

Berezin-Li-Yau bounds with additional term for the Stokes operator and the Dirichlet Laplacian

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General theme

Let $\{\varphi_i\}_{i=1}^m$ be an orthonormal system (o.n.s.) in $L_2(\Omega)$:

$$\int_{\Omega} \varphi_i(x) \varphi_j(x) dx = \delta_{ij},$$

where $\Omega \subset \mathbb{R}^n$, $|\Omega| = \text{Vol}(\Omega) < \infty$ and $\varphi_j \in H_0^\gamma(\Omega)$. We are interested in lower bounds of the form

$$\sum_{j=1}^m \|(-\Delta)^{\gamma/2} \varphi_j\|^2 \geq \dots,$$

where $(-\Delta)^{\gamma/2} \varphi = \mathcal{F}^{-1}(|\xi|^\gamma \widehat{\varphi}(\xi))$.

Constant coefficients and Dirichlet boundary conditions.

Beresin–Li–Yau estimates for Dirichlet Laplacian, $\gamma = 1$

Theorem (P.Li–S.-T.Yau, 1983). Let $\{\varphi_i\}_{i=1}^m \in H_0^1(\Omega)$ be an o.n.s. in $L_2(\Omega)$.

$$\sum_{j=1}^m \|\nabla \varphi_j\|^2 \geq \frac{n}{2+n} \left(\frac{(2\pi)^n}{\omega_n |\Omega|} \right)^{2/n} m^{1+2/n}, \quad m = 1, 2, \dots$$

Here $\omega_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)}$ denotes the volume of the unit ball in \mathbb{R}^n .

This bound is sharp, since letting $\{\varphi_i\}_{i=1}^m$ be the first m orthonormal eigenfunctions $-\Delta \varphi_k = \mu_k \varphi_k$, $\|\nabla \varphi_j\|^2 = \mu_j$, we have the Weyl asymptotic formula

$$\mu_k \sim \left(\frac{(2\pi)^n}{\omega_n |\Omega|} \right)^{2/n} k^{2/n} \text{ as } k \rightarrow \infty.$$

Hence,

$$\sum_{j=1}^m \|\nabla \varphi_j\|^2 = \sum_{k=1}^m \mu_k \sim \frac{n}{2+n} \left(\frac{(2\pi)^n}{\omega_n |\Omega|} \right)^{2/n} m^{1+2/n} \text{ as } m \rightarrow \infty.$$

Theorem (F.A.Beresin, 1972). Let $\{\varphi_i\}_{i=1}^m \in H_0^1(\Omega)$ be an o.n.s. in $L_2(\Omega)$. For $\mu \geq 0$ the following inequality holds:

$$\sum_k (\mu - \mu_k)_+ \leq \frac{2}{2+n} \frac{\omega_n}{(2\pi)^n} |\Omega| \mu^{1+n/2}, \quad (1)$$

where, as before, $\|\nabla\varphi_j\|^2 = \mu_j$ and $\mu_1 \leq \mu_2 \leq \dots$.

A.Laptev, T.Weidl (2000): the Li–Yau bound

$$\sum_{j=1}^m \|\nabla\varphi_j\|^2 = \sum_{j=1}^m \mu_j \geq \frac{n}{2+n} \left(\frac{(2\pi)^n}{\omega_n |\Omega|} \right)^{2/n} m^{1+2/n}, \quad m = 1, 2, \dots \quad (2)$$

follows from (1) by means of the Legendre transform in μ ; while (2) \Rightarrow (1) by the Legendre transform in m .

[$f(x)$ is defined on \mathbb{R}_+ and is convex (concave up). The Legendre transform $\tilde{f}(p) := \sup_{x \geq 0} (px - f(x))$. Clearly, if $f(x) \leq g(x)$, then $\tilde{f}(p) \geq \tilde{g}(p)$.]

Bounds for the Fourier transforms of the $\{\varphi_k\}$

$$\widehat{\varphi}_k(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \varphi_k(x) e^{-i\xi x} dx = (2\pi)^{-n/2} \int_{\Omega} \varphi_k(x) e^{-i\xi x} dx.$$

For

$$h_\xi(x) = (2\pi)^{-n/2} e^{-i\xi x} \Big|_{x \in \Omega} \quad \text{with} \quad \|h_\xi\|_{L_2(\Omega)}^2 = (2\pi)^{-n} |\Omega|$$

Bessel's inequality gives (uniformly in ξ and m):

$$F(\xi) := \sum_{k=1}^m |\widehat{\varphi}_k(\xi)|^2 = \sum_{k=1}^m |(h_\xi, \varphi_k)_{L_2(\Omega)}|^2 \leq \|h_\xi\|_{L_2(\Omega)}^2 = (2\pi)^{-n} |\Omega| =: M.$$

In addition

$$\int_{\mathbb{R}^n} F(\xi) d\xi = \sum_{j=1}^m \|\varphi_j\|^2 = m \quad \text{and} \quad \int_{\mathbb{R}^n} |\xi|^2 F(\xi) d\xi = \sum_{j=1}^m \|\nabla \varphi_j\|^2.$$

As a result,

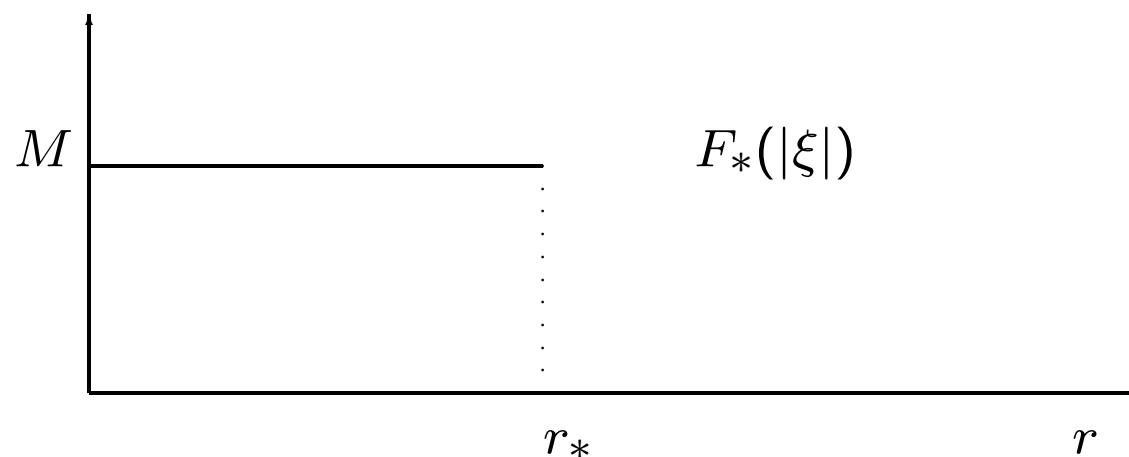
$$\sum_{j=1}^m \|\nabla \varphi_j\|^2 \geq \inf \left\{ \int_{\mathbb{R}^n} |\xi|^2 f(\xi) d\xi : 0 \leq f \leq M, \int_{\mathbb{R}^n} f(\xi) d\xi = m \right\}.$$

Minimization problem : find $\Sigma_M(m)$

$$\int_{\mathbb{R}^n} |\xi|^2 f(\xi) d\xi \rightarrow \inf =: \Sigma_M(m), \quad \text{under conditions}$$

$$1) 0 \leq f(\xi) \leq M, \quad 2) \int_{\mathbb{R}^n} f(\xi) d\xi = m.$$

Solution: $\Sigma_M(m) = \frac{n}{n+2} \left(\frac{1}{\omega_n M} \right)^{2/n} m^{1+2/n}$. $M = (2\pi)^{-n} |\Omega| \Rightarrow$ Berezin–Li–Yau.



r_* is found from the condition : $\int_{\mathbb{R}^n} F_*(|\xi|) d\xi = \omega_n M r_*^n = m$.

Vector functions and Stokes operator

We set $\mathcal{V} = \{u : \Omega \rightarrow \mathbb{R}^n, u \in C_0^\infty(\Omega), \operatorname{div} u = 0\}$ and denote by H and V the closure of \mathcal{V} in $\mathbf{L}_2(\Omega)$ and $\mathbf{H}^1(\Omega)$, respectively. The Helmholtz–Leray orthogonal projection P maps $\mathbf{L}_2(\Omega)$ onto H , $P : \mathbf{L}_2(\Omega) \rightarrow H$

$$\mathbf{L}_2(\Omega) = H \oplus H^\perp,$$

$$H = \{u \in \mathbf{L}_2(\Omega), \operatorname{div} u = 0, u \cdot \nu|_{\partial\Omega} = 0\}, \quad H^\perp = \{u \in \mathbf{L}_2(\Omega), u = \nabla p, p \in L_2^{\text{loc}}(\Omega)\},$$

$$V \subseteq \{u \in \mathbf{H}_0^1(\Omega), \operatorname{div} u = 0\},$$

where the last inclusion becomes equality for a bounded Ω with $\partial\Omega \in \text{Lip}$.

The Stokes operator $A : V \rightarrow V'$ is defined by the relation

$$\langle Au, v \rangle_{V' \times V} = (\nabla u, \nabla v) \quad \text{for all } u, v \text{ in } V$$

and is an isomorphism between V and V' . For a sufficiently smooth u

$$Au = -P\Delta u.$$

The Stokes operator A is an unbounded self-adjoint positive operator in H with compact inverse. It has a complete in H system of orthonormal

eigenfunctions $\{v_k\}_{k=1}^{\infty} \in V$ with corresponding eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$, $\lambda_k \rightarrow \infty$ as $k \rightarrow \infty$:

$$Av_k = \lambda_k v_k, \quad 0 < \lambda_1 \leq \lambda_2 \leq \dots .$$

By orthonormality, $\lambda_k = \|\nabla v_k\|^2$.

In case when Ω is a bounded domain with smooth boundary the eigenvalue problem goes over to

$$\begin{aligned} -\Delta v_k + \nabla p_k &= \lambda_k v_k, \\ \operatorname{div} v_k &= 0, \quad v_k|_{\partial\Omega} = 0, \end{aligned}$$

where the p_k 's are the 'pressures'.

Lemma. (2009) Let $\{u_k\}_{k=1}^m$ be an o.n.s. of vector functions in $\mathbf{L}_2(\Omega)$: $u_k = (u_k^1, \dots, u_k^n)$ and $\int_{\Omega} u_i(x) \cdot u_j(x) dx = \delta_{ij}$. Then

$$\sum_{k=1}^m |\hat{u}_k(\xi)|^2 \leq (2\pi)^{-n} n |\Omega|.$$

If $\operatorname{div} u_k = 0$, more precisely $u_k \in H = \{u \in \mathbf{L}_2(\Omega), \operatorname{div} u = 0, u \cdot \nu|_{\partial\Omega} = 0\}$, then

$$\sum_{k=1}^m |\hat{u}_k(\xi)|^2 \leq (2\pi)^{-n} (n-1) |\Omega|.$$

Sketch of proof.

Step 1. Let $\{\varphi_k\}_{k=1}^m$ be an o.n.s. $L_2(\Omega)$. Then (as shown before) Bessel's inequality gives

$$\sum_{k=1}^m |\hat{\varphi}_k(\xi)|^2 \leq (2\pi)^{-n} |\Omega|.$$

Alternatively, denote by $*$ the complex conjugate. Then by orthonormality

$$\begin{aligned} 0 &\leq \int \left((2\pi)^{-n/2} e^{-i\xi x} - \sum_{k=1}^m \hat{\varphi}_k(\xi) \varphi_k(x) \right) \left((2\pi)^{-n/2} e^{-i\xi x} - \sum_{l=1}^m \hat{\varphi}_l(\xi) \varphi_l(x) \right)^* dx = \\ &= (2\pi)^{-n} |\Omega| - \sum_{k=1}^m |\hat{\varphi}_k(\xi)|^2. \end{aligned}$$

Step 2. Let $\{\varphi_k\}_{k=1}^m$ be a suborthonormal system in L_2 :

$$\sum_{k,l=1}^m \zeta_k \zeta_l^*(\varphi_k, \varphi_l) \leq \sum_{k=1}^m |\zeta_k|^2.$$

Then, as before,

$$\sum_{k=1}^m |\hat{\varphi}_k(\xi)|^2 \leq (2\pi)^{-n} |\Omega|.$$

In fact,

$$\begin{aligned} 0 &\leq \int \left((2\pi)^{-n/2} e^{-i\xi x} - \sum_{k=1}^m \hat{\varphi}_k(\xi) \varphi_k(x) \right) \left((2\pi)^{-n/2} e^{-i\xi x} - \sum_{l=1}^m \hat{\varphi}_l(\xi) \varphi_l(x) \right)^* dx = \\ &= (2\pi)^{-n} |\Omega| - 2 \sum_{k=1}^m |\hat{\varphi}_k(\xi)|^2 + \sum_{k,l=1}^m \hat{\varphi}_k(\xi) \hat{\varphi}_l(\xi)^*(\varphi_k, \varphi_l) \leq (2\pi)^{-n} |\Omega| - \sum_{k=1}^m |\hat{\varphi}_k(\xi)|^2. \end{aligned}$$

Step 3. If a family of vector functions $\{u_k\}_{k=1}^m$ is orthonormal in $L_2(\Omega)$, then for each j the family of the j 's components $\{u_k^j\}_{k=1}^m$ is suborthonormal in $L_2(\Omega)$, and, hence,

$$\sum_{k=1}^m |\hat{u}_k(\xi)|^2 = \sum_{k=1}^m \sum_{j=1}^n |\hat{u}_k^j(\xi)|^2 \leq (2\pi)^{-n} n |\Omega|.$$

Step 4. $\operatorname{div} u_k = 0 \Rightarrow n \rightarrow n - 1.$ For all $\xi \neq 0$

$$\xi \cdot \widehat{u}_k(\xi) = \xi \cdot \int e^{-i\xi x} u_k(x) dx = i \int u_k \cdot \nabla_x e^{-i\xi x} dx = -i \int e^{-i\xi x} \operatorname{div} u_k dx = 0.$$

Therefore for $\xi_0 = (a, 0, \dots, 0)$, $a \neq 0$ $\widehat{u}_k^1(\xi_0) = 0$ for $k = 1, \dots, m$ and at ξ_0

$$\sum_{k=1}^m |\widehat{u}_k(\xi_0)|^2 = \sum_{j=2}^n \sum_{k=1}^m |\widehat{u}_k^j(\xi_0)|^2 \leq (2\pi)^{-n} (n-1) |\Omega|.$$

Now for $\xi \neq 0$ we set $\xi_0 = (|\xi|, 0, \dots, 0)$ and let ρ be the rotation such that

$$\xi = \rho^{-1} \xi_0.$$

We set

$$u_\rho(x) := \rho u(\rho^{-1} x), \quad x \in \rho\Omega.$$

The family $\{(u_k)_\rho\}_{k=1}^m \in \mathbf{H}_0^1(\rho\Omega)$ is orthonormal in $\mathbf{L}_2(\rho\Omega)$ and divergence-free. In addition

$$\widehat{u}_\rho(\xi) = \rho \widehat{u}(\rho^{-1} \xi).$$

Hence,

$$\sum_{k=1}^m |\widehat{u}_k(\xi)|^2 = \sum_{k=1}^m |(\widehat{u}_k)_\rho(\xi_0)|^2 \leq (2\pi)^{-n} (n-1) |\Omega|.$$

Theorem (2009)(Berezin–Li–Yau bounds for the Stokes operator).

Let $\{u_k\}_{k=1}^m$ be an o.n.s. in $\mathbf{L}_2(\Omega)$, $\{u_k\}_{k=1}^m \in \mathbf{H}_0^1(\Omega)$ and $\operatorname{div} u_k = 0$. For $\lambda_k = \|\nabla u_k\|^2$ we have Li–Yau-type bound

$$\sum_{k=1}^m \lambda_k \geq \frac{n}{2+n} \left(\frac{(2\pi)^n}{\omega_n(n-1)|\Omega|} \right)^{2/n} m^{1+2/n},$$

or, equivalently, Berezin-type bound

$$\sum_k (\lambda - \lambda_k)_+ \leq \frac{2}{2+n} \frac{\omega_n}{(2\pi)^n} (n-1)|\Omega| \lambda^{1+n/2}.$$

Proof. Set $M = (2\pi)^{-n}(n-1)|\Omega|$ in

$$\Sigma_M(m) = \frac{n}{2+n} \left(\frac{1}{\omega_n M} \right)^{2/n} m^{1+2/n}.$$

Remark. The lower bound

$$\sum_{k=1}^m \|\nabla u_k\|^2 \geq \frac{n}{2+n} \left(\frac{(2\pi)^n}{\omega_n(n-1)|\Omega|} \right)^{2/n} m^{1+2/n}$$

is sharp. If the u_k 's are the orthonormal eigenfunctions of the Stokes operator, then $\lambda_k = \|\nabla u_k\|^2$ are the corresponding eigenvalues and (G.Metivier, 1978; K.I.Babenko, 1983)

$$\lambda_k \sim \left(\frac{(2\pi)^n}{\omega_n(n-1)|\Omega|} \right)^{2/n} k^{2/n} \quad \text{as } k \rightarrow \infty.$$

Remark. Without condition $\operatorname{div} u_k = 0$ we have

$$\sum_{k=1}^m \|\nabla u_k\|^2 \geq \frac{n}{2+n} \left(\frac{(2\pi)^n}{\omega_n n |\Omega|} \right)^{2/n} m^{1+2/n}.$$

Corollary. Since $m\lambda_m \geq \sum_{k=1}^m \lambda_k$, the lower bound for sums implies lower bounds for individual Stokes eigenvalues

$$\lambda_k \geq \frac{n}{2+n} \left(\frac{(2\pi)^n}{\omega_n(n-1)|\Omega|} \right)^{2/n} k^{2/n}, \quad \text{in addition, } \lambda_1 > \mu_1 \geq \frac{n}{2+n} \left(\frac{(2\pi)^n}{\omega_n|\Omega|} \right)^{2/n}.$$

Two-term lower bounds: $F(\xi) \leq M$, $|\nabla F(\xi)| \leq L$

1) Scalar case: $\{\varphi_j\}_{j=1}^m$ is an o.n.s. in $L_2(\Omega)$ and as before

$$F(\xi) = \sum_{k=1}^m |\hat{\varphi}_k(\xi)|^2.$$

The identity

$$\partial_{\xi_j} \hat{\varphi}_k(\xi) = -(2\pi)^{-n/2} i \int_{\Omega} \varphi_k(x) x_j e^{-i\xi x} dx,$$

and Bessel's inequality give (observing that $\|x_j e^{-i\xi x}\|_{L_2(\Omega)}^2 = \int_{\Omega} x_j^2 dx$)

$$\sum_{k=1}^m |\partial_{\xi_j} \hat{\varphi}_k(\xi)|^2 \leq (2\pi)^{-n} \int_{\Omega} x_j^2 dx, \text{ and } \sum_{k=1}^m |\nabla \hat{\varphi}_k(\xi)|^2 \leq (2\pi)^{-n} \int_{\Omega} x^2 dx =: (2\pi)^{-n} I,$$

$$I := \int_{\Omega} x^2 dx.$$

We obtain (A.Melas, 2002)

$$|\nabla F(\xi)| \leq 2 \left(\sum_{k=1}^m |\hat{\varphi}_k(\xi)|^2 \right)^{1/2} \left(\sum_{k=1}^m |\nabla \hat{\varphi}_k(\xi)|^2 \right)^{1/2} \leq 2(2\pi)^{-n} (|\Omega| I)^{1/2}.$$

As a result, $F(\xi) = \sum_{k=1}^m |\hat{\varphi}_k(\xi)|^2$ satisfies $F(\xi) \leq M$, $|\nabla F(\xi)| \leq L$:

$$F(\xi) \leq M := (2\pi)^{-n}|\Omega|, \quad |\nabla F(\xi)| \leq L := 2(2\pi)^{-n}(|\Omega|I)^{1/2}.$$

2) Stokes operator: $\{u_j\}_{j=1}^m \in H$ ($\operatorname{div} u_k = 0$) is an o.n.s. in $\mathbf{L}_2(\Omega)$. Then

$$F(\xi) = \sum_{k=1}^m |\hat{u}_k(\xi)|^2$$

satisfies $F(\xi) \leq M$, $|\nabla F(\xi)| \leq L$:

$$F(\xi) \leq M := (2\pi)^{-n}(n-1)|\Omega|, \quad |\nabla F(\xi)| \leq L := 2(2\pi)^{-n}(n(n-1))^{1/2}(|\Omega|I)^{1/2}.$$

In both cases

$$\int_{\mathbb{R}^n} F(\xi) d\xi = m.$$

We have to find a lower bound for

$$\int_{\mathbb{R}^n} |\xi|^2 F(\xi) d\xi = \sum_{k=1}^m \mu_k \quad (\text{or } \sum_{k=1}^m \lambda_k).$$

Minimization problem II : find $\Sigma_{M,L}^\gamma(m)$ (for second-order operators $\gamma = 1$)

$$\int_{\mathbb{R}^n} |\xi|^{2\gamma} F(\xi) d\xi \rightarrow \inf =: \Sigma_{M,L}^\gamma(m),$$

under conditions

$$1) 0 \leq F(\xi) \leq M, \quad 2) \int_{\mathbb{R}^n} F(\xi) d\xi = m, \quad 3) |\nabla F(\xi)| \leq L.$$

$$\left[\text{Clearly, } \Sigma_{M,L}(m) \geq \Sigma_M(m). \right]$$

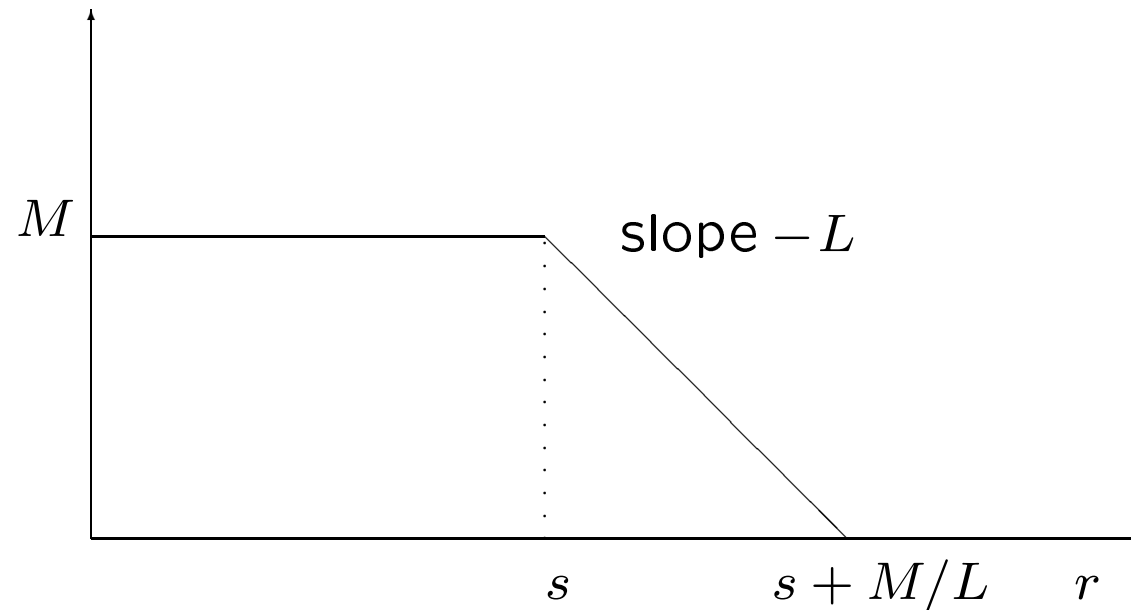
Symmetrization: an equivalent 1-D problem for a non-increasing function $F(r)$:

$$\sigma_n \int_0^\infty r^{n-1+2\gamma} F(r) dr \rightarrow \inf =: \Sigma_{M,L}^\gamma(m),$$

under conditions

$$1) 0 \leq F(r) \leq M, \quad 2) \sigma_n \int_0^\infty r^{n-1} F(r) dr = m, \quad 3) -F'(r) \leq L.$$

Minimizer



$s = s(m)$ is found from the condition (independent of γ)

$$\int_{\mathbb{R}^n} F_s(|\xi|) d\xi = \sigma_n \int_0^\infty r^{n-1} F_s(r) dr = m.$$

Exact solution:

$$\Sigma_{M,L}^\gamma(m) = \sigma_n \int_0^\infty r^{n-1+2\gamma} F_{s(m)}(r) dr.$$

Formula for $\Sigma_{M,L}^\gamma(m)$

Proposition. For $\beta > -1$

$$\int_0^\infty r^\beta F_s(r) dr = \frac{M^{\beta+2}}{(\beta+1)(\beta+2)L^{\beta+1}} \left((t+1)^{\beta+2} - t^{\beta+2} \right), \quad s = \frac{tM}{L}.$$

Theorem. The exact solution is given by

$$\Sigma_{M,L}^\gamma(m) = \frac{\sigma_n M^{n+1+2\gamma}}{(n+2\gamma)(n+1+2\gamma)L^{n+2\gamma}} \left((t(m_*)+1)^{n+1+2\gamma} - t(m_*)^{n+1+2\gamma} \right),$$

where $t(m_*)$ is the unique positive root of the n th-order equation

$$(t+1)^{n+1} - t^{n+1} = m \frac{(n+1)L^n}{\omega_n M^{n+1}} =: m_*.$$

Asymptotic expansion

$$\begin{aligned} \Sigma_{M,L}^\gamma(m) &= \frac{n}{n+2\gamma} \left(\frac{1}{\omega_n M} \right)^{2\gamma/n} m^{1+2\gamma/n} + \\ &+ \frac{n\gamma}{12\omega_n^{(2\gamma-2)/n}} \frac{M^{2+(2-2\gamma)/n}}{L^2} m^{1+2\gamma/n-2/n} - O(m^{1+2\gamma/n-4/n}). \end{aligned}$$

Theorem (2010). The eigenvalues of the Dirichlet Laplacian (μ_k) and the Stokes operator (λ_k) satisfy, respectively:

$$\begin{aligned} \sum_{k=1}^m \mu_k &\geq \frac{n}{n+2} \left(\frac{(2\pi)^n}{\omega_n |\Omega|} \right)^{2/n} m^{1+2/n} + \frac{n}{48} \frac{|\Omega|}{I} m (1 - \varepsilon_n(m)), \\ \sum_{k=1}^m \lambda_k &\geq \frac{n}{n+2} \left(\frac{(2\pi)^n}{\omega_n (n-1) |\Omega|} \right)^{2/n} m^{1+2/n} + \frac{(n-1)}{48} \frac{|\Omega|}{I} m (1 - \varepsilon_n(m)), \end{aligned}$$

where $\varepsilon_n(m) \geq 0$, $\varepsilon_n(m) = O(m^{-2/n})$. Recall that $|\Omega| = \text{Vol}(\Omega)$ and $I = \int_{\Omega} x^2 dx$.

Remark. In fact, for $n = 2, 3, 4$ we show that $\varepsilon_n(m) \leq 0.02$ for all $m \geq 1$.

From the previous results (A.Melas, 2002) it follows that

$$\Sigma_{M,L}^{\gamma=1}(m) \geq \frac{n}{n+2} \left(\frac{1}{\omega_n M} \right)^{2/n} m^{1+2/n} + \frac{1}{6(n+2)} \frac{M^2}{L^2} m,$$

which gives for the Dirichlet Laplacian ($M = (2\pi)^{-n} |\Omega|$, $L = 2(2\pi)^{-n} (|\Omega|I)^{1/2}$)

$$\sum_{k=1}^m \mu_k \geq \frac{n}{2+n} \left(\frac{(2\pi)^n}{\omega_n |\Omega|} \right)^{2/n} m^{1+2/n} + \frac{1}{24(n+2)} \frac{|\Omega|}{I} m.$$

We have the coefficient of the second term (linear with respect to m) that is

$$\left(\frac{n}{12} \right) / \left(\frac{1}{6(n+2)} \right) = \frac{n(n+2)}{2}$$

times greater.

Examples $n = 2, \gamma = 1$

Find $\Sigma_{M,L}^{\gamma=1}(m)$, where

$$\int_{\mathbb{R}^2} |\xi|^2 f(\xi) d\xi \rightarrow \inf =: \Sigma_{M,L}(m), \quad \text{under conditions}$$

$$1) 0 \leq F(\xi) \leq M, \quad 2) |\nabla F(\xi)| \leq L, \quad 3) \int_{\mathbb{R}^2} F(\xi) d\xi = m.$$

Lemma. For $n = 2$ and $\gamma = 1$ the exact solution is

$$\Sigma_{M,L}^1(m) = \frac{1}{2\pi M} m^2 + \frac{M^2}{6L^2} m - \frac{\pi M^5}{90L^4}.$$

Proof. The positive root $t(m_*)$ of the quadratic equation $(t+1)^3 - t^3 = m_*$ is

$$t(m_*) = \left(\frac{m_*}{3} - \frac{1}{12} \right)^{1/2} - \frac{1}{2}$$

and substitution into the 4-th order equation gives

$$(t(m_*) + 1)^5 - t(m_*)^5 = \frac{5}{9} m_*^2 + \frac{5}{9} m_* - \frac{1}{9}.$$

Recall that

$$m_* = m \frac{(n+1)L^n}{\omega_n M^{n+1}} \Big|_{n=2} = m \frac{3L^2}{\pi M^3}.$$

Theorem. (2010) For the Dirichlet Laplacian and the Stokes operator in the two-dimensional case the following lower bounds hold (where $I = \int_{\Omega} x^2 dx$):

$$\sum_{k=1}^m \mu_k \geq \frac{2\pi}{|\Omega|} m^2 + \frac{1}{24} \frac{|\Omega|}{I} m \left(1 - \frac{1}{120m} \right) \geq \frac{2\pi}{|\Omega|} m^2 + \frac{1}{24} \frac{119}{120} \frac{|\Omega|}{I} m,$$

$$\sum_{k=1}^m \lambda_k \geq \frac{2\pi}{|\Omega|} m^2 + \frac{1}{48} \frac{|\Omega|}{I} m \left(1 - \frac{1}{240m} \right) \geq \frac{2\pi}{|\Omega|} m^2 + \frac{1}{48} \frac{239}{240} \frac{|\Omega|}{I} m.$$

Proof. Previous Lemma, formulas for M and L + inequality $|\Omega|^2/I \leq 2\pi$.

Remark. These estimates are entirely new for the Stokes operator (λ_k) and improve the previously known for the Dirichlet Laplacian: (A.Melas, 2002) and (Kovařík H., Vugalter S. and Weidl T., 2008), respectively,

$$\sum_{k=1}^m \mu_k \geq \frac{2\pi}{|\Omega|} m^2 + \frac{1}{96} \frac{|\Omega|}{I} m, \quad \sum_{k=1}^m \mu_k \geq \frac{2\pi}{|\Omega|} m^2 + \frac{1}{32} \frac{|\Omega|}{I} m.$$

$$n = 2, \gamma = 2, \Delta^2 \varphi_k = \nu_k \varphi_k$$

Find $\Sigma_{M,L}^2(m)$, where

$$\int_{\mathbb{R}^2} |\xi|^4 f(\xi) d\xi \rightarrow \inf =: \Sigma_{M,L}^4(m), \quad \text{under conditions}$$

$$1) 0 \leq F(\xi) \leq M, \quad 2) |\nabla F(\xi)| \leq L, \quad 3) \int_{\mathbb{R}^2} F(\xi) d\xi = m.$$

Lemma. The exact solution is

$$\Sigma_{M,L}^2(m) = \frac{1}{3\pi^2 M^2} m^3 + \frac{M}{3\pi L^2} m^2 - \frac{\pi M^7}{7 \cdot 3^4 L^6}.$$

Proof. Substitute the root $t(m_*)$,

$$t(m_*) = \left(\frac{m_*}{3} - \frac{1}{12} \right)^{1/2} - \frac{1}{2},$$

of the equation $(t+1)^3 - t^3 = m_*$ into

$$(t(m_*) + 1)^7 - t(m_*)^7 = \frac{7}{27} m_*^3 + \frac{7}{9} m_*^2 - \frac{1}{27}.$$

$$m_* = m \frac{(n+1)L^n}{\omega_n M^{n+1}} \Big|_{n=2} = m \frac{3L^2}{\pi M^3}.$$

Theorem. Biharmonic operator: $\Delta^2 \varphi_k = \nu_k \varphi_k$, $\varphi_k \in H_0^2(\Omega)$, $\Omega \subset \mathbb{R}^2$

$$\sum_{k=1}^m \nu_k \geq \frac{16\pi^2}{3|\Omega|^2} m^3 + \frac{\pi}{3I} m^2 \left(1 - \frac{1}{7 \cdot 27 \cdot 64 m^2} \right) \geq \frac{16\pi^2}{3|\Omega|^2} m^3 + \frac{\pi}{3I} \frac{12095}{12096} m^2,$$

where, as before, $I = \int_{\Omega} x^2 dx$.

$$n = 2, \gamma = 3$$

Find $\Sigma_{M,L}^3(m)$, where

$$\int_{\mathbb{R}^2} |\xi|^6 f(\xi) d\xi \rightarrow \inf =: \Sigma_{M,L}^3(m),$$

under conditions

$$1) 0 \leq F(\xi) \leq M, \quad 2) |\nabla F(\xi)| \leq L, \quad 3) \int_{\mathbb{R}^2} F(\xi) d\xi = m.$$

Lemma. The exact solution is given below:

$$\Sigma_{M,L}^3(m) = \frac{1}{4\pi^3 M^3} m^4 + \frac{1}{2\pi^2} \frac{1}{L^2} m^3 + \frac{1}{12\pi} \frac{M^3}{L^4} m^2 - \frac{1}{108} \frac{M^6}{L^6} m.$$

Theorem. $(-\Delta)^3 \varphi_k = \nu_k \varphi_k$, $\varphi_k \in H_0^3(\Omega)$, $\Omega \subset \mathbb{R}^2$:

$$\sum_{k=1}^m \nu_k \geq \frac{16\pi^2}{|\Omega|^3} m^4 + \frac{\pi^2}{2|\Omega|I} m^3 + \frac{\pi}{4872} \frac{|\Omega|}{I^2} m^2.$$

The coefficient of the leading power of m is sharp.

Applications

2D Navier–Stokes system. We write the two-dimensional Navier–Stokes system as an evolution equation in H

$$\partial_t u + \nu A u + B(u, u) = f, \quad u(0) = u_0,$$

where $A = -P\Delta$ is the Stokes operator and $B(u, v) = P(\sum_{i=1}^2 u^i \partial_i v)$. The equation generates the semigroup $S_t : H \rightarrow H$, $S_t u_0 = u(t)$, which has a compact global attractor $\mathcal{A} \Subset H$. The attractor \mathcal{A} is the maximal strictly invariant compact set.

Theorem. The fractal dimension of \mathcal{A} satisfies the following estimate:

$$\dim_F \mathcal{A} < \frac{1}{4\pi 3^{1/4}} \frac{\|f\| |\Omega|}{\nu^2}.$$

History

1. Existence of \mathcal{A} — O.A.Ladyzhenskaya 1972, C.Foias and R.Temam 1979.

2. $\dim_F \mathcal{A} \leq c(\Omega) \left(\frac{\|f\| \|\Omega\|}{\nu^2} \right)^2$ — A.V.Babin, M.I.Vishik, Yu.S.Ilyashenko, P.Constantin, C.Foias and others 1980-1985.

3. $\dim_F \mathcal{A} \leq c(\Omega) \frac{\|f\| \|\Omega\|}{\nu^2}$ — R.Temam 1985 (E.Lieb 1983).

4. $c(\Omega) \leq \frac{1}{\pi}$ — A.A.Ilyin 1996.

5. $c(\Omega) \leq \frac{1}{2\pi^{3/2}}$ — V.V.Chepyzhov and A.A.Ilyin 2004.

6. $c(\Omega) < \frac{1}{4\pi^{3^{1/4}}}$ — .

The improvement of the previous result (that is, 5. \rightarrow 6.) is by the factor $(2 \cdot (2/R))^{1/2} = 1.485\dots$, where $R = \pi/3^{1/2} = 1.8138\dots$ improves the previous estimate of R : $R = 2$ (D.Hundertmark, A.Laptev, T.Weidl (2000).)

$$\dim_F \mathcal{A} \leq \left(\frac{c_{\text{LT}}}{4\pi c_{\text{sp}}} \right)^{1/2} \frac{\|f\| |\Omega|}{\nu^2}, \quad (\text{V.V.Chepyzhov, A.A.Ilyin 2004}),$$

where c_{sp} comes from our lower bound for the Stokes operator in 2D

$$\sum_{k=1}^m \|\nabla v_k\|^2 \geq \frac{c_{\text{sp}} m^2}{|\Omega|} = \frac{2\pi m^2}{|\Omega|}, \quad c_{\text{sp}} = 2\pi \quad (\text{sharp}),$$

(the previous estimate of c_{sp} was $n/(n-1)|_{n=2} = 2$ times worse; this is the first factor "2" in $(2 \cdot (2/R))^{1/2} = 1.485\dots$) and c_{LT} comes from the Lieb–Thirring inequality

$$\int_{\Omega} \left(\sum_{j=1}^m |v_j(x)|^2 \right)^2 dx \leq c_{\text{LT}} \sum_{j=1}^m \|\nabla v_j\|^2, \quad c_{\text{LT}} \leq \frac{1}{2 \cdot 3^{1/2}}.$$

As before $\{v_k\}_{k=1}^m \in \mathbf{H}(\Omega)_0^1$ is an o.n.s. in $\mathbf{L}_2(\Omega)$, $\Omega \in \mathbb{R}^2$, and $\text{div } v_k = 0$.

The constant c_{LT} satisfies (Ch–I,2004)

$$c_{\text{LT}} \leq 4L_{1,2},$$

where the constant $L_{1,2}$ comes from the Lieb–Thirring spectral estimate

$$\sum_{\mu_j < 0} |\mu_j|^\gamma \leq L_{\gamma,n} \int_{\mathbb{R}^n} f(x)^{\gamma+n/2} dx$$

for the negative eigenvalues of the scalar Schrödinger operator $-\Delta - f$ in \mathbb{R}^n , $f \geq 0$. For $L_{\gamma,n}$ we always have the following “classical” lower bound

$$L_{\gamma,n} \geq L_{\gamma,n}^{\text{cl}} := \frac{\Gamma(\gamma + 1)}{(4\pi)^{n/2} \Gamma(\gamma + n/2 + 1)}.$$

It was recently shown by J.Dolbeault, A.Laptev and M.Loss 2008 that

$$L_{\gamma,n} \leq R \cdot L_{\gamma,n}^{\text{cl}}, \quad R = \pi/3^{1/2} = 1.8138\dots, \quad \gamma \geq 1.$$

Hence $c_{\text{LT}} \leq 4 R L_{1,2}^{\text{cl}} = 1/(2 \cdot 3^{1/2})$. Finally,

$$\frac{1}{2\pi} \leq c_{\text{LT}} \leq \frac{1}{2 \cdot 3^{1/2}}.$$

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