Linear Fokker Planck

Coupling to Stokes system

Local existence, global existence for small data

A Regularization

Baby Oldroyd B

Oldroyd B Models

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The University of Chicago

Edinburgh, September 10

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Overview

1. Linear Fokker-Planck equation, harmonic potential
2. Coupling to Stokes system
3. Local Existence
4. Global existence, small data
5. A Regularization
6. Baby Oldroyd B

Linear Fokker Planck

$$f = f(x, m, t) : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}_+$$

$$\partial_t f + \mathrm{div}_x(uf) + \mathrm{div}_m((\nabla_x u)mf) = \epsilon \mathrm{div}_m(f\nabla_m(\log f + U(m)))$$

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$$\widehat{f}(x, t, \xi) = \int_{\mathbb{R}^d} e^{-im\cdot\xi} f(x, m, t)dm$$

Lagrangian solution

$$D_t\widehat{f}(x,\xi,t) + \left[\frac{\epsilon}{R^2}\mathbb{I} - (\nabla_x u)^T\right]\xi\cdot\nabla_\xi\widehat{f} = -\epsilon|\xi|^2\widehat{f}(x,\xi,t)$$

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$$g(a,t) = (\nabla_x u)(X(a,t),t)$$

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and, integrating we obtain

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$$\widehat{f}(X(a, t), \xi, t) = e^{-\epsilon \int_0^t |\xi(a, \eta(a, \xi, t), s)|^2 ds} \widehat{f_0}(a, e^{-\frac{\epsilon t}{R^2}} \Psi(a, t)\xi)$$

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where $\widehat{f_0}(x, \xi)$ is the Fourier transform in m of the initial data $f_0(x, m) = f(x, m, 0)$.

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$$\widehat{f}(x,\xi,t) = e^{-\epsilon\int_0^t |\xi(A(x,t),\eta(A(x,t),\xi,t),s)|^2 ds} \times \widehat{f_0}(A(x,t), e^{-\frac{\epsilon t}{R^2}} M(x,t)\xi).$$

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$$\xi(A(x,t),\eta(A(x,t),\xi,t),s) = e^{-\frac{\epsilon(t-s)}{R^2}} Q(x,t,s)\xi$$

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where

$$Q(x,t,s) = q(A(x,t),t,s)$$

and

$$q(a,t,s) = \Phi(a,s)\Psi(a,t)$$

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and

$$M_{ki}(x,t) = \frac{\partial X^i}{\partial a_k}(A(x,t),t).$$

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$$\sigma^{ij}(x,t) = -\frac{1}{R^2}\frac{\partial^2 \widehat{f}}{\partial\xi_i\partial\xi_j}(x,\xi,t)_{|\xi=0}$$

$$\begin{array}{c}\sigma^{ij}(x,t) = \frac{2\epsilon}{R^2}\rho(x,t)\int_0^t e^{-\frac{2\epsilon(t-s)}{R^2}} \\ \times\left[Q^T(x,t,s)Q(x,t,s)\right]_{ij} ds \\ -\frac{e^{-\frac{2\epsilon t}{R^2}}}{R^2} M_{ki}(x,t)M_{lj}(x,t)\frac{\partial^2\widehat{f_0}}{\partial\xi_k\partial\xi_l}(A(x,t),\xi)_{|\xi=0}\end{array}$$

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where

$$Q(x,t,s) = [(\nabla_a X(a,s))^{-1}]^T(\nabla_a X(a,t))^T_{|a=A(x,t)}$$

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Passing to Lagrangian variables, integrating by parts and returning to Eulerian variables:

$$\tau(x,t) = \sigma(x,t) - \rho(x,t)\mathbb{I} =$$
$$2\rho(x,t)\int_0^t e^{-\frac{2\epsilon(t-s)}{R^2}} Q^T(x,t,s)S(X(A(x,t),s),s)Q(x,t,s)ds$$
$$+e^{-\frac{2\epsilon t}{R^2}}(\nabla_a X(A(x,t),t))\tau_0(A(x,t))(\nabla_a X(A(x,t),t))^T$$

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where

$$S(x,t) = \frac{1}{2}\left[(\nabla_x u(x,t)) + (\nabla_x u(x,t))^T\right]$$

is the rate of strain.

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$$[Q(\cdot,t,s)]_\alpha \le Ce^{(1+2\alpha)\int_0^t\gamma(z)dz}\int_s^t [\nabla_x u(\cdot,z)]_\alpha dz.$$

Theorem

Let $u \in L^1(0, T; C^{1,\alpha}(\mathbb{R}^d)$. Then $\tau = \sigma - \rho\mathbb{I}$ obeys

$$\begin{array}{c} \|\tau(\cdot,t)\|_{L^\infty} \\ \le C\|\rho_0\|_{L^\infty} \int_0^t \exp\left\{-\frac{2\epsilon(t-s)}{R^2} + 2\int_s^t \gamma(z)dz\right\}\gamma(s)ds \\ +C\|\tau_0\|_{L^\infty} \exp\left\{-\frac{2\epsilon t}{R^2} + 2\int_0^t \gamma(s)ds\right\}, \end{array}$$

$$\begin{array}{c} [\tau(\cdot,t)]_\alpha \le Ce^{(2+2\alpha)\int_0^t \gamma(s)ds}\left\{[\rho_0]_\alpha \int_0^t \exp\left(-\frac{2\epsilon(t-s)}{R^2}\right)\gamma(s)ds \right. \\ +\|\rho_0\|_{L^\infty} \int_0^t \exp\left(-\frac{2\epsilon(t-s)}{R^2}\right) \\ \times\left(\gamma(s)\int_s^t [\nabla_x u(\cdot,z)]_\alpha dz + [\nabla_x u(\cdot,s)]_\alpha\right) ds \\ \left. + \left[[\tau_0]_\alpha + \|\tau_0\|_{L^\infty}\int_0^t [\nabla_x u(\cdot,s)]_\alpha dt\right] \exp\left(-\frac{2\epsilon t}{R^2}\right)\right\} \end{array}$$

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$$\partial_j u^i = kR_j\left(R_l(\tau^{il}) + R_i R_m R_n(\tau^{mn})\right)$$

$$\nabla_x u = k\mathcal{R}\tau$$

Theorem

There exists a constant C depending only on d and α such that, for any $\widetilde{\sigma}$,

$$\|\mathcal{R}\widetilde{\sigma}\|_{L^\infty(dx)} \le C\|\widetilde{\sigma}\|_{L^\infty(dx)}\left\{1+\log\left[1+\frac{\|\widetilde{\sigma}\|_{L^1(\mathbb{R}^d)}^{\frac{\alpha}{d+\alpha}}[\widetilde{\sigma}]_\alpha^{\frac{d}{d+\alpha}}}{\|\widetilde{\sigma}\|_{L^\infty(dx)}}\right]\right\}$$

where

$$[\widetilde{\sigma}]_\alpha = \max_{mn}\sup_{x\neq y}\frac{|\widetilde{\sigma}^{mn}(x)-\widetilde{\sigma}^{mn}(y)|}{|x-y|^\alpha}$$

and $0<\alpha<1$. There exists a constant C depending only on d and α such that

$$[\mathcal{R}\widetilde{\sigma}]_\alpha \le C[\widetilde{\sigma}]_\alpha$$

Energetics

$$
\partial_t \int_{\mathbb{R}^d} Tr\,\sigma dx + \frac{2}{k}\int_{\mathbb{R}^d} |\nabla_x u|^2 dx
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$$\|\tau(\cdot,t)\|_{L^1(\mathbb{R}^d)} \le C[\|\tau_0\|_{L^1(\mathbb{R}^d)} + \|\rho_0\|_{L^1(\mathbb{R}^d)}]$$

$$M_1 = [\|\rho_0\|_{L^1(\mathbb{R}^d)} + \|\tau_0\|_{L^1(\mathbb{R}^d)}]$$

$$M_\infty = \|\rho_0\|_{L^\infty(\mathbb{R}^d)} + \|\tau_0\|_{L^\infty(\mathbb{R}^d)}$$

and

$$M_\alpha = [\rho_0]_\alpha + [\tau_0]_\alpha$$

Deborah number

$$D := \frac{k}{\kappa_0} = \frac{kR^2}{\epsilon},$$

$$\kappa_0 = \frac{\epsilon}{R^2}.$$

Theorem

Let $(\tau_0, \rho_0) \in L^1(\mathbb{R}^d)^2 \cap C^\alpha(\mathbb{R}^d)^2$. There exist constants $\varepsilon > 0$, $\Gamma \geq 2$ and there exists a time $T_0 > 0$ and a weak solution $(\rho, \tau) \in C([0, T_0], W^{-1,p}(\mathbb{R}^d)^2) \cap L^\infty([0, T_0]; (C^\alpha(\mathbb{R}^d) \cap L^1(\mathbb{R}^d))^2)$, $1 < p < \infty$, satisfiying the equations in weak sense, and such that

$$Dk_0 T_0 M_\infty \left\{ 1 + \log\left(1 + M_\alpha^{\frac{d}{d+\alpha}} M_1^{\frac{\alpha}{d+\alpha}} M_\infty^{-1} \right) \right\} \geq \varepsilon$$

and

$$\|\rho(\cdot, t)\|_{L^1(\mathbb{R}^d)} + \|\tau(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} \leq \Gamma M_1,$$

$$\|\rho(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} + \|\tau(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} \leq \Gamma M_\infty$$

and

$$[\rho(\cdot, t)]_\alpha + [\tau(\cdot, t)]_\alpha \leq \Gamma M_\alpha$$

hold on $[0, T_0]$.

Theorem

Let $(\tau_0, \rho_0) \in (L^1(\mathbb{R}^d))^2 \cap (C^\alpha(\mathbb{R}^d))^2$, $0 < \alpha < 1$. *There exists* ε_1 *such that, if*

$$DM_\infty \left\{1 + \log\left(1 + M_\infty^{-1} M_\alpha^{\frac{d}{d+\alpha}} M_1^{\frac{\alpha}{d+\alpha}}\right)\right\} \le \varepsilon_1$$

then there exists a unique global weak solution of $(\tau, \rho) \in L^\infty([0,\infty), (L^1(\mathbb{R}^d) \cap C^\alpha(\mathbb{R}^d))^2) \cap C([0,\infty), (W^{-1,p}(\mathbb{R}^d))^2)$, $p < \infty$.

$$\frac{1}{\kappa_0}\|\nabla_x u(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \le Ke^{-\kappa_0 t}$$

$$\frac{1}{\kappa_0}[\nabla_x u(\cdot,t)]_\alpha \le Ke^{-\kappa_0 t}$$

$$[\tau(\cdot,t)]_\alpha \le Ke^{-\kappa_0 t},$$

$$\|\tau(\cdot,t)\|_{L^p(\mathbb{R}^d)} \le K_p e^{-\kappa_0 t}$$

$$\|\rho(\cdot,t)\|_{L^p} = \|\rho_0\|_{L^p}, \qquad [\rho(\cdot,t)]_\alpha \le K.$$

Uniqueness

$$\frac{d}{dt}X = F[X]$$

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with $X(a,0) = a$, where $X = X(a,t)$ is viewed as an element of $\mathcal{X} := C([0,T], C^{1,\alpha}(\mathbb{R}^d)^d)$.

$$\begin{array}{c}\sigma[X](a,t) \\ = 2\kappa_0\rho_0(a)\int_0^t e^{-2\kappa_0(t-s)}q(a,t,s)q(a,t,s)^T ds \\ +e^{-2\kappa_0 t}(\nabla_a X(a,t))\sigma_0(a)(\nabla_a X(a,t))^T\end{array}$$

$$q(a,t,s) = q[X](a,t,s) = (\nabla_a X(a,t))(\nabla_a X(a,s))^{-1}$$

$$\tau[X](a,t) = \sigma[X](a,t) - \rho_0(a)\mathbb{I},$$

$$\tau_X(x,t) = \tau[X](X^{-1}(x,t),t),$$

$$u_X^i(x,t) = k\Lambda^{-1}(R_l\tau_X^{il} + R_iR_mR_n(\tau_X^{mn})).$$

We write symbolically

$$u_X = k\Lambda^{-1}\mathbb{H}(\tau[X]\circ X^{-1}) = k\Lambda^{-1}\mathbb{H}(\tau_X)$$

where $\mathbb{H}$ stands for the combinations of Riesz transforms

$$\mathbb{H}_{imn} = \delta_{im}R_n + R_iR_mR_n$$

Finally, we compose with $X(a,t)$

$$F^i[X](a,t) = u_X^i(X(a,t),t).$$

Thus, $F[X]$ is obtained via the succession of compositions

$$X \mapsto \tau[X] \mapsto \tau_X = \tau[X]\circ X^{-1}$$
$$\mapsto u_X = k\Lambda^{-1}\mathbb{H}(\tau_X) \mapsto F = u_X\circ X$$

The norm in $\mathcal{X}$ is

$$\|X\|_{\mathcal{X}} := \sup_{0\le t\le T} \|X(\cdot, t)\|_{C^{1,\alpha}(\mathbb{R}^d)}$$

We consider a fixed constant M and the set

$$\mathcal{D} := \{X \in \mathcal{X} \mid X(0,a) = a, \\ \frac{1}{2} \le \det \nabla_a X(a,t) \le \frac{3}{2}, \ \|X\|_{\mathcal{X}} \le M\}$$

The initial data for the PDE serve as parameters in the definition of F. We wish to show that two solutions $X_1 \in \mathcal{D}$ and $X_2 \in \mathcal{D}$ correspondingto the same ρ_0, τ_0, are identical. In order to do so we establish

$$\|(DF[X])Y\|_{\mathcal{X}} \le L\|Y\|_{\mathcal{X}}$$

with a uniform constant L that depends on M.

$$(DF(X)Y)(a,t) = k\left\{\Lambda^{-1}\mathbb{H}((D\tau[X]Y)\circ A)\right\}(X(a,t),t) \\ +(K[X]Y)(a,t)$$

$$(K[X]Y)(a,t) = (\nabla_x u_X)(X(a,t),t)Y(a,t) \\ -k(\Lambda^{-1}\mathbb{H}((\nabla_x \tau_X)(Y\circ X^{-1})))(X(a,t),t)$$

$Y \mapsto K[X]Y$ is a bounded linear operator in $C(0,T;(C^{1,\alpha})^d)$ with norm uniformly bounded for $X \in \mathcal{D}$. Becuse composition with X^{-1} and composition with X are both bounded linear operators $C^{1,\alpha} \to C^{1,\alpha}$, with norms controlled by M, the boundedness of $K[X]Y$ is equivalent to the boundedness of the map

$$\phi \mapsto L[X]\phi$$

where

$$\phi(x,t) = Y(X^{-1}(x,t),t)$$

and

$$(L[X]\phi)(x,t) = k(\nabla_x \Lambda^{-1}\mathbb{H}\tau_X)(x,t)\phi(x,t) \\ -k(\Lambda^{-1}\mathbb{H}((\nabla_x \tau_X)\phi))(x,t).$$

$$(L[X]\phi)^i = k\left[\phi^p\partial_p\Lambda^{-1}\mathbb{H}_{imn}\tau_X^{mn} - \Lambda^{-1}\mathbb{H}_{imn}(\phi^p\partial_p\tau_X^{mn})\right]$$

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is bounded as a linear map from $C(0,T;[C^{1,\alpha}]^d)$ to itself.

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is bounded as a linear map from $C(0,T;[C^{1,\alpha}]^d)$ to itself. For the undifferentiated part, $\tau_X \in L^1 \cap L^\infty$, $(\partial_p\phi^p)\tau_X \in L^1 \cap L^\infty$. The commutators

$$\phi \mapsto \phi^p\mathcal{R}_{pimn}(\tau_X^{mn}) - \mathcal{R}_{pimn}(\phi_p\tau_X^{mn})$$

are bounded as operators from $C(0,T;[C^{1,\alpha}]^d)$ to itself. This quite obvious for smooth τ_X but a little less obvious for $\tau_X \in \Sigma = C(0,T;[C^\alpha(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)]^{d^2})$.

The commutators are

$$\int_{\mathbb{R}^d} K(x-y)(\phi(x)-\phi(y))\tau_X(y)dy$$

The kernel K is smooth away from the origin and is homogeneous of order $-d$. Differentiating in some direction and writing K' for the singular (of order $d+1$) kernel obtained by differentiating K, we have

$$T[X]\phi = P.V. \int K'(x-y)(\phi(x)-\phi(y))\tau_X(y)dy$$

The kernel $(x-y)K'(x-y)$ is homogeneous of order $-d$. It might have nonzero average on the unit sphere. Nevertheless, we subtract the value $\nabla_x\phi(x)$:

$$\begin{array}{c} T_1[X]\phi = \int_0^1 d\lambda \left\{ P.V. \int_{\mathbb{R}^d}(x-y)K'(x-y) \right. \\ \left. \times \left[\nabla_x\phi(x+\lambda((y-x))) - \nabla_x\phi(x)\right]\tau_X(y)dy \right\} \end{array}$$

The contributions left from the average on the unit sphere, when nonzero, are a constant multiple of $(\nabla_x\phi(x))\tau_X(x)$ and $\nabla_x\phi(x)T_2\tau_X(x)$,

$$T_2(\tau_X)(x) = \int_{\mathbb{R}^d}(x-y)K'(x-y)(\tau_X(x)-\tau_X(y))dy$$

both bounded with values in C^α. The fact that $T_1[X]\phi$ is bounded in C^α, and similarly, that $T_2(\tau_X)$ is a Hölder continuous function are classical. We have one more term in $DF[X]$, namely

$$Y \mapsto k(\Lambda^{-1}\mathbb{H}((D\tau[X]Y)\circ X^{-1}))\circ X$$

Its boundedness is equivalent to the boundedness of maps

$$\phi \mapsto \Lambda^{-1}\mathbb{H}(g_X\nabla\phi)$$

in $C^{1,\alpha}$ where g_X is in Σ. Derivatives:

$$\phi \mapsto \mathcal{R}(g_X\nabla\phi)$$

No derivatives: the L^∞ boundedness follows as above from $g_X\nabla\phi \in L^1\cap L^\infty$.

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conserved on particle paths.

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For the case of compact manifolds, $\sigma \in L^\infty$ is automatic.For the co-rotational case there is a maximum principle.

Global existence with locally growing R

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$$|\delta'(g)| \le 2C_0$$

for all $g \ge 0$.

Theorem

(P.C., W. Sun) Assume that the initial distribution f_0 *and* R_0 *satisfy*

$$\left\{\begin{array}{c} \sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d} f_0(x,m)(1+|m|^2)dm < \infty \\ R_0(x) \geq R_{min} > 0, \\ \int_{\mathbb{R}^d} |\nabla_x R_0(x)|^p dx < \infty, \\ \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}(1+|m|^2)|\nabla_x f_0(x,m)|dm\right)^p dx < \infty \end{array}\right.$$

with $p > d$. *Then the solution exists for all time, is unique and obeys the a priori bounds*

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with $p > d$. Then the solution exists for all time, is unique and obeys the a priori bounds

$$\sup_x Tr(\sigma(x,t)) \leq e^{2c\kappa t}\left[\sup_x Tr(\sigma_0(x)) + \frac{d\epsilon}{c\kappa R_{min}^2}\|\rho_0\|_{L^\infty(\mathbb{R}^d)}\right]$$

$$\|\nabla_x \sigma(\cdot,t)\|_{L^p(dx)} \leq C$$

Idea of Proof

Improved dissipation:

$$D_t\sigma = (\nabla_x u)\sigma + \sigma(\nabla_x u)^T - \frac{2\epsilon}{R^2}\sigma + \frac{2\epsilon}{R^2}\rho\mathbb{I} - \frac{2D_tR}{R}\sigma$$

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Results in

$$\sup_x Tr(\sigma(x,t)) \le e^{2c\kappa t}\left[\sup_x Tr(\sigma_0(x)) + \frac{d\epsilon}{c\kappa R_{min}^2}\|\rho_0\|_{L^\infty(\mathbb{R}^d)}\right]$$

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Improved dissipation:

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$$\frac{|\nabla_x R(x,t)|}{R^3(x,t)} \le Ce^{c\kappa t}\times$$
$$\left(\frac{|\nabla_a R_0(A(x,t))|}{R_{min}^3} + \frac{1}{R_{min}^2}\int_0^t |\nabla_x\nabla_x u(X(A(x,t),s),s)|ds\right)$$

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$A(x,t)$ "back-to-labels".

$$D_t(y(x,t)) \le 3c\kappa y(x,t)$$
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Scalar, 1D. Does not obey the energy principle.

Dissipative structure, 2D

$$\left\{\begin{array}{c} a(x,t) = \frac{1}{2}\left(\sigma^{11}(x,t) - \sigma^{22}(x,t)\right), \\ b(x,t) = \sigma^{12}(x,t) = \sigma^{21}(x,t), \\ c(x,t) = \sigma^{11}(x,t) + \sigma^{22}(x,t) = Tr\left(\sigma(x,t)\right) \end{array}\right.$$

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$$\omega = \partial_1 u^2 - \partial_2 u^1, \quad \lambda = \frac{1}{2}(\partial_1 u^1 - \partial_2 u^2), \quad \mu = \frac{1}{2}(\partial_1 u^2 + \partial_2 u^1)$$

2D relations

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$$A = \frac{1}{2}\left(\partial_1^2 - \partial_2^2\right)(-\Delta)^{-1} = \frac{1}{2}(R_1^2 - R_2^2)$$

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General considerations

$$D_t\left(\frac{c^2}{4} - a^2 - b^2\right) = -\frac{4\epsilon}{R^2}\left(\frac{c^2}{4} - a^2 - b^2\right) + \frac{2\epsilon}{R^2}\rho c$$

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The positivity of the matrix is equivalent (in this case) to the positivity of the determinant, i.e. to

$$\frac{c^2}{4}-a^2-b^2 > 0.$$

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$$-\frac{\epsilon}{R^2}\left(\frac{c^2}{4}-a^2-b^2\right)+\frac{2\epsilon}{R^2}\rho c \le 0$$

on the particle path (because ρ is bounded) and then, we would arrive at the contradiction that c remains bounded. From its PDE we can write

$$c(x,t) = e^{-\frac{2\epsilon}{R^2}t}c_0(A(x,t)) + \\ \int_0^t e^{-\frac{2\epsilon}{R^2}(t-s)}\left(\lambda a+\mu b-\frac{\epsilon}{R^2}\rho_0\right)(X(A(x,t),s),s)ds$$

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Thus, exponential growth or blowup of c is possible only if $(\lambda a+\mu b)$ has time integrals on particle paths that are positive and grow exponentially or stronger, without bound.

Stokes relations

$$\omega = 2k\,(Ab - Ba)$$

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$$\int_{\mathbb{R}^2} c(x,t)dx + 4\int_0^t e^{-\frac{t-s}{\varepsilon}} \int_{\mathbb{R}^2} |B(a)(x,s) - A(b)(x,s)|^2 dx$$
$$= e^{-\frac{t}{\varepsilon}} \int_{\mathbb{R}^2} c_0(x)dx + 2(1 - e^{-\frac{t}{\varepsilon}}) \int_{\mathbb{R}^2} \rho_0(x)dx$$

Baby Oldroyd B

$$\tau(x,t) = \tau_0(t) + \sum_{k\in\mathbb{Z}^2\setminus\{0\}} \tau_k(t)e^{ik\cdot x}$$

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$$\frac{d\tau_l}{dt} = -\sum_{k+j=l} \tau_k \alpha^2(j)\tau_j$$

Theorem

(P.C. W. Sun) Let $\tau(x,0) = \sum_k \tau_k(0)e^{ik\cdot x}$ *satisfy*

$$\tau_{-k}(0) = \overline{\tau_k(0)}, \quad \sum_k (1+|k|)^s|\tau_k(0)| \le C_s(0) < \infty, \quad s > 0,$$

and $\tau_0(0) \ge \sum_{k\neq 0}|\tau_k(0)|$. *Then the solution exists for all time and obeys*

$$\tau_{-k}(t) = \overline{\tau_k(t)}$$

$$\sum_k (1+|k|)^s|\tau_k(t)| \le C_s(0)e^{2^{s+1}\tau_0(0)\Gamma^2 t} < \infty$$

and

$$\tau_0(t) \ge \sum_{k\neq 0}|\tau_k(t)|$$

which implies that $\tau(x,t)$ *remains smooth and positive.*

Idea of proof

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Calculation... Results in:

$$\frac{d}{dt}\left(\sum_{l\neq 0} |\tau_l| - \tau_0\right) \leq \left(\sum_{j\neq 0} \alpha^2(j)|\tau_j|\right)\left(\sum_{l\neq 0} |\tau_l| - \tau_0\right)$$

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If the initial data is non-positive, the quantity remains non positive. Thus we have the invariance of this cone.

$$\sum_{l\neq 0} |\tau_l(t)| \leq \tau_0(t) \leq \tau_0(0)$$

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$$\frac{d}{dt}\sum_{l\neq 0}(1+|l|)^s|\tau_l| \leq 2^{s+1}\tau_0(0)\Gamma^2\left(\sum_{l\neq 0}(1+|l|)^s|\tau_l|\right)$$