Strong and weak trajectory attractors for ill-posed problems: Dissipative reaction-diffusion systems

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Plan of the lecture

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§1. INTRODUCTION

The method of trajectory dynamical systems and trajectory attractors helps to study effectively the behaviour of solutions as $t \to +\infty$ of non-linear partial differential equations for which the corresponding boundary-value problem is ill-posed or it well-posedness is not proved yet. For example, for the 3D Navier–Stokes system, we know how to construct global weak solutions but the uniqueness result remains an open problem for years. At the same time, as a role, for meaningful equations and problems, we were able to establish the attraction to the corresponding trajectory attractors only in the "maximal" possible weak topology of the problem we are dealing with.

In the present report, for a general dissipative reaction-diffusion system (RD-system), we prove that its trajectory (global solutions) tend to the trajectory attractor in the "maximal" strong topology of the corresponding trajectory phase space. The considered RD-systems are ill-posed since they do not satisfy conditions that provide the unique solvability of the corresponding Cauchy problem.

To prove the strong attraction result, we use the general method of energy identities that was applied earlier in the works of Ghydaglia (J.Diff.Eq., 1994) and Moise–Rosa–Wang (Nonlinearity, 1998) and by other authors in the study of global attractors for well-posed problems in unbounded domains when the standard compactness embedding theorems in Sobolev spaces do not work.

§1. DISSIPATIVE RD-SYSTEM AND ITS WEAK SOLUTIONS

We consider the following PDE system:

$$\partial_t u = \mathbf{d}\Delta u - f(u) + g(x), \quad x \in \Omega \Subset \mathbb{R}^3,\ t \geq 0, \tag{1}$$

$$u|_{\partial\Omega} = 0. \tag{2}$$

The vector function $u = (u^1(x,t), \ldots, u^N(x,t))$ is unknown. The function $f(u) = (f^1(u), \ldots, f^N(u))$ describes a non-linear interaction of components u. A non-homogeneous "external force" $g(x) = (g^1(x), \ldots, g^N(x))$ is a known function. The symbol $\mathbf{d}$ denotes an $N \times N$-matrix with positive symmetric part $(\mathbf{d} + \mathbf{d}^*)/2 \geq \beta\mathbf{I}$, $\beta > 0$. On the boundary of the domain $\partial\Omega$, we consider the Dirichlet conditions (for simplicity).

Systems of the form (1) model complex multi-component chemical reactions in non-homogeneous environments, when diffusion and cross-diffusion (non-diagonal matrix $\mathbf{d}$) of chemical components are admissible. The non-homogeneous action $g(x)$ is also possible, for instance, in a form of radiation. We note also that the complex Ginzburg–Landau equation can be written in the form (1).

We define spaces: $\mathbf{H} := [L_2(\Omega)]^N$ and $\mathbf{V} := [H_0^1(\Omega)]^N$. The norms in these spaces are denoted, respectively, by

$$|v|^2 := \int_\Omega \sum_{i=1}^N |v^i(x)|^2 dx \quad \text{and} \quad \|v\|^2 := \int_\Omega \sum_{i=1}^N |\nabla v^i(x)|^2 dx.$$

Besides, $\mathbf{V}' = [H^{-1}(\Omega)]^N$ denotes the dual space for $\mathbf{V}$.

We assume that $g(\cdot) \in \mathbf{V}'$, $f(\cdot) \in C(\mathbb{R}^N; \mathbb{R}^N)$ and the following inequalities hold

$$\sum_{i=1}^N |f^i(v)|^{\frac{p_i}{p_i-1}} \leq C_0 (\sum_{i=1}^N |v^i|^{p_i} + 1), \tag{3}$$

$$\sum_{i=1}^N \gamma_i |v^i|^{p_i} - C_1 \leq \sum_{i=1}^N f^i(v) v^i, \ \forall v \in \mathbb{R}^N. \tag{4}$$

where $\gamma_i > 0, i = 1, \ldots, N$. For definiteness, we assume that $2 \leq p_1 \leq \cdots \leq p_N$.

Inequality (3) is caused by the fact that in real RD-systems the non-linear functions $f^i(u)$ are polynomials.

Inequality (4) is called the dissipation condition for RD-system (1).

We set $q_i = p_i/(p_i - 1),\ 1/p_i + 1/q_i = 1,\ 1 < q_i \leq 2, i = 1,\ \ldots, N.$

We shall use the notations $\mathbf{p} = (p_1, \ldots, p_N),\ \mathbf{q} = (q_1, \ldots, q_N)$ and the spaces:

$$\mathbf{L_p}(\Omega) := L_{p_1}(\Omega) \times \cdots \times L_{p_N}(\Omega),$$
$$\mathbf{L_p}(0, M; \mathbf{L_p}(\Omega)) := L_{p_1}(0, M; L_{p_1}(\Omega)) \times \cdots \times L_{p_N}(0, M; L_{p_N}(\Omega)).$$

We also denote the space

$$\mathcal{L}(0, M) := \mathbf{L}_2(0, M; \mathbf{V}') + \mathbf{L_q}(0, M; \mathbf{L_q}(\Omega)).$$

If a function $u(x, t) \in \mathbf{L_p}(0, M; \mathbf{L_p}(\Omega))$, then condition (3) implies, that the function $f(u(x, t)) \in \mathbf{L_q}(0, M; \mathbf{L_q}(\Omega))$ and moreover, we have the inequality

$$\sum_{i=1}^{N} \|f_i(u(\cdot))\|^{q_i}_{L_{q_i}(0,M;L_{q_i})} \leq C_0 \left(\sum_{i=1}^{N} \|u(\cdot)\|^{p_i}_{L_{p_i}(0,M;L_{p_i})} + |\Omega| M \right). \tag{5}$$

Besides, if $u(x, t) \in \mathbf{L}_2(0, M; \mathbf{V})$, then $\mathbf{d}\Delta u(x, t) + g(x) \in \mathbf{L}_2(0, M; \mathbf{V}')$.

DEFINITION 1. A function $u(x,t)$, $x \in \Omega$, $t \geq 0$, is called (*global*) *weak* solution to (1), if

$$u(x,t) \in \mathbf{L_p}(0,M;\mathbf{L_p}(\Omega)) \cap \mathbf{L_2}(0,M;\mathbf{V})$$

for each $M > 0$, and, for every (test) function $\varphi(x) \in \mathbf{L_p}(\Omega) \cap \mathbf{V}$, the following identity holds (in the distribution sense):

$$\frac{d}{dt}\int_\Omega u(x,t)\cdot\varphi(x)dx + \int_\Omega \{\mathbf{d}\nabla u(x,t)\cdot\nabla\varphi(x) + f(u(x,t))\cdot\varphi(x)\}\,dx = \langle g,\varphi\rangle\,.$$

It follows from equation (1) that

$$\partial_t u(x,t) \in \mathcal{L}(0,M) := \mathbf{L_2}(0,M;\mathbf{V}') + \mathbf{L_q}(0,M;\mathbf{L_q}(\Omega)).$$

Using the Sobolev embedding theorem, we conclude that

$$\mathcal{L}(0,M) \subset \mathbf{L_q}(0,M;\mathbf{H}^{-\mathbf{r}}(\Omega)),$$

where

$$\mathbf{H}^{-\mathbf{r}}(\Omega) = H^{-r_1}(\Omega)\times\cdots\times H^{-r_N}(\Omega),\ \ \mathbf{r} = (r_1,\ldots,r_N),$$
$$r_i \equiv \max\{1, 3/q_i - 3/2\}\,,\ i = 1,\ldots,N.$$

Consequently, for any weak solution to (1)

$$\partial_t u(x,t) \in \mathbf{L_q}(0, M; \mathbf{H^{-r}}(\Omega)).$$

If $u(t)$ is a weak solution to system (1), then

$$u(\cdot) \in \mathbf{C}([0, M]; \mathbf{H^{-r}}(\Omega)).$$

Besides, if it is known that $u(\cdot) \in \mathbf{L}_\infty(0, M; \mathbf{H})$, then the classical Lions–Magenes lemma implies that $u(\cdot) \in \mathbf{C}_w([0, M]; \mathbf{H})$. Hence, for equation (1) we can set the initial data

$$u|_{t=0} = u_0, \quad \text{where } u_0 \in \mathbf{H}. \tag{6}$$

Using the Galerkin approximation method, we prove

<u>PROPOSITION 1.</u> For any $u_0 \in \mathbf{H}$, system (1) has a global weak solution

$$u(x,t) \in \mathbf{L}_\infty(\mathbb{R}_+; \mathbf{H}) \cap \mathbf{L}_2^{\mathrm{loc}}(\mathbb{R}_+; \mathbf{V}) \cap \mathbf{L}_{\mathbf{p}}^{\mathrm{loc}}(\mathbb{R}_+; \mathbf{L_p}(\Omega)),$$

that satisfy initial data (6).

<u>REMARK 1.</u> Under conditions (3)–(4), a weak solution to problem (1) and (6) can be non-unique, since the function $f(u)$ is not necessary Lipschitz. The uniqueness theorem is true, for example, under the extra condition that

$$f(u) \in C^1(\mathbb{R}^N; \mathbb{R}^N),$$

and Jacobi matrix $\mathbf{J}(v) = \partial f(v)/\partial v$ satisfies the inequality

$$\mathbf{J}(v) + \alpha \mathbf{E} \geq 0, \ \forall v \in \mathbb{R}^N \quad (\alpha > 0).$$

However this condition looks very restrictive, so it is not assumed, when we construct the trajectory attractor for RD-system (1).

It is known that any weak solution $u(x,t), t \geq 0$, to RD-system (1) is a strongly continuous function in $\mathbf{C}(\mathbb{R}_+; \mathbf{H})$ and real-valued norm function $|u(\cdot,t)|^2,\ t \geq 0$, is absolutely continuous and the following energy equality holds:

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + (\mathbf{d}\nabla u(t), \nabla u(t)) + (f(u(t)), u(t)) = \langle g, u(t)\rangle\ ,\ t \geq 0. \qquad (7)$$

It follows from (7) that any weak solution $u(\cdot,t)$ to (1) satisfies the following inequalities:

$$|u(t)|^2 \leq |u(0)|^2 e^{-\lambda_1 \beta t} + R_1^2, \qquad (8)$$

$$\beta \int_t^{t+1} \|u(s)\|^2 ds + 2\sum_{i=1}^{N} \gamma_i \int_t^{t+1} \|u_i(s)\|_{L_{p_i}}^{p_i} ds \leq |u(t)|^2 + R_2^2, \qquad (9)$$

where λ_1 is the first eigenvalue of the scalar operator $-\Delta$ with Dirichlet boundary conditions. The values R_1 and R_2 depend on $\|g\|_{\mathbf{V}'}$ and are independent of $u(0)$.

To construct the trajectory attractor for RD-system (1), we have to describe the trajectory space (the phase space) for this system.

<u>DEFINITION 2</u>. <u>*Trajectory space*</u> $\mathcal{K}^+$ of RD-system (1) is the union of all its global weak solutions $u(t), t \geq 0$, belonging to the class

$$u(\cdot) \in \mathbf{L}_\infty(\mathbb{R}_+; \mathbf{H}) \cap \mathbf{L}_2^{\mathrm{loc}}(\mathbb{R}_+; \mathbf{V}) \cap \mathbf{L}_{\mathbf{p}}^{\mathrm{loc}}(\mathbb{R}_+; \mathbf{L}_{\mathbf{p}}(\Omega)).$$

We conclude from PROPOSITION 1 that, for every $u_0 \in \mathbf{H}$, there is a trajectory $u(\cdot) \in \mathcal{K}^+$, such that $u(0) = u_0$. Therefore, the system (1) has plenty of trajectories.

§3. WEAK TRAJECTORY ATTRACTOR CONSTRUCTION

Consider a linear space $\mathcal{F}_+^{\text{loc}}$, consisting of functions $v(x,t), x \in \Omega, t \geq 0$,

$$v(\cdot) \in L_\infty(0,M;\mathbf{H}) \cap \mathbf{L_2}(0,M;\mathbf{V}) \cap \mathbf{L_p}(0,M;\mathbf{L_p}(\Omega)), \tag{10}$$

$$\partial_t v(\cdot) \in \mathcal{L}(0,M) = \mathbf{L_2}(0,M;\mathbf{V}') + \mathbf{L_q}(0,M;\mathbf{L_q}(\Omega)),\ \forall M > 0. \tag{11}$$

In the space $\mathcal{F}_+^{\text{loc}}$, we consider the topology $\Theta_+^{\text{s,loc}}$, that is the topology of local <u>*strong*</u> convergence of sequences $\{v_m(\cdot)\}$ and $\{\partial_t v_m(\cdot)\}$ as $m \to \infty$, in the norms of Banach spaces (10) and (11), respectively, for every $M > 0$. It is easy to verify that the topology $\Theta_+^{\text{s,loc}}$ is metrizable and the corresponding metric space $\mathcal{F}_+^{\text{s,loc}} := \mathcal{F}_+^{\text{loc}} \cap \Theta_+^{\text{s,loc}}$ is complete.

Similarly, in the space $\mathcal{F}_+^{\text{loc}}$, we define the topology $\Theta_+^{\text{w,loc}}$ of local <u>*weak*</u> convergence, which is generated by the weak and $*$-weak convergence of sequences $\{v_m(\cdot)\}$ and $\{\partial_t v_m(\cdot)\}$ in the same Banach spaces for every $M > 0$. The space $\mathcal{F}_+^{\text{w,loc}} := \mathcal{F}_+^{\text{loc}} \cap \Theta_+^{\text{w,loc}}$ is already not metrizable; it is a Hausdorff and Frechet–Urysohn space with countable topology base. It is clear that the topology $\Theta_+^{\text{s,loc}}$ is stronger than $\Theta_+^{\text{w,loc}}$.

In the space $\mathcal{F}_+^{\text{loc}}$, we consider a linear subspace $\mathcal{F}_+^{\text{b}}$ of elements $v \in \mathcal{F}_+^{\text{loc}}$ for which

$$v \in \mathbf{L}_\infty(\mathbb{R}_+;\mathbf{H}) \cap \mathbf{L}_2^{\text{b}}(\mathbb{R}_+;\mathbf{V}) \cap \mathbf{L}_{\mathbf{p}}^{\text{b}}(\mathbb{R}_+;\mathbf{L}_{\mathbf{p}}(\Omega)),$$
$$\partial_t v \in \mathcal{L}^{\text{b}}(\mathbb{R}_+) := \mathbf{L}_2^{\text{b}}(\mathbb{R}_+;\mathbf{V}') + \mathbf{L}_{\mathbf{q}}^{\text{b}}(\mathbb{R}_+;\mathbf{L}_{\mathbf{q}}(\Omega))$$

with finite norm

$$\|v\|_{\mathcal{F}_+^b} := \|v\|_{\mathbf{L}_\infty(\mathbb{R}_+;\mathbf{H})} + \|v\|_{\mathbf{L}_2^{\text{b}}(\mathbb{R}_+;\mathbf{V})} + \|v\|_{\mathbf{L}_{\mathbf{p}}^{\text{b}}(\mathbb{R}_+;\mathbf{L}_{\mathbf{p}}(\Omega))} + \|\partial_t v\|_{\mathcal{L}^{\text{b}}(\mathbb{R}_+)}. \quad (12)$$

Recall that the norm in the space $L_p^{\text{b}}(\mathbb{R}_+;X)$, where X is a Banach space and $p \geq 1$, is defined by the formula

$$\|\chi\|_{L_p^{\text{b}}(\mathbb{R}_+;X)} := \left[\sup_{t\in\mathbb{R}_+} \int_t^{t+1} \|\chi(s)\|_X^p ds\right]^{1/p}.$$

$\mathcal{F}_+^{\text{b}}$ is a Banach space and we use it norm (12) to define bounded set of trajectories from $\mathcal{K}^+$.

We note that any ball $\mathcal{B}_r := \left\{v \in \mathcal{F}_+^{\text{b}} \mid \|v\|_{\mathcal{F}_+^{\text{b}}} \leq r\right\}$ in the space $\mathcal{F}_+^{\text{b}}$ is a compact set in the weak topology $\Theta_+^{\text{w,loc}}$. We prove

<u>PROPOSITION 2.</u> The trajectory space $\mathcal{K}^+$ of RD-system (1) is closed in the weak topology $\Theta_+^{\mathsf{w,loc}}$ (and is also closed in the strong topology $\Theta_+^{\mathsf{s,loc}}$).

We now consider the translation semigroup $\{T(\tau)\} := \{T(\tau),\ \tau \geq 0\}$, acting in the space $\mathcal{F}_+^{\mathsf{loc}}$ by the formula:

$$T(\tau)v(t) = v(\tau + t),\ t \geq 0.$$

We note that $T(\tau) : \mathcal{K}^+ \mapsto \mathcal{K}^+$, for all $\tau \geq 0$. From the definition of local topology we deduce that the semigroup $\{T(\tau)\}$ is continuous.

<u>PROPOSITION 3.</u> The semigroup $\{T(t)\}$ is continuous in the weak topology $\Theta_+^{\mathsf{w,loc}}$ (as well as in the strong topology $\Theta_+^{\mathsf{s,loc}}$).

Inequalities (8), (9), and (5) imply the following property.

<u>PROPOSITION 4.</u> The space $\mathcal{K}^+$ belongs to $\mathcal{F}_+^{\mathsf{b}}$, and, for every function $u(\cdot) \in \mathcal{K}^+$, the following inequality holds:

$$\|T(\tau)u(\cdot)\|^2_{\mathcal{F}_+^{\mathsf{b}}} \leq C|u(0)|^2 \mathsf{e}^{-\sigma t} + R_0^2,\ \forall \tau \geq 0, \tag{13}$$

where $\sigma = \beta\lambda_1$, and the value R_0 depends on λ_1 and $\|g\|_{V'}$.

It follows from inequality (13) that the set

$$\mathcal{P}_0 = \left\{ u \in \mathcal{K}^+ \mid \|u\|_{\mathcal{F}_+^{\mathrm{b}}} \leq 2R_0 \right\} \tag{14}$$

is *absorbing* for the semigroup $\{T(\tau)\}$ on $\mathcal{K}^+$, that is, for every set of trajectories $\mathfrak{B} \subset \mathcal{K}^+$, that is bounded in $\mathcal{F}_+^{\mathrm{b}}$, the is a time $\tau_1 = \tau_1(\mathfrak{B}) \geq 0$ such that $T(\tau)\mathfrak{B} \subseteq \mathcal{P}_0$ for all $\tau \geq \tau_1$. Besides, this set is invariant with respect to the semigroup $\{T(\tau)\}$:

$$T(\tau)\mathcal{P}_0 \subseteq \mathcal{P}_0, \ \forall \tau \geq 0.$$

We now construct the weak trajectory attractor for the translation semigroup $\{T(\tau)\}$ acting on the trajectory space $\mathcal{K}^+$.

Recall the definition of a (weakly) attracting set. A set $\mathcal{P} \subseteq \mathcal{K}^+$ is called *weakly attracting* (in the topology $\Theta_+^{\mathrm{w,loc}}$) for the semigroup $\{T(\tau)\}$, if, any neighbourhood $\mathcal{O}(\mathcal{P})$ of the set $\mathcal{P}$ in the (weak) topology $\Theta_+^{\mathrm{w,loc}}$ is an absorbing set, that is, for any bounded set $\mathfrak{B} \subset \mathcal{K}^+$ there is $\tau_1 = \tau_1(\mathfrak{B}, \mathcal{O}) \geq 0$, such that $T(\tau)\mathfrak{B} \subseteq \mathcal{O}(\mathcal{P})$ for all $\tau \geq \tau_1$.

We now define the (weak) trajectory attractor.

DEFINITION 3. A set $\mathfrak{A} \subseteq \mathcal{K}^+$ is called the (weak) *trajectory attractor* for the translation semigroup $\{T(\tau)\}$ actin on $\mathcal{K}^+$, iff

(i) it is bounded in the norm $\mathcal{F}_+^{\mathsf{b}}$, compact in the topology $\Theta_+^{\mathsf{w,loc}}$,

(ii) strictly invariant w.r.t. $\{T(\tau)\}$, that is, $T(\tau)\mathfrak{A} = \mathfrak{A}$, for all $t \geq 0$,

(iii) and $\mathfrak{A}$ is a weakly attracting set for the semigroup $\{T(\tau)\}$ on $\mathcal{K}^+$ in the topology $\Theta_+^{\mathsf{w,loc}}$.

We are ready to present the weak trajectory attractor for RD-system (1).

THEOREM 1. The set

$$\mathfrak{A} = \bigcap_{\tau \geq 0} T(\tau)\mathcal{P}_0$$

is the weak trajectory attractor for the semigroup $\{T(\tau)\}$ acting on the trajectory space $\mathcal{K}^+$.

§4. STRONG TRAJECTORY ATTRACTOR

To define *strong trajectory attractor* we just take the strong topology $\Theta_+^{\text{s,loc}}$ in place of the weak topology $\Theta_+^{\text{w,loc}}$ in DEFINITION 3.

We now formulate the main theorem of the lecture.

THEOREM 2. The weak trajectory attractor $\mathfrak{A}$ constructed in THEOREM 1 is in fact the strong trajectory attractor of the semigroup $\{T(\tau)\}$ acting on the trajectory space $\mathcal{K}^+$ of RD-system (1) under the study.

PROOF. It is sufficient to establish that the set $T(1)\mathcal{P}_0$ is compact in the strong topology $\mathbf{L}_2(0, M; \mathbf{V}) \cap \mathbf{L}_\mathbf{p}(0, M; \mathbf{L}_\mathbf{p}(\Omega))$ for any $M > 0$. Here $\mathcal{P}_0$ is the absorbing set from (14). Without lose of generality we may prove this property only for $M = 1$.

Thus we verify that any sequence of trajectories $\{u_m(\cdot)\} \subset \mathcal{P}_0$ has a subsequence that converges strongly in the space $\mathbf{L}_2(1, 2; \mathbf{V}) \cap \mathbf{L}_\mathbf{p}(1, 2; \mathbf{L}_\mathbf{p}(\Omega))$.

For the simplicity sake we consider only the case of homogeneous RD-system (1) with $g \equiv 0$.

The set $\mathcal{P}_0$ is compact in the topology $\Theta_+^{\mathrm{w,loc}}$. Then we may assume that

$$u_m(\cdot) \rightharpoonup \hat{u}(\cdot) \quad \text{as} \quad m \to \infty$$

weakly in the spaces $\mathbf{L}_2(0,2;\mathbf{V})$ and $\mathbf{L}_\mathbf{p}(0,2;\mathbf{L}_\mathbf{p}(\Omega))$;

$$\partial_t u_m(\cdot) \rightharpoonup \partial_t \hat{u}(\cdot) \quad \text{as} \quad m \to \infty$$

weakly in $\mathbf{L}_2(0,2;\mathbf{V}') + \mathbf{L}_\mathbf{q}(0,2;\mathbf{L}_\mathbf{q}(\Omega))$.

Here $\hat{u}(\cdot)$ is a solution of RD-system (1) belonging to $\mathcal{P}_0$.

Equation (1) and Lions-Magenes lemma imply that

$$u_m(t) \rightharpoonup \hat{u}(t) \quad \text{as} \quad m \to \infty$$

weakly in $\mathbf{H}$ for each time $t \in [0,2]$.

From the Aubin embedding theorem, it follows that

$$u_m(\cdot) \to \hat{u}(\cdot) \quad \text{as} \quad m \to \infty$$

strongly in the space $\mathbf{L}_2(0,2;\mathbf{H}) = \mathbf{L}_2(\Omega \times [0,2])$, and, therefore,

$$u_m(x,t) \to \hat{u}(x,t) \quad \text{as} \quad m \to \infty$$

for almost all $(x,t) \in \Omega \times [0,2]$.

Observe, if there is a sequence in a Banach space X such that

$$\chi_m \rightharpoonup \hat{\chi} \quad \text{as} \quad m \to \infty$$

weakly in X, then

$$\|\hat{\chi}\|_X \leq \liminf_{m\to\infty} \|\chi_m\|_X.$$

Hence, having the above weak convergence results for the sequence $\{u_m(\cdot)\}$, we obtain the following inequalities

$$|\hat{u}(2)|^2 \leq \liminf_{m\to\infty} |u_m(2)|^2, \tag{15}$$

$$\int_0^2 t(\mathbf{d}\nabla\hat{u}(t), \nabla\hat{u}(t))dt \leq \liminf_{m\to\infty} \int_0^2 t(\mathbf{d}\nabla u_m(t), \nabla u_m(t))dt, \tag{16}$$

$$\int_0^2 t\|\hat{u}^i(t)\|^{p_i}_{L_{p_i}(\Omega\times[0,2])}dt \leq \liminf_{m\to\infty} \int_0^2 t\|u^i_m(t)\|^{p_i}_{L_{p_i}(\Omega\times[0,2])}dt, \ i = 1, \dots N. \tag{17}$$

The norms in (16) are (17) correspond to weighted spaces $\mathbf{L}_{2,t}(0,2;\mathbf{V})$ and $\mathbf{L}_{\mathbf{p},t}(0,2;\mathbf{L}_{\mathbf{p}}(\Omega))$ with weight t. Clearly the weak convergence $u_m(\cdot) \rightharpoonup \hat{u}(\cdot)$ $(m \to \infty)$ preserves in these weighted spaces.

Consider the scalar function

$$F(v) = \sum_{i=1}^{N} f^i(v)v^i - \sum_{i=1}^{N} \gamma_i |v^i|^{p_i}, \; v \in \mathbb{R}^N.$$

We claim that

$$\int_0^2 \int_\Omega tF(\hat{u}(x,t))dxdt \le \liminf_{m\to\infty} \int_0^2 \int_\Omega tF(u_m(x,t))dxdt. \qquad (18)$$

To prove this inequality we use inequality $F(u_m(x,t)) + C_1 \ge 0$ that follows from (4), convergence $u_m(x,t)$ for almost all $(x,t) \in \Omega \times [0,2]$, and the Fatu theorem on the estimate of integrals over a sequence of positive functions.

We now recall that weak solutions $u_m(\cdot)$ and $\hat{u}(\cdot)$ to system (1) satisfy the energy equality

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + (\mathbf{d}\nabla u(t), \nabla u(t)) + (f(u(t)), u(t)) = 0,\ t \geq 0.$$

We multiply this equation by t, integrate over $[0,2]$ and use the notation $F(u)$. Then we obtain

$$\frac{1}{2}|u_m(2)|^2 + \int_0^2 t(\mathbf{d}\nabla u_m(t), \nabla u_m(t))dt + \int_0^2 t\sum_{i=1}^N \gamma_i \|u_m^i(t)\|^{p_i}_{L_{p_i}(\Omega\times[0,2])}dt +$$
$$+\int_0^2\int_\Omega tF(u_m(x,t))dxdt = \frac{1}{2}\int_0^2 |u_m(t)|^2 dt, \qquad (19)$$
$$\frac{1}{2}|\hat{u}(2)|^2 + \int_0^2 t(\mathbf{d}\nabla \hat{u}(t), \nabla \hat{u}(t))dt + \int_0^2 t\sum_{i=1}^N \gamma_i \|\hat{u}^i(t)\|^{p_i}_{L_{p_i}(\Omega\times[0,2])}dt +$$
$$+\int_0^2\int_\Omega tF(\hat{u}(x,t))dxdt = \frac{1}{2}\int_0^2 |\hat{u}(t)|^2 dt. \qquad (20)$$

Recall that $u_m(\cdot) \to \hat{u}(\cdot)$ strongly in $\mathbf{L}_2([0,2];\mathbf{H})$ as $m \to \infty$ and, therefore, the right-hand side of (19) tends to the right-hand side of (20). Consequently, we obtain the convergence of the right-hand sides as $m \to \infty$. Thus inequalities (15)–(18) imply that the number sequences in the right-hand sides have the limits as $m \to \infty$ which coincides with values in the left-hand sides of (15)–(18), respectively. In particular,

$$\lim_{m\to\infty}\int_0^2 t(\mathbf{d}\nabla u_m(t), \nabla u_m(t))dt = \int_0^2 t(\mathbf{d}\nabla \hat{u}(t), \nabla \hat{u}(t))dt,$$

$$\lim_{m\to\infty}\int_0^2 t\|u_m^i(t)\|^{p_i}_{L_{p_i}(\Omega\times[0,2])}dt = \int_0^2 t\|\hat{u}^i(t)\|^{p_i}_{L_{p_i}(\Omega\times[0,2])}dt,\ i = 1,\dots N.$$

It is known that, in a uniformly convex Banach space X, the weak convergence $\chi_m \rightharpoonup \hat{\chi}$ and the convergence of norms $\|\chi_m\|_X \to \|\hat{\chi}\|_X$ imply the strong convergence $\|\chi_m - \hat{\chi}\|_X \to 0$ as $m \to \infty$ (this fact follows from the Mazur theorem). The weighted spaces $\mathbf{L}_{2,t}(0,2;\mathbf{V})$ and $\mathbf{L}_{\mathbf{p},t}(0,2;\mathbf{L}_\mathbf{p}(\Omega))$ are uniformly convex.

Therefore, weak convergencies of the sequence $\{u_m(\cdot)\}$ and the convergencies of their norms in the space

$$\mathbf{L}_{2,t}(0,2;\mathbf{V}) \cap \mathbf{L}_{\mathbf{p},t}(0,2;\mathbf{L}_\mathbf{p}(\Omega))$$

imply the strong convergence $u_m(\cdot) \to \hat{u}(\cdot)$ $(m \to \infty)$ in these spaces as well as in the space $\mathbf{L}_{2,t}(1,2;\mathbf{V}) \cap \mathbf{L}_{\mathbf{p},t}(1,2;\mathbf{L}_\mathbf{p}(\Omega))$, which clearly is equivalent to the space $\mathbf{L}_2(1,2;\mathbf{V}) \cap \mathbf{L}_\mathbf{p}(1,2;\mathbf{L}_\mathbf{p}(\Omega))$.

We have proved the convergence $u_m(\cdot) \to \hat{u}(\cdot)$ $(m \to \infty)$ in the strong topology of the space

$$\mathbf{L}_2^{\mathsf{loc}}(1,2;\mathbf{V}) \cap \mathbf{L}_{\mathbf{p}}^{\mathsf{loc}}(1,2;\mathbf{L}_{\mathbf{p}}(\Omega)).$$

Recall that $u_m(\cdot)$ and $\hat{u}(\cdot)$ satisfy the equations

$$\begin{aligned} \partial_t u_m &= \mathbf{d}\boldsymbol{\Delta} u_m - f(u_m), \\ \partial_t \hat{u} &= \mathbf{d}\boldsymbol{\Delta} \hat{u} - f(\hat{u}). \end{aligned}$$

The convergence of derivatives $\partial_t u_m$ to $\partial_t \hat{u}$ in the strong topology of

$$\mathbf{L}_2^{\mathsf{loc}}(1,2;\mathbf{V}') + \mathbf{L}_{\mathbf{q}}^{\mathsf{loc}}(1,2;\mathbf{L}_{\mathbf{q}}(\Omega))$$

follows directly from the equation and from the continuity of the Nemytsky operator

$$u \longmapsto f(u),$$

which, due to inequality (5), acts from $\mathbf{L}_{\mathbf{p}}$ to $\mathbf{L}_{\mathbf{q}}$.

Finally, we have to demonstrate the convergence $u_m(\cdot) \to \hat{u}(\cdot)$ $(m \to \infty)$ in the space $C([1,2];\mathbf{H})$. This fact follows from the known embedding

$$\{\mathbf{L}_2(1,2;\mathbf{V}) \cap \mathbf{L}_{\mathbf{p}}(1,2;\mathbf{L}_{\mathbf{p}}(\Omega))\} \cap \Big\{\partial_t v \in \mathbf{L}_2(1,2;\mathbf{V}') + \mathbf{L}_{\mathbf{q}}(1,2;\mathbf{L}_{\mathbf{q}}(\Omega))\Big\} \subset C([1,2];\mathbf{H}),$$

§5. MORE EXAMPLES

0. Well-posed problems have the strong trajectory attractors

1. Elliptic equations in cylindrical domains (M.Vishik, S.Zelik)

2. Dissipative 2D Euler equation (V.Chepyzhov, M.Vishik, S.Zelik)

3. Cahn-Hilliard equation in cylindrical domain (A.Eden, K.Kalantarov, S.Zelik)

4. 3D Navier–Stokes system (???) Open problem.