Continuous Families of Exponential Attractors for Singularly Perturbed Equations with Memory

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joint work with A. Miranville, V. Pata, S. Zelik

Dissipative PDEs in Bounded and Unbounded Domains
and Related Attractors
ICMS workshop 2010, Edinburgh

Problem

Given a family (P_ε) of evolutionary problems $\varepsilon \in (0, 1]$
if formally $(P_\varepsilon) \to (P_0)$ as $\varepsilon \to 0$
and also $(P_\varepsilon) \to (P_{\varepsilon_1})$ as $\varepsilon \to \varepsilon_1 \in (0, 1]$
What can ensure the asymptotic closeness in all limits?
Can we render and measure the distance of the asymptotics?
Which objects are more suitable to pursue our scopes?

Too generic! $\varepsilon \to 0$ and $\varepsilon \to \varepsilon_1$ are different! Maybe $\varepsilon \to 0$ is singular limit

- How may singular perturbations influence our choices? It depends
- How may this interact with global asymptotic closeness?

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An example

(H) Hyperbolic relaxation

$$\varepsilon \partial_{tt} u + \partial_t u - \Delta u + f(u) = 0 \quad k_\varepsilon(s) = \frac{e^{-s/\varepsilon}}{\varepsilon}$$

As $\varepsilon \to \varepsilon_1$ $\quad \varepsilon \partial_{tt} u + \partial_t u - \Delta u + f(u) = 0 \to \varepsilon_1 \partial_{tt} u + \partial_t u - \Delta u + f(u) = 0$

(M) Memory relaxation:

$$\partial_t u + \int_0^\infty k_\varepsilon(s)[-\Delta u(t-s)ds + f(u(t-s))]ds = 0$$

As $\varepsilon \to 0$ if $k_\varepsilon \to \delta$
the Allen-Cahn or Reaction-Diffusion equation

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Impact of singular perturbations (alone)

For any $\varepsilon \in (0,1]$ we have a dynamical system $(S_\varepsilon(t), \mathcal{H}_\varepsilon^0 = X \times ?)$ possibly singular perturbation with memory of $(S(t), X)$ for $\varepsilon = 0$

- When $\mathcal{H}_\varepsilon^0 = X \times Y$ does not depend on $\varepsilon \leftarrow$ regular perturbation
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Getting Global (asymptotic closeness)

Globally Hölder continuous family of exponential attractors

For each $\varepsilon \in [0,1]$ $\quad \exists \mathcal{E}_\varepsilon \subset \mathcal{H}^0_\varepsilon$ compact in $\mathcal{H}^0_\varepsilon$ and $S_\varepsilon(t)\mathcal{E}_\varepsilon \subset \mathcal{E}_\varepsilon, \forall t \geq 0$, satisfying

(i) $\mathcal{E}_\varepsilon$ are uniformly exponentially attracting
$\exists \omega > 0, \exists F : [0,\infty) \to [0,\infty) \uparrow$ such that, for every bounded $\mathcal{B} \subset \mathcal{H}^0_\varepsilon$,

$$\mathrm{dist}_{\mathcal{H}^0_\varepsilon}(S_\varepsilon(t)\mathcal{B}, \mathcal{E}_\varepsilon) \leq F\big(\|\mathcal{B}\|_{\mathcal{H}^0_\varepsilon}\big)e^{-\omega t}$$

(ii) uniformly bounded fractal dimensions

$$\exists C > 0 \quad \text{such that} \quad \dim_{\mathcal{H}^0_\varepsilon}\mathcal{E}_\varepsilon \leq C$$

(iii) global continuity
$\exists C \geq 0, \exists \vartheta \in (0,1)$ such that $\mathrm{dist}^{\mathrm{sym}}_{\mathcal{H}^0_\varepsilon}(\mathcal{E}_\varepsilon, \mathcal{E}_0) \leq C\varepsilon^\vartheta$ and

$$\mathrm{dist}^{\mathrm{sym}}_{\mathcal{H}^0_{\varepsilon_2}}(\mathcal{E}_{\varepsilon_1}, \mathcal{E}_{\varepsilon_2}) \leq C(\varepsilon_1 - \varepsilon_2)^\vartheta, \quad \forall \varepsilon_1 \geq \varepsilon_2 > 0$$

Literature on globally continuous exponential attractors

- Efendiev, Miranville, Zelik (2005) henceforth EMZ
  autonomous and non-autonomous problems
  regular perturbation $\to$ same phase-space

$$\partial_t u - \Delta u + f(u) = g(t) \rightsquigarrow U_{g(t)}(t,\tau)$$

  outline of several extensions
- Miranville, Pata, Zelik (2007) henceforth MPZ
  hyperbolic relaxation $\to$ singular perturbation

$$\varepsilon^2 \partial_{tt} u + \partial_t u - \Delta u + f(u) = g \quad \to \quad \partial_t u - \Delta u + f(u) = g \text{ as } \varepsilon \to 0$$

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$$\begin{aligned} \mathfrak{I}_\varepsilon : \quad \mathcal{H}_\varepsilon^0 &\longrightarrow \mathcal{H}_1^0 \\ (u,v) &\mapsto (u,\varepsilon v) \end{aligned} \qquad \widehat{S}_\varepsilon(t) = \begin{cases} \mathfrak{I}_\varepsilon S_\varepsilon(t) \mathfrak{I}_\varepsilon^{-1} & \text{if } \varepsilon > 0 \\ S_0 \Pi & \text{if } \varepsilon = 0 \end{cases}$$

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  autonomous and non-autonomous problems
  regular perturbation $\rightarrow$ same phase-space

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1. Introduce the memory variable η and translate (P_ε) into the History Space Setting $\rightarrow$ generation of a semigroup

   η lives in $L^2_{\mu_\varepsilon}((0,\infty);Y^0) \rightarrow$ COMPACTNESS Pata, Zucchi (2000)
   $\varepsilon-$comparison Conti, Pata, Squassina (2006) AND

2. Find a way to take all the solution operators to the same phase-space $\widehat{S}_\varepsilon(t) \rightarrow$ SCALING

3. Detect suitable assumptions to construct Hölder continuous discrete exponential attractors $\widehat{\mathcal{D}}_\varepsilon$ for $\widehat{S}_\varepsilon(t^\star)$ ($\leftarrow$ Apply EMZ Theorem) $\rightarrow$

$$\mathcal{D}_\varepsilon = S_\varepsilon(t_\star)\mathfrak{I}_\varepsilon^{-1}\widehat{\mathcal{D}}_\varepsilon \qquad (\mathfrak{I}_0^{-1} = \Pi)$$

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- **Original problem**

$k : \mathbb{R}^+ \to \mathbb{R}^+$ convex $\int_0^\infty k(s)ds = 1 \quad k_\varepsilon(s) = \frac{1}{\varepsilon} k\left(\frac{s}{\varepsilon}\right), \ \varepsilon \in (0,1]$

$$\begin{cases} \partial_t u + \int_0^\infty k_\varepsilon(s)\big[-\Delta u(t-s) + \phi(u(t-s)) + f\big]ds = 0 \\ u_{|\partial\Omega} = 0 \\ u(t) = u_0(-t), \quad t \le 0 \end{cases}$$

- **History setting:** let $\eta^t(s) = \int_0^s [-\Delta u(t-y) + \phi(u(t-y)) + f]dy$

$0 \le \mu \doteq -k' \in W^{1,1}(\mathbb{R}^+) \downarrow: \ \exists \delta > 0 : \ \mu'(s) + \delta\mu(s) \le 0, \ \text{a.e.in } \mathbb{R}^+$

Rescaled kernels: $\mu_\varepsilon(s) = \frac{d}{ds}(k_\varepsilon(s)) = \frac{1}{\varepsilon^2}\mu\left(\frac{s}{\varepsilon}\right) \to \mathcal{Y}^0_\varepsilon = L^2_{\mu_\varepsilon}(\mathbb{R}^+; Y^0)$

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Scaling

- $\forall \varepsilon \in [0,1] \quad \exists S_\varepsilon(t) : \mathcal{H}^0_\varepsilon \to \mathcal{H}^0_\varepsilon$ solution operator

$$\mathcal{H}^0_\varepsilon = X^0 \times \underbrace{Y^0_\varepsilon = L^2_{\mu_\varepsilon}(\mathbb{R}^+; Y^0)}_{\text{singular component}} \qquad Y^0_0 = \{0\}$$

In our case $X^0 = H^1_0(\Omega)$ and $Y^0 = L^2(\Omega)$

$$\widehat{S}_\varepsilon(t) : \mathcal{H}^0_1 \to \mathcal{H}^0_1 \quad \widehat{S}_\varepsilon(t) = \begin{cases} \mathfrak{I}_\varepsilon S_\varepsilon(t) \mathfrak{I}_\varepsilon^{-1} & \text{if } \varepsilon > 0 \\ S_0 \Pi & \text{if } \varepsilon = 0 \end{cases} \quad \Pi(x,\eta) = (x,0)$$

- define a suitable isometric isomorphism $J_\varepsilon : Y^0_\varepsilon \to Y^0_1$
  $\to$ define $\mathfrak{I}_\varepsilon : \mathcal{H}^0_\varepsilon \to \mathcal{H}^0_1 \quad \mathfrak{I}_\varepsilon(x,\eta) = (x, J_\varepsilon \eta)$

What should scaling do?

Together with the "singular" component of the phase-space, it should allow to compare dynamics corresponding to $\varepsilon_1 > \varepsilon_2$ and to go back and forth between $S_\varepsilon(t)$ and $\widehat{S}_\varepsilon(t)$

Scaling with memory

- Nested spaces: if $\varepsilon_1 \geq \varepsilon_2 \quad \Rightarrow \quad L^2_{\mu_{\varepsilon_1}}(\mathbb{R}^+; Y^0) \subset L^2_{\mu_{\varepsilon_2}}(\mathbb{R}^+; Y^0)$
- Scaling map: $J_\varepsilon : Y^0_\varepsilon = L^2_{\mu_\varepsilon}(\mathbb{R}^+; Y^0) \to Y^0_1 = L^2_{\mu}(\mathbb{R}^+; Y^0)$

$$(J_\varepsilon \eta)(s) = \frac{\eta(\varepsilon s)}{\sqrt{\varepsilon}}, \quad (J_\varepsilon^{-1}\eta)(s) = \sqrt{\varepsilon}\eta\left(\frac{s}{\varepsilon}\right)$$

- $\exists F, F_0 : [0,\infty) \to [0,\infty) \uparrow, \quad F_0(s) \leq c s^\kappa, \; \forall s \in [0,1], \quad \exists a \geq 0 :$

$$\|\eta - J_{\varepsilon_2} J_{\varepsilon_1}^{-1}\eta\|_{Y^0_1} \leq F(\|\eta\|_{Y^2_1}) F_0\left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_2^a}\right), \; \forall \eta \in Y^2_1, \; \forall \varepsilon_1 \geq \varepsilon_2 > 0$$

where $Y^2_\varepsilon \subset Y^1_\varepsilon \Subset Y^0_\varepsilon$ for any $\varepsilon \in (0,1]$

Main Theorem

For any $\varepsilon \in [0,1]$ $\quad \mathcal{H}^2_\varepsilon \subset \mathcal{H}^1_\varepsilon \Subset \mathcal{H}^0_\varepsilon, \quad S_\varepsilon(t) : \mathcal{H}^0_\varepsilon \to \mathcal{H}^0_\varepsilon, \quad \mathbb{B}_\varepsilon(r_0) \subset \mathcal{H}^2_\varepsilon$

(H1) $\exists r_0 > 0 : \mathbb{B}_\varepsilon(r_0)$ uniformly exponentially attracting in $\mathcal{H}^0_\varepsilon$

(H2) $\exists r_1 > r_0, \exists t_\star > 0 : S_\varepsilon(t)\mathbb{B}_\varepsilon(r_1) \subset \mathbb{B}_\varepsilon(r_0), \forall t \geq t_\star$

(H3) $\exists \kappa \in (0,1], \exists F \in \mathcal{N}\mathcal{D} : \forall z_1 \in \mathbb{B}_\varepsilon(r_1)$ and $\|z_2\|_{\mathcal{H}^0_\varepsilon} \leq r$

$$\|S_\varepsilon(t)z_1 - S_\varepsilon(t)z_2\|_{\mathcal{H}^0_\varepsilon} \leq F(r)e^{F(r)t}\|z_1 - z_2\|^\kappa_{\mathcal{H}^0_\varepsilon}$$

(H4) $\exists \kappa \in (0,1], \exists K = K(\varepsilon) \geq 0 :$

$$\sup_{z \in \mathbb{B}_\varepsilon(r_0)} \|S_\varepsilon(t_1)z - S_\varepsilon(t_2)z\|_{\mathcal{H}^0_\varepsilon} \leq K|t_1 - t_2|^\kappa, \quad \forall t_1, t_2 \in [t_\star, 2t_\star].$$

(H5) $\exists \Lambda \geq 0, \exists \lambda \in (0,1) : S_\varepsilon = S_\varepsilon(t_\star)$ satisfies

$$S_\varepsilon z_1 - S_\varepsilon z_2 = L_\varepsilon(z_1, z_2) + K_\varepsilon(z_1, z_2), \quad \forall z_1, z_2 \in \mathbb{B}_\varepsilon(r_1)$$

$$\|L_\varepsilon(z_1, z_2)\|_{\mathcal{H}^0_\varepsilon} \leq \lambda\|z_1 - z_2\|_{\mathcal{H}^0_\varepsilon}, \; \|K_\varepsilon(z_1, z_2)\|_{\mathcal{H}^1_\varepsilon} \leq \Lambda\|z_1 - z_2\|_{\mathcal{H}^0_\varepsilon},$$

(H6) $\sup_{t\in[t_\star,2t_\star]} \sup_{z\in\mathbb{B}_{\varepsilon_1}(r_1)} \|S_{\varepsilon_1}(t)z - S_0(t)\mathfrak{I}_0^{-1}z\|_{\mathcal{H}^0_{\varepsilon_2}} \le C\varepsilon_1^b$, $\forall \varepsilon_1 \ge \varepsilon_2 > 0$

(H7) I. $\sup_{t\in[t_\star,2t_\star]} \sup_{z\in\mathbb{B}_{\varepsilon_1}(r_1)} \|S_{\varepsilon_1}(t)z - S_{\varepsilon_2}(t)z\|_{\mathcal{H}^0_{\varepsilon_2}} \le F_0\left(\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_2^a}\right)$
$\forall \varepsilon_1 \ge \varepsilon_2 > 0$,

or

II. $\sup_{t\in[t_\star,2t_\star]} \sup_{z\in\mathbb{B}_1(r_1)} \|\widehat{S}_{\varepsilon_1}(t)z - \widehat{S}_{\varepsilon_2}(t)z\|_{\mathcal{H}^0_1} \le F_0\left(\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_2^a}\right)$
$\forall \varepsilon_1 \ge \varepsilon_2 > 0$

$\Rightarrow \exists \mathcal{E}_\varepsilon \subset \mathbb{B}_\varepsilon(r_0)$ such that $\{\mathcal{E}_\varepsilon\}_\varepsilon$ is (Hölder) continuous family of exponential attractors for $\{S_\varepsilon(t)\}_\varepsilon$

Usually (H7)II is easier to check than (H7)I

- $$\begin{cases} \partial_t u + \int_0^\infty k_\varepsilon(s)\big[-\Delta u(t-s) + \phi(u(t-s)) + f\big]ds = 0 \\ u_{|\partial\Omega} = 0 \\ u(t) = u_0(-t), \quad t \le 0 \end{cases}$$

- History setting: set $z \doteq (u_0(0), \int_0^s [-\Delta u_0(-y) + \phi(u_0(-y)) + f]dy)$

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Preliminary remark

(H3) $S_\varepsilon(t)$ Hölder in z at fixed $t \Rightarrow$
(H7)I $S_\varepsilon(t)$ Hölder in ε at fixed $t \Leftrightarrow$ (H7)II $\widehat{S}_\varepsilon(t)$ Hölder in ε at fixed t

Lemma (1^{st} technical lemma)

Let $\varepsilon_1 \geq \varepsilon_2 > 0$ *and* $z \in \mathbb{B}_{\varepsilon_1}(r)$, *for some* $r > 0$ *(*$\Rightarrow z \in \mathcal{H}^0_{\varepsilon_2}$*)*
$\Rightarrow \exists c_0 > 0$ *such that*

$$\|z\|_{\mathcal{H}^0_{\varepsilon_2}} \leq c_0 r + F(r) F_0\left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_2^a}\right),$$

with a, F *and* F_0 *as in*

$$\|z - \mathfrak{I}_{\varepsilon_2}\mathfrak{I}^{-1}_{\varepsilon_1} z\|_{\mathcal{H}^0_1} \leq F(\underbrace{\|\mathbb{Q}z\|_{Y^2_1}}_{\text{memory component}}) F_0\left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_2^a}\right), \quad \forall \varepsilon_1 \geq \varepsilon_2 > 0$$

whenever $z \in X^0 \times Y^2_1$

Apply EMZ+MPZ to $(\widehat{S}_\varepsilon(t_\star), \mathbb{B}_1(r_1))$

- $[(H2),(H3),(H5),(H7)]_{t=t_\star}$ give all the assumptions of EMZ but one.
  $\widehat{S}_\varepsilon = \widehat{S}_\varepsilon(t_\star) : \mathbb{B}_1(r_1) \to \mathbb{B}_1(r_0)$
  $\sup_{x\in\mathbb{B}_1(r_1)} \|\widehat{S}_{\varepsilon_1}x - \widehat{S}_{\varepsilon_2}x\|_{\mathcal{H}_1^0} \le c|\varepsilon_1 - \varepsilon_2|^b$ for $\varepsilon_1 > \varepsilon_2 > 0$?
- $\sup_{z\in\mathbb{B}_1(r_1)} \|\widehat{S}_{\varepsilon_1}z - \widehat{S}_{\varepsilon_2}z\|_{\mathcal{H}_1^0} \le F_0\left(\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_2^a}\right), \quad \forall \varepsilon_1 \ge \varepsilon_2 > 0,$
- $\|\widehat{S}_{\varepsilon_1}z - \widehat{S}_{\varepsilon_2}z\|_{\mathcal{H}_1^0} \le \|\widehat{S}_{\varepsilon_1}z - \widehat{S}_0 z\|_{\mathcal{H}_1^0} + \|\widehat{S}_{\varepsilon_2}z - \widehat{S}_0 z\|_{\mathcal{H}_1^0} \le C(\varepsilon_1^b + \varepsilon_2^b) \le C\varepsilon_1^b,$

Lemma (2^{nd} technical lemma)

Given $\alpha \ge 0$, $\beta > 0$, $F_0 \in \mathcal{N}\mathcal{D}_0$
($\Leftrightarrow F_0 : [0,\infty) \to [0,\infty) \nearrow$ *Hölder at* 0 *with* $F_0(0) = 0$)
$\Rightarrow \exists K > 0, \gamma > 0 : \min\left\{F_0\left(\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_2^\alpha}\right), \varepsilon_1^\beta\right\} \le K(\varepsilon_1 - \varepsilon_2)^\gamma,\ 1 \ge \varepsilon_1 \ge \varepsilon_2 > 0$

$$k : [0,\infty) \to [0,\infty) \text{ convex with } \int_0^\infty k(s)ds = 1$$

$$\varepsilon \in (0,1] \quad k_\varepsilon(s) = \frac{1}{\varepsilon} k\left(\frac{s}{\varepsilon}\right) \Rightarrow k_\varepsilon \to \delta \text{ as } \varepsilon \to 0$$

- $$\partial_t u + \int_0^\infty k_\varepsilon(s) \underbrace{\left[-\Delta u(t-s) + \phi(u(t-s)) \right]}_{B_0(x(t-s))} ds = 0$$
- $$\partial_t u \underbrace{-\alpha \Delta u + \phi(u)}_{B_1(x(t))} + \int_0^\infty k_\varepsilon(s) \underbrace{-\Delta u(t-s)}_{B_0(x(t-s))} ds = 0$$

$$\begin{cases} \partial_t x(t) + \int_0^\infty k_\varepsilon(s) B_0(x(t-s)) ds + B_1(x(t)) = 0, \ t > 0 \\ x(t) = \tilde{x}(t), \quad t \le 0 \end{cases}$$

General problems

Given a Banach space $X(\mathbb{R})$, the function $x : \mathbb{R} \to X$ solves

$$\begin{cases} \partial_t x(t) + \int_0^\infty k_\varepsilon(s) B_0(x(t-s))ds + B_1(x(t)) = 0, \ t > 0 \\ x(t) = \tilde{x}(t), \quad t \leq 0 \end{cases}$$

$$\downarrow \quad \varepsilon \to 0$$

$$\begin{cases} \partial_t x(t) + B_0(x(t)) + B_1(x(t)) = 0, \quad t > 0 \\ x(0) = \tilde{x}(0) \end{cases}$$

B_0 and B_1 (nonlinear) densely defined operators on X, often perturbations of strictly positive linear operators

The compactness problem with memory: $\mathcal{H}^2_\varepsilon \subset \mathcal{H}^1_\varepsilon \Subset \mathcal{H}^0_\varepsilon$

Natural choice: $\mathcal{H}^\imath_\varepsilon = \underbrace{X^\imath}_{x} \times \underbrace{Y^\imath_\varepsilon}_{\eta} \quad Y^\imath_\varepsilon = L^2_{\mu_\varepsilon}(\mathbb{R}^+; Y^\imath)$

Reflexive Banach: $X^2 \subset X^1 \Subset X^0 \quad Y^2 \subset Y^1 \Subset Y^0 \subset Y^{-1}$

$Y^1_\varepsilon \subset Y^0_\varepsilon$ not compact

Recover compactness: Pata-Zucchi 2000 $\mathcal{K}^\imath_\varepsilon \Subset Y^0_\varepsilon$, $\imath = 1, 2$

$\mathcal{K}^\imath_\varepsilon = \{\eta \in Y^\imath_\varepsilon : -\partial_s \eta \in Y^{\imath-2}_\varepsilon, \ \eta(0) = 0, \ \sup_{\tau \geq 1} \tau \mathbb{T}^\imath_\varepsilon(\tau; \eta) < \infty\}$

$\mathbb{T}^\imath_\varepsilon(\tau; \eta) = \int_{(0,\varepsilon/\tau)\cup(\varepsilon\tau,\infty)} \mu_\varepsilon(s) \|\eta(s)\|^2_{Y^{\imath-2}} ds \quad \tau \geq 1$

$$\|\eta\|^2_{\mathcal{K}^\imath_\varepsilon} = \|\eta\|^2_{Y^\imath_\varepsilon} + \varepsilon^2 \|\partial_s \eta\|^2_{Y^{\imath-2}_\varepsilon} + \sup_{\tau \geq 1} \tau \mathbb{T}^\imath_\varepsilon(\tau; \eta)$$

$Y^0_\varepsilon = L^2_{\mu_\varepsilon}(\mathbb{R}^+; Y^0)$, $Y^\imath_\varepsilon = \mathcal{K}^\imath_\varepsilon \Subset Y^0_\varepsilon$, $\imath = 1, 2$, $Y^\imath_{\varepsilon_1} \subset Y^\imath_{\varepsilon_2}$, $\varepsilon_1 \geq \varepsilon_2 > 0$

Can we avoid $\mathcal{K}_\varepsilon^\imath$? YES

Phase-spaces: $\mathcal{H}_\varepsilon^0$ and $\mathcal{V}_\varepsilon^\imath = X^\imath \times Y_\varepsilon^\imath, \quad \imath = 1, 2$
First basins of attractions: $\mathbb{D}_\varepsilon(r) = \{z \in \mathcal{V}_\varepsilon^2 \; : \; \|z\|_{\mathcal{V}_\varepsilon^2} \leq r\}$
Semigroups $S_\varepsilon(t) : \mathcal{H}_\varepsilon^0 \to \mathcal{H}_\varepsilon^0$ such that

(A1) $\exists \kappa \in (0,1], \exists F \in \mathcal{ND} : \; z_1 \in \mathbb{D}_\varepsilon(r), \|z_2\|_{\mathcal{H}_\varepsilon^0} \leq r$

$$\|S_\varepsilon(t)z_1 - S_\varepsilon(t)z_2\|_{\mathcal{H}_\varepsilon^0} \leq F(r)e^{F(r)t}\|z_1 - z_2\|_{\mathcal{H}_\varepsilon^0}^\kappa$$

$\Rightarrow$ (H3)

(A2) $\exists F \in \mathcal{ND} : \; S_\varepsilon(t)\mathbb{D}_\varepsilon(r) \subset \mathbb{D}_\varepsilon(F(r)), \quad \forall t \geq 0$

(A3) $\exists \rho_0 > 0 : \; \mathbb{D}_\varepsilon(\rho_0)$ uniformly exponentially attracting

(A4) $\exists \rho_1 > 0 : \forall r \geq 0 \, \exists t_r \geq 0 : \; S_\varepsilon(t)\mathbb{D}_\varepsilon(r) \subset \mathbb{D}_\varepsilon(\rho_1), \quad \forall t \geq t_r$

(A5) The operator B_0 maps bounded subsets of X^2 into bounded subsets of Y^0.

$$\partial_t \eta^t = -\partial_s \eta^t + B_0(x(t))$$

How do we recover compactness?

$$\mathbb{H}^\imath_\varepsilon(\eta) = \varepsilon^2 \|\partial_s \eta\|^2_{Y^{\imath-2}_\varepsilon} + \sup_{\tau \geq 1} \tau \mathbb{T}^\imath_\varepsilon(\tau;\eta)$$

Lemma (3^{rd} Technical Lemma, CPS 2006)

For $\imath = 1, 2$, *let* $\eta_0 \in Y^0_\varepsilon : \eta_0(0) = 0$ *in* $Y^{\imath-2}$ *and* $\mathbb{H}^\imath_\varepsilon(\eta_0) < \infty$
Given $t_0 \geq 0$, *assume* $\sup_{t \geq t_0} \|w(t)\|^2_{Y^{\imath-2}} = K < \infty$.

If $\eta = \eta^t(s)$ *solve* $\begin{cases} \partial_t \eta^t = -\partial_s \eta^t + w(t), & t > t_0 \\ \eta^{t_0} = \eta_0, \end{cases}$

$\Rightarrow \exists M_1, M_2 > 0 : \forall t \geq t_0$, $\eta^t(0) = 0$ *in* $Y^{\imath-2}$ *and*

$$\mathbb{H}^\imath_\varepsilon(\eta^t) \leq M_1 \mathbb{H}^\imath_\varepsilon(\eta_0) e^{-\frac{\delta}{2}(t-t_0)} + M_2 K$$

Why do our weaker assumptions allow to obtain Hölder continuity?

Apply EMZ+MPZ to $(\widehat{S}_\varepsilon(t_\star), \mathbb{B}_1(r_1))$

- $[(H2), (H3), (H5), (H7)]_{t=t_\star}$ give all the assumptions of EMZ but one.
  $\widehat{S}_\varepsilon = \widehat{S}_\varepsilon(t_\star) : \mathbb{B}_1(r_1) \to \mathbb{B}_1(r_0)$
  $\sup_{x\in\mathbb{B}_1(r_1)} \|\widehat{S}_{\varepsilon_1}x - \widehat{S}_{\varepsilon_2}x\|_{\mathcal{H}_1^0} \le c|\varepsilon_1 - \varepsilon_2|^b$ for $\varepsilon_1 > \varepsilon_2 > 0$?
- $\sup_{z\in\mathbb{B}_1(r_1)} \|\widehat{S}_{\varepsilon_1}z - \widehat{S}_{\varepsilon_2}z\|_{\mathcal{H}_1^0} \le F_0\left(\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_2^a}\right), \quad \forall\varepsilon_1 \ge \varepsilon_2 > 0,$
- $\|\widehat{S}_{\varepsilon_1}z - \widehat{S}_{\varepsilon_2}z\|_{\mathcal{H}_1^0} \le \|\widehat{S}_{\varepsilon_1}z - \widehat{S}_0 z\|_{\mathcal{H}_1^0} + \|\widehat{S}_{\varepsilon_2}z - \widehat{S}_0 z\|_{\mathcal{H}_1^0} \le C(\varepsilon_1^b + \varepsilon_2^b) \le C\varepsilon_1^b,$

Lemma (2nd technical lemma)

Given $\alpha \ge 0,\ \beta > 0,\ F_0 : [0,\infty) \to [0,\infty)\ \uparrow, \quad F_0(s) \le cs^\kappa,\ \forall s \in [0,1]$

$$\Rightarrow\ \exists K > 0,\ \gamma > 0 : \min\left\{F_0\left(\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_2^\alpha}\right), \varepsilon_1^\beta\right\} \le K(\varepsilon_1 - \varepsilon_2)^\gamma,\ 1 \ge \varepsilon_1 \ge \varepsilon_2 > 0$$

Regular perturbation: EMZ's Theorem
discrete globally Hölder exponential attractors

X^0 and X^1 Banach spaces such that $X^1 \Subset X^0$
$(\widehat{S}_\varepsilon(t), X^0)$ family of strongly continuous semigroups
$\exists t^\star > 0 \quad \exists r_1 > r_0 > 0$ such that $\widehat{S}_\varepsilon = \widehat{S}_\varepsilon(t^\star) : \mathbb{B}^1(r_1) \to \mathbb{B}^1(r_0)$

- $\exists \Lambda \geq 0,\ \exists \lambda < 1 : \widehat{S}_\varepsilon z_1 - \widehat{S}_\varepsilon z_2 = L_\varepsilon(z_1, z_2) + K_\varepsilon(z_1, z_2),\ z_1, z_2 \in \mathbb{B}^1(r_1)$

$$\|L_\varepsilon(z_1, z_2)\|_{X^0} \leq \lambda \|z_1 - z_2\|_{X^0}, \quad \|K_\varepsilon(z_1, z_2)\|_{X^1} \leq \Lambda \|z_1 - z_2\|_{X^0}$$

- $\exists C > 0,\ \exists \vartheta \in (0, 1] :\ \sup_{z \in \mathbb{B}^1(r_1)} \|\widehat{S}_{\varepsilon_1} z - \widehat{S}_{\varepsilon_2} z\|_{X^0} \leq C|\varepsilon_1 - \varepsilon_2|^\vartheta$

$$\Rightarrow \forall \varepsilon \in [0, 1] \quad \exists \widehat{\mathcal{D}}_\varepsilon \subset \mathbb{B}^1(r_0) \subset X^1 \text{ such that } \widehat{S}_\varepsilon \widehat{\mathcal{D}}_\varepsilon \subset \widehat{\mathcal{D}}_\varepsilon$$

- $\exists \omega > 0,\ \exists C > 0 : \quad \mathrm{dist}_{X^0}(\widehat{S}_\varepsilon{}^n \mathbb{B}^1(r_0), \widehat{\mathcal{D}}_\varepsilon) \leq C e^{-\omega n}$
- $\dim_{X^0}[\widehat{\mathcal{D}}_\varepsilon] \leq C$
- $\mathrm{dist}_{X^0}^{sym}(\widehat{\mathcal{D}}_{\varepsilon_1}, \widehat{\mathcal{D}}_{\varepsilon_2}) \leq C|\varepsilon_1 - \varepsilon_2|^\vartheta$

Hyperbolic singular perturbation: MPZ's Theorem
discrete globally Hölder exponential attractors

For $\varepsilon \in (0,1]$ phase-space $\mathcal{H}^0_\varepsilon = H^1_0(\Omega) \times \varepsilon L^2(\Omega)$ semigroup $(S_\varepsilon(t), \mathcal{H}^0_\varepsilon)$
For $\varepsilon = 0$ phase-space $\mathcal{H}^0_0 = H^1_0(\Omega) \times \{0\}$ semigroup $(S_0(t), H^1_0(\Omega))$
$\exists t^\star > 0$ such that $S_\varepsilon = S_\varepsilon(t_\star)$ and $\widehat{S}_\varepsilon = \widehat{S}_\varepsilon(t_\star)$ satisfy

1. $\exists r_1 > r_0 > 0$ such that $S_\varepsilon \mathbb{B}_\varepsilon(r_1) \subset \mathbb{B}_\varepsilon(r_0)$
2. Decomposition of the difference $S_\varepsilon z_1 - S_\varepsilon z_2$ on $\mathbb{B}_\varepsilon(r_1)$
3. Continuous dependence of $\widehat{S}_\varepsilon$ on the parameter in $\mathbb{B}_1(r_1)$
4. $\sup_{z \in \mathbb{B}_{\varepsilon_1}(r_1)} \|S_{\varepsilon_1} z - S_0 \Pi z\|_{\mathcal{H}^0_{\varepsilon_2}} \le C\varepsilon_1^b, \quad \forall \varepsilon_1 \ge \varepsilon_2 > 0$

EMZ Theorem $\Rightarrow \exists \widehat{\mathcal{D}}_\varepsilon$ discrete exponential attractor for $\widehat{S}_\varepsilon(t_\star)$ with basin $\mathbb{B}_1(r_0)$ in $\mathcal{H}^0_1$ such that $\{\widehat{\mathcal{D}}_\varepsilon\}_\varepsilon$ globally Hölder
$\Rightarrow \exists \mathcal{D}_\varepsilon$ discrete exponential attractor for $S_\varepsilon(t_\star)$ with basin $\mathbb{B}_\varepsilon(r_0)$ in $\mathcal{H}^0_\varepsilon$ such that $\{\mathcal{D}_\varepsilon\}_\varepsilon$ globally Hölder

$$\mathcal{D}_\varepsilon = S_\varepsilon(t_\star)\mathfrak{I}_\varepsilon^{-1}\widehat{\mathcal{D}}_\varepsilon \qquad (\mathfrak{I}_0^{-1} = \Pi)$$

- The functional setting: $\mathcal{H}_\varepsilon^2 \subset \mathcal{H}_\varepsilon^1 \Subset \mathcal{H}_\varepsilon^0 \quad \mathcal{H}_\varepsilon^\imath = X^\imath \times Y_\varepsilon^\imath, \ \imath = 0, 1, 2$
  Real reflexive Banach: $X^2 \subset X^1 \Subset X^0, \ Y_\varepsilon^2 \subset Y_\varepsilon^1 \Subset Y_\varepsilon^0, \quad \forall \varepsilon \in (0, 1]$
  $$Y_{\varepsilon_1}^\imath \subset Y_{\varepsilon_2}^\imath, \quad \varepsilon_1 \geq \varepsilon_2 > 0$$
- The scaling map $J_\varepsilon : Y_\varepsilon^0 \to Y_1^0$ is an isometric isomorphism and

$$\exists F \in \mathcal{ND} = \{F : [0, \infty) \to [0, \infty) \uparrow\} \quad \exists a \geq 0$$
$$\exists F_0 \in \mathcal{ND}_0 = \{F_0 \in \mathcal{ND} : F_0(s) \leq cs^\kappa, \ \forall s \in [0, 1]\} :$$
$$\|\eta - J_{\varepsilon_2} J_{\varepsilon_1}^{-1} \eta\|_{Y_1^0} \leq F(\|\eta\|_{Y_1^2}) F_0\left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_2^a}\right), \ \forall \eta \in Y_1^2, \ \forall \varepsilon_1 \geq \varepsilon_2 > 0$$

- The functional setting: $\mathcal{H}^2_\varepsilon \subset \mathcal{H}^1_\varepsilon \Subset \mathcal{H}^0_\varepsilon \quad \mathcal{H}^\imath_\varepsilon = X^\imath \times Y^\imath_\varepsilon, \ \imath = 0, 1, 2$
  Real reflexive Banach: $X^2 \subset X^1 \Subset X^0$, $Y^2_\varepsilon \subset Y^1_\varepsilon \Subset Y^0_\varepsilon, \quad \forall \varepsilon \in (0, 1]$
  $$Y^\imath_{\varepsilon_1} \subset Y^\imath_{\varepsilon_2}, \quad \varepsilon_1 \geq \varepsilon_2 > 0$$
- The scaling map $J_\varepsilon : Y^0_\varepsilon \to Y^0_1$ is an isometric isomorphism and

$$\exists F \in \mathcal{ND} = \{F : [0, \infty) \to [0, \infty) \uparrow\} \quad \exists a \geq 0$$
$$\exists F_0 \in \mathcal{ND}_0 = \{F_0 \in \mathcal{ND} : F_0(s) \leq c s^\kappa, \ \forall s \in [0, 1]\} :$$
$$\|\eta - J_{\varepsilon_2} J^{-1}_{\varepsilon_1} \eta\|_{Y^0_1} \leq F(\|\eta\|_{Y^2_1}) F_0\left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_2^a}\right), \ \forall \eta \in Y^2_1, \ \forall \varepsilon_1 \geq \varepsilon_2 > 0$$

$$\Rightarrow \exists \mathfrak{I}_\varepsilon : \mathcal{H}^0_\varepsilon \to \mathcal{H}^0_1 \quad \mathfrak{I}_\varepsilon(x,\eta) = (x, J_\varepsilon \eta) \text{ satisfies}$$

$$\|z - \mathfrak{I}_{\varepsilon_2}\mathfrak{I}^{-1}_{\varepsilon_1} z\|_{\mathcal{H}^0_1} \le F(\underbrace{\|\mathbb{Q}z\|_{Y^2_1}}_{\text{singular component}}) F_0\left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_2^a}\right), \ \forall \varepsilon_1 \ge \varepsilon_2 > 0, \ \forall z \in X^0 \times Y^2_1$$

- $\mathbb{B}_\varepsilon(r) = \{z \in \mathcal{H}^2_\varepsilon : \|z\|_{\mathcal{H}^2_\varepsilon} \le r\} = \begin{cases} \mathfrak{I}^{-1}_\varepsilon \mathbb{B}_1(r) & \text{if } \varepsilon > 0 \\ \Pi \mathbb{B}_1(r) & \text{if } \varepsilon = 0 \end{cases}$ closed in $\mathcal{H}^0_\varepsilon$
- The best constants in the embeddings $Y^2_\varepsilon \subset Y^1_\varepsilon \subset Y^0_\varepsilon$ are $\varepsilon-$ independent

Main Theorem

(H1) $\exists r_0 > 0 : \mathbb{B}_\varepsilon(r_0)$ uniformly exponentially attracting in $\mathcal{H}^0_\varepsilon$

(H2) $\exists r_1 > r_0, \exists t_\star > 0 : S_\varepsilon(t)\mathbb{B}_\varepsilon(r_1) \subset \mathbb{B}_\varepsilon(r_0), \forall t \geq t_\star$

(H3) $\exists \kappa \in (0, 1], \exists F \in \mathcal{N}\mathcal{D} : \forall z_1 \in \mathbb{B}_\varepsilon(r_1)$ and $\|z_2\|_{\mathcal{H}^0_\varepsilon} \leq r$

$$\|S_\varepsilon(t)z_1 - S_\varepsilon(t)z_2\|_{\mathcal{H}^0_\varepsilon} \leq F(r)e^{F(r)t}\|z_1 - z_2\|^\kappa_{\mathcal{H}^0_\varepsilon}$$

(H4) $\exists \kappa \in (0, 1], \exists K = K(\varepsilon) \geq 0 :$

$$\sup_{z \in \mathbb{B}_\varepsilon(r_0)} \|S_\varepsilon(t_1)z - S_\varepsilon(t_2)z\|_{\mathcal{H}^0_\varepsilon} \leq K|t_1 - t_2|^\kappa, \quad \forall t_1, t_2 \in [t_\star, 2t_\star].$$

(H5) $\exists \Lambda \geq 0, \exists \lambda \in (0, 1) : S_\varepsilon = S_\varepsilon(t_\star)$ satisfies

$$S_\varepsilon z_1 - S_\varepsilon z_2 = L_\varepsilon(z_1, z_2) + K_\varepsilon(z_1, z_2), \quad \forall z_1, z_2 \in \mathbb{B}_\varepsilon(r_1)$$

$$\|L_\varepsilon(z_1, z_2)\|_{\mathcal{H}^0_\varepsilon} \leq \lambda\|z_1 - z_2\|_{\mathcal{H}^0_\varepsilon}, \ \|K_\varepsilon(z_1, z_2)\|_{\mathcal{H}^1_\varepsilon} \leq \Lambda\|z_1 - z_2\|_{\mathcal{H}^0_\varepsilon},$$

(H6) $\sup_{t\in[t_\star,2t_\star]}\sup_{z\in\mathbb{B}_{\varepsilon_1}(r_1)}\|S_{\varepsilon_1}(t)z - S_0(t)\mathfrak{I}_0^{-1}z\|_{\mathcal{H}^0_{\varepsilon_2}} \leq C\varepsilon_1^b,\ \forall\varepsilon_1\geq\varepsilon_2>0$

(H7) I. $\sup_{t\in[t_\star,2t_\star]}\sup_{z\in\mathbb{B}_{\varepsilon_1}(r_1)}\|S_{\varepsilon_1}(t)z - S_{\varepsilon_2}(t)z\|_{\mathcal{H}^0_{\varepsilon_2}} \leq F_0\left(\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_2^a}\right)$
$\forall\varepsilon_1\geq\varepsilon_2>0$,

or

II. $\sup_{t\in[t_\star,2t_\star]}\sup_{z\in\mathbb{B}_1(r_1)}\|\widehat{S}_{\varepsilon_1}(t)z - \widehat{S}_{\varepsilon_2}(t)z\|_{\mathcal{H}^0_1} \leq F_0\left(\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_2^a}\right)$
$\forall\varepsilon_1\geq\varepsilon_2>0$

$\Rightarrow \exists\mathcal{E}_\varepsilon\subset\mathbb{B}_\varepsilon(r_0)$ such that $\{\mathcal{E}_\varepsilon\}_\varepsilon$ is (Hölder) continuous family of exponential attractors for $\{S_\varepsilon(t)\}_\varepsilon$

Memory variable: $\eta^t(s) = \int_0^t B_0(x(t-\tau))d\tau$

$0 \not\equiv \mu \doteq -k' \in L^1(\mathbb{R}^+) \downarrow$: $\exists\Theta \geq 1,\ \exists\delta > 0$: $\mu(s) \leq \Theta\mu(\sigma)e^{-\delta(s-\sigma)},\ s \geq \sigma > 0$

Rescaled memory kernel: $\mu_\varepsilon(s) = \frac{d}{ds}(k_\varepsilon(s)) = \frac{1}{\varepsilon^2}\mu\left(\frac{s}{\varepsilon}\right)$

$$\begin{cases} \partial_t x(t) + \int_0^\infty k_\varepsilon(s) B_0(x(t-s))ds + B_1(x(t)) = 0,\ t > 0 \\ x(t) = \tilde{x}(t), \quad t \leq 0 \end{cases}$$

$\downarrow$ integrating by parts

$$\begin{cases} \partial_t x(t) + \int_0^\infty \mu_\varepsilon(s)\eta^t(s)ds + B_1(x(t)) = 0 \\ \partial_t \eta^t = -\partial_s \eta^t + B_0(x(t)) \\ x(0) = x_0 = \tilde{x}(0) \\ \eta^0(s) = \eta_0(s) \doteq \int_0^s B_0(\tilde{x}(-\sigma))d\sigma \end{cases}$$

The extended phase-space

For each $\varepsilon \in (0,1]$ the memory variable $\eta \in Y^0_\varepsilon = L^2_{\mu_\varepsilon}(\mathbb{R}^+; Y^0)$

Y^0 is reflexive Banach space

$$\|\eta\|^2_{Y^0_\varepsilon} = \int_0^\infty \mu_\varepsilon(s)\|\eta(s)\|^2_{Y^0} ds$$

For each $\varepsilon \in [0,1]$, the first component $x \in X^0$ reflexive Banach space

For $\varepsilon \in [0,1]$ $\exists$ solution operator $S_\varepsilon(t) : \mathcal{H}^0_\varepsilon \to \mathcal{H}^0_\varepsilon$ where

$$\mathcal{H}^0_\varepsilon = \begin{cases} X^0 \times Y^0_\varepsilon & \varepsilon > 0 \\ X^0 \times \{0\} & \varepsilon = 0 \end{cases} \quad S_\varepsilon(t)z = \begin{cases} (u(t), \eta^t) & \varepsilon > 0 \\ (u(t), 0) & \varepsilon = 0 \end{cases}$$

Scaling (in the memory variable)

$$J_\varepsilon : Y^0_\varepsilon = L^2_{\mu_\varepsilon}(\mathbb{R}^+; Y^0) \to Y^0_1 = L^2_\mu(\mathbb{R}^+; Y^0)$$

$$(J_\varepsilon \eta)(s) = \frac{\eta(\varepsilon s)}{\sqrt{\varepsilon}}, \quad (J_\varepsilon^{-1}\eta)(s) = \sqrt{\varepsilon}\,\eta\left(\frac{s}{\varepsilon}\right)$$

Preliminary remark

(H3) $S_\varepsilon(t)$ Hölder in z at fixed $t \Rightarrow$
(H7)I $S_\varepsilon(t)$ Hölder in ε at fixed $t \Leftrightarrow$ (H7)II $\widehat{S}_\varepsilon(t)$ Hölder in ε at fixed t

Lemma (1st technical lemma)

Let $\varepsilon_1 \geq \varepsilon_2 > 0$ *and* $z \in \mathbb{B}_{\varepsilon_1}(r)$, *for some* $r > 0$ *(*$\Rightarrow z \in \mathcal{H}^0_{\varepsilon_2}$*)*
$\Rightarrow \exists c_0 > 0$ *such that*

$$\|z\|_{\mathcal{H}^0_{\varepsilon_2}} \leq c_0 r + F(r) F_0\left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_2^a}\right),$$

with a, F *and* F_0 *as in*

$$\|z - \mathfrak{I}_{\varepsilon_2}\mathfrak{I}_{\varepsilon_1}^{-1} z\|_{\mathcal{H}^0_1} \leq F(\underbrace{\|\mathbb{Q}z\|_{Y_1^2}}_{\text{memory component}}) F_0\left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_2^a}\right), \quad \forall \varepsilon_1 \geq \varepsilon_2 > 0$$

whenever $z \in X^0 \times Y_1^2$

Apply EMZ+MPZ to $(\widehat{S}_\varepsilon(t_\star), \mathbb{B}_1(r_1))$

- $[(H2), (H3), (H5), (H7)]_{t=t_\star}$ give all the assumptions of EMZ but one.
  $\widehat{S}_\varepsilon = \widehat{S}_\varepsilon(t_\star) : \mathbb{B}_1(r_1) \to \mathbb{B}_1(r_0)$
  $\sup_{x\in\mathbb{B}_1(r_1)} \|\widehat{S}_{\varepsilon_1}x - \widehat{S}_{\varepsilon_2}x\|_{\mathcal{H}_1^0} \le c|\varepsilon_1 - \varepsilon_2|^b$ for $\varepsilon_1 > \varepsilon_2 > 0$?
- $\sup_{z\in\mathbb{B}_1(r_1)} \|\widehat{S}_{\varepsilon_1}z - \widehat{S}_{\varepsilon_2}z\|_{\mathcal{H}_1^0} \le F_0\left(\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_2^a}\right), \quad \forall \varepsilon_1 \ge \varepsilon_2 > 0,$
- $\|\widehat{S}_{\varepsilon_1}z - \widehat{S}_{\varepsilon_2}z\|_{\mathcal{H}_1^0} \le \|\widehat{S}_{\varepsilon_1}z - \widehat{S}_0 z\|_{\mathcal{H}_1^0} + \|\widehat{S}_{\varepsilon_2}z - \widehat{S}_0 z\|_{\mathcal{H}_1^0} \le C(\varepsilon_1^b + \varepsilon_2^b) \le C\varepsilon_1^b,$

Lemma (2^{nd} technical lemma)

Given $\alpha \ge 0$, $\beta > 0$, $F_0 \in \mathcal{N}\mathcal{D}_0$
($\Leftrightarrow F_0 : [0,\infty) \to [0,\infty) \nearrow$ *Hölder at* 0 *with* $F_0(0) = 0$)
$\Rightarrow \exists K > 0, \gamma > 0 : \min\left\{F_0\left(\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_2^\alpha}\right), \varepsilon_1^\beta\right\} \le K(\varepsilon_1 - \varepsilon_2)^\gamma, 1 \ge \varepsilon_1 \ge \varepsilon_2 > 0$

EMZ Theorem $\Rightarrow$

- $\exists\{\widehat{\mathcal{D}}_\varepsilon\}_{\varepsilon\in[0,1]}$, $\widehat{\mathcal{D}}_\varepsilon \subset \mathbb{B}_1(r_0)$ compact in $\mathcal{H}_1^0$
- $\widehat{S}_\varepsilon \widehat{\mathcal{D}}_\varepsilon \subset \widehat{\mathcal{D}}_\varepsilon$ and $\dim_{\mathcal{H}_1^0}(\widehat{\mathcal{D}}_\varepsilon) \leq C$
- $\exists\omega > 0 : \mathrm{dist}_{\mathcal{H}_1^0}(\widehat{S}_\varepsilon^n \mathbb{B}_1(r_0), \widehat{\mathcal{D}}_\varepsilon) \leq Ce^{-\omega n}$
- $\exists\vartheta \in (0,1) : \mathrm{dist}^{\mathrm{sym}}_{\mathcal{H}_1^0}(\widehat{\mathcal{D}}_{\varepsilon_1}, \widehat{\mathcal{D}}_{\varepsilon_2}) \leq C|\varepsilon_1 - \varepsilon_2|^\vartheta$

$\mathcal{D}_\varepsilon$ for $S_\varepsilon = S_\varepsilon(t_\star)$ such that $\{\mathcal{D}_\varepsilon\}_\varepsilon$ globally Hölder

1. $\mathcal{D}_\varepsilon = S_\varepsilon \mathfrak{I}_\varepsilon^{-1} \widehat{\mathcal{D}}_\varepsilon \subset \mathbb{B}_\varepsilon(r_0)$ $(\mathfrak{I}_0^{-1} = \Pi)$
   $\widehat{\mathcal{D}}_\varepsilon \subset \mathbb{B}_1(r_0)$ and $\mathcal{D}_\varepsilon = S_\varepsilon \mathfrak{I}_\varepsilon^{-1} \widehat{\mathcal{D}}_\varepsilon \Rightarrow$
   $\mathcal{D}_\varepsilon \subset S_\varepsilon \mathfrak{I}_\varepsilon^{-1} \mathbb{B}_1(r_0) = S_\varepsilon \mathbb{B}_\varepsilon(r_0) \subset \mathbb{B}_\varepsilon(r_0)$
2. $S_\varepsilon \mathcal{D}_\varepsilon = S_\varepsilon \mathfrak{I}_\varepsilon^{-1} \widehat{S}_\varepsilon \widehat{\mathcal{D}}_\varepsilon \subset S_\varepsilon \mathfrak{I}_\varepsilon^{-1} \widehat{\mathcal{D}}_\varepsilon = \mathcal{D}_\varepsilon$
3. $\mathcal{D}_\varepsilon$ compact in $\mathcal{H}_\varepsilon^0$ and $\dim_{\mathcal{H}_\varepsilon^0}(\mathcal{D}_\varepsilon) \le C$
   $(\bigstar)$ $S_\varepsilon \mathfrak{I}_\varepsilon^{-1} : \mathbb{B}_1(r_0) \to \mathbb{B}_\varepsilon(r_0)$ satisfy
   $$\|S_\varepsilon \mathfrak{I}_\varepsilon^{-1} z_1 - S_\varepsilon \mathfrak{I}_\varepsilon^{-1} z_2\|_{\mathcal{H}_\varepsilon^0} = \|\widehat{S}_\varepsilon z_1 - \widehat{S}_\varepsilon z_2\|_{\mathcal{H}_1^0} \le C\|z_1 - z_2\|_{\mathcal{H}_1^0}$$
   $(\bigstar) \Rightarrow \mathcal{D}_\varepsilon$ compact in $\mathcal{H}_\varepsilon^0$
   $(\bigstar) \Rightarrow \dim_{\mathcal{H}_\varepsilon^0}(\mathcal{D}_\varepsilon) = \dim_{\mathcal{H}_\varepsilon^0}(S_\varepsilon \mathfrak{I}_\varepsilon^{-1} \widehat{\mathcal{D}}_\varepsilon) \le \dim_{\mathcal{H}_1^0}(\widehat{\mathcal{D}}_\varepsilon) \le C$
4. $\exists \omega > 0 :\ \mathrm{dist}_{\mathcal{H}_\varepsilon^0}(S_\varepsilon^n \mathbb{B}_\varepsilon(r_0), \mathcal{D}_\varepsilon) \le C e^{-\omega n}$, $S_\varepsilon^n = \underbrace{S_\varepsilon \circ \cdots \circ S_\varepsilon}_{n \text{ times}}$
5. $\exists \vartheta \in (0,1) :$ $\begin{cases} \mathrm{dist}^{\mathrm{sym}}_{\mathcal{H}_\varepsilon^0}(\mathcal{D}_\varepsilon, \mathcal{D}_0) \le C\varepsilon^\vartheta \\ \mathrm{dist}^{\mathrm{sym}}_{\mathcal{H}_{\varepsilon_2}^0}(\mathcal{D}_{\varepsilon_1}, \mathcal{D}_{\varepsilon_2}) \le C(\varepsilon_1 - \varepsilon_2)^\vartheta, \ \forall \varepsilon_1 \ge \varepsilon_2 > 0 \end{cases}$

$\mathcal{E}_\varepsilon$ continuous exponential attractors for $S_\varepsilon(t)$ such that $\{\mathcal{E}_\varepsilon\}_\varepsilon$ globally Hölder

1. $\mathcal{E}_\varepsilon \subset \mathbb{B}_\varepsilon(r_0)$ $\quad (\Leftarrow S_\varepsilon(t)\mathbb{B}_\varepsilon(r_1) \subset \mathbb{B}_\varepsilon(r_0), \forall t \geq t_\star)$
2. $S_\varepsilon(t)\mathcal{E}_\varepsilon = \bigcup_{\tau\in[t_\star,2t_\star]} S_\varepsilon(t+\tau)\mathcal{D}_\varepsilon \subset \mathcal{E}_\varepsilon$
3. $\mathcal{E}_\varepsilon$ compact in $\mathcal{H}^0_\varepsilon$ and $\dim_{\mathcal{H}^0_\varepsilon}\mathcal{E}_\varepsilon \leq C$
   $\Leftarrow \mathcal{S}_\varepsilon : [t_\star, 2t_\star] \times \mathbb{B}_\varepsilon(r_0) \to \mathbb{B}_\varepsilon(r_0)$ $\quad \mathcal{S}_\varepsilon(\tau, z) = S_\varepsilon(\tau)z$ $\kappa-$Hölder $\forall\varepsilon$
4. $\mathcal{E}_\varepsilon$ exponentially attracting in $\mathcal{H}^0_\varepsilon$ uniformly in ε

Transitivity of exponential attraction: FGMZ 2004

(H1)= $\mathbb{B}_\varepsilon(r_0)$ uniformly exponentially attracting $\mathcal{H}^0_\varepsilon$
$\mathrm{dist}_{\mathcal{H}^0_\varepsilon}(S_\varepsilon(t)\mathbb{B}_\varepsilon(r_0), \mathcal{E}_\varepsilon) \leq Ce^{-\kappa\omega t}$
(H3)= $S_\varepsilon(t)$ Hölder continuous in the initial data

5. $\mathcal{E}_\varepsilon$ Hölder continuous in ε $\Leftarrow$ same property for $\mathcal{D}_\varepsilon$ and
   If $\varepsilon_1 = \varepsilon > \varepsilon_2 = 0$, (H3) (H6)
   If $\varepsilon_1 > \varepsilon_2 > 0$, (H3) (H6) (H7)+ 1^{st} and 2^{nd} technical Lemmata