only have one hill. Further, by analiticity of τ^0, Ω_{h_0} is a disk if and only if Ω is a disk.

The variational principle

- Let $v : \Omega \to \mathbb{R}^2$ be smooth (not necessarily divergence free), with $v \cdot \hat{n} = 0$ on $\partial\Omega$. *('Direction' of the variation.)*

- Let $\dfrac{dX_\varepsilon}{d\varepsilon} = v(X_\varepsilon)$ with $X_0(x) = x$.

- Compute $V(\psi, v) = \dfrac{d}{d\varepsilon}\Big|_{\varepsilon=0} I(\psi \circ X_\varepsilon)$.

- Some suffering shows $V(\psi, v) = 0$ for all v if and only if

$$-2\triangle\phi(x) = 1 + |\nabla\phi(x)|^2 \left(\int_{\{\phi=\phi(x)\}} \frac{d\sigma}{|\nabla\phi|}\right) \left(\int_{\{\phi=\phi(x)\}} |\nabla\phi| \, d\sigma\right)^{-1}$$

where $\phi = \bar{\tau}^{\nabla^\perp \psi}$.

- From above $V(\psi, v) = 0$ for all v if and only if

$$-2\triangle\phi(x) = 1 + |\nabla\phi(x)|^2 \left(\int_{\{\phi=\phi(x)\}} \frac{d\sigma}{|\nabla\phi|}\right)\left(\int_{\{\phi=\phi(x)\}} |\nabla\phi|\, d\sigma\right)^{-1}$$

  where $\phi = \bar{\tau}^{\nabla^\perp\psi}$.

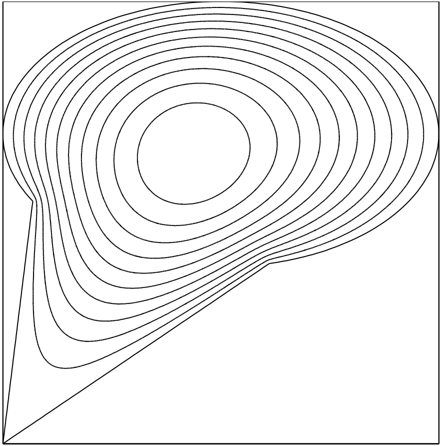
- If $V(\tau^0, v) = 0$ for all v, then

$$2 = 1 + \left|\nabla\tau^0\right|^2 M(\tau^0)$$

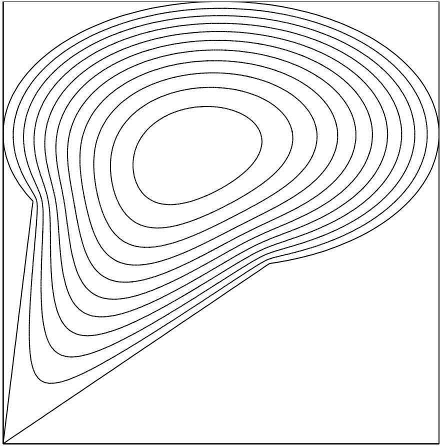
  and so τ^0 solves the eikonal equation.

- If Ω is not a disk, the eikonal equation necessarily has interior singularities. However τ^0 is analytic.

Simulations

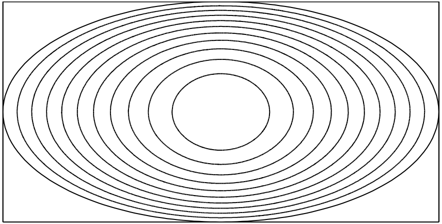


(a) Maximiser ψ

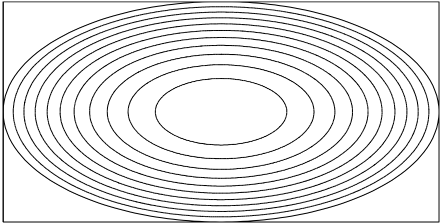


(b) Expected exit time τ_0

Simulations



(c) Maximiser ψ



(d) Expected exit time τ_0

Open questions

- Existence/uniqueness of solutions to the previous PDE.
- Are such solutions indeed maximisers?
- An understanding of why such stirring increases the exit time.
- Maximising other norms (L^p). Other constraints (e.g. finite power).
- Characterize flows that increase the explosion threshold.