

Slide 1

Truth values algebras and proof normalization

Gilles Dowek

What does the formula A and B are equivalent mean ?

A has a proof π iff B has a proof π' (e.g. Py and Th)

A has proof π iff B has proof π

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$A \Leftrightarrow B$ v.s. $A \equiv B$

Examples of \equiv

- definitional equality in Martin-Löf type theory (and in the Calculus of Constructions, PTS, ...)
- computational equality in **Deduction modulo** (a theory = axioms + computation rules)
- (the symmetric closure of) a sub-typing relation

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Link with distinction \wedge/\wedge of conjunctive types ?

What impact on the notion of model ?

Slide 4 Can a model distinguish \Leftrightarrow from \equiv ?

{0,1}, Boole and Heyting

{0,1}-models: in a given model CH is valid (1) or not (0),
excluded middle valid, $\llbracket Py \rrbracket = \llbracket Th \rrbracket$

Slide 5 Replace {0,1} by a boolean algebra (*e.g.* $\mathcal{P}(\{\pi, e\})$): CH may
have an intermediate value, excluded middle valid, $\llbracket Py \rrbracket = \llbracket Th \rrbracket$

Replace boolean by Heyting: CH may have an intermediate value,
excluded middle not always valid, $\llbracket Py \rrbracket = \llbracket Th \rrbracket$

What is a Heyting algebra ?

Slide 6 Like a boolean algebra: an ordered set
with *inf* (for \vee , \exists and \top) and *sup* (for \wedge , \forall and \perp)

But a weaker complement

What are the key elements of a Heyting algebra ?

The order ? **Not really**

Correction: Definition of $\tilde{\wedge}, \tilde{\Rightarrow}, \dots$

$(a \tilde{\wedge} b) \tilde{\Rightarrow} a$ always in $\{Max\}$

Slide 7 ... (closed by deduction rules)

Thus provable formulas are valid (structural recursion)

Completeness:

Γ , build a model that validates exactly $Thm(\Gamma)$

Truth values algebras

$\langle \mathcal{B}, \mathcal{B}^+, \tilde{\wedge}, \tilde{\Rightarrow}, \dots \rangle$

$(a \tilde{\wedge} b) \tilde{\Rightarrow} a$ always in \mathcal{B}^+, \dots

Generalization of Heyting algebras

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\mathcal{B} -models: $\mathcal{M} = \langle \mathcal{M}, \mathcal{B}, \hat{f}_i, \hat{P}_j \rangle$

Completeness: for free (but simpler)

Correction: the closure conditions of \mathcal{B}^+ are sufficient (minimal)

An alternative presentation (due to Thierry Coquand)

In a truth values algebra one can define

$$a \leq b \text{ iff } a \Rightarrow b \in \mathcal{B}^+$$

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Verifies all properties of Heyting algebras except one:

antisymmetry

A simple remark: **antisymmetry is useless in the definition of Heyting algebras, it can be dropped**

Nothing else can be dropped: no more closure by deduction rules

What is the relation with \Leftrightarrow / \equiv ?

In a Heyting algebra, because of **antisymmetry**

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$$A \Leftrightarrow B \text{ valid}$$

iff

$$\llbracket A \rrbracket = \llbracket B \rrbracket$$

But not in a TVA

$$\vdash (A \Leftrightarrow B) \text{ iff } \llbracket A \rrbracket \leq \llbracket B \rrbracket \text{ and } \llbracket B \rrbracket \leq \llbracket A \rrbracket$$

$$A \equiv B \text{ iff } \llbracket A \rrbracket = \llbracket B \rrbracket$$

An example of TVA that is not a Heyting algebra: $\{0, 1, I\}$

$\tilde{\vee}$	0	I	1
0	0	1	1
I	1	1	1
1	1	1	1

Slide 11 same for $\tilde{\exists}, \tilde{\top}, \tilde{\wedge}, \tilde{\vee}, \tilde{\perp}$, but

$\tilde{\Rightarrow}$	0	I	1
0	1	1	1
I	0	1	1
1	0	I	I

$$\mathcal{B}^+ = \{I, 1\}, 0 \leq 1, 0 \leq I, I \leq 1, 1 \leq I$$

Consistency

Slide 12 If there exists a (non trivial) TVA \mathcal{B} such that \mathcal{T}, \equiv has a \mathcal{B} -model $(\mathcal{M} = \langle \mathcal{M}, \mathcal{B}, \hat{f}_i, \hat{P}_j \rangle)$ then \mathcal{T}, \equiv is consistent

From consistency to super-consistency

$$P \longrightarrow (Q \Rightarrow Q)$$

has a $\{0, 1\}$ -model $\llbracket Q \rrbracket = 1, \llbracket P \rrbracket = 1$

Slide 13 and a $\{0, I, 1\}$ -model $\llbracket Q \rrbracket = 1, \llbracket P \rrbracket = 1 \not\Rightarrow 1$

Actually: a \mathcal{B} -model for all \mathcal{B}

From consistency to super-consistency

$$P \longrightarrow (P \Rightarrow Q)$$

a $\{0, 1\}$ -model $\llbracket P \rrbracket = \llbracket Q \rrbracket = 1$

Slide 14 but no $\{0, I, 1\}$ -model

$\not\Rightarrow$	0	I	1
0	1	1	1
I	0	1	1
1	0	I	I

Super-consistency

Slide 15 If for all truth values algebra \mathcal{B} , \mathcal{T}, \equiv has a \mathcal{B} -model then \mathcal{T}, \equiv is said to be **super-consistent**

Examples

$P \longrightarrow P \Rightarrow Q$ is consistent but not super-consistent

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$P \longrightarrow Q \Rightarrow Q$, $P \longrightarrow Q \Rightarrow P$, **arithmetic**, simple type theory, ...
are super-consistent

Arithmetic

Arithmetic can be presented in deduction modulo (10 rules)

Slide 17 An arbitrary TVA \mathcal{B}

$$\mathcal{M}_\iota = \mathbb{N}$$

$$\mathcal{M}_\kappa = \mathcal{B}^{\mathbb{N}}$$

...

The theorem

Slide 18 In Deduction modulo, if \mathcal{T}, \equiv is super-consistent
then all proof in \mathcal{T}, \equiv **strongly normalize**

The truth values algebra of reducibility candidates

Sets of proofs

Slide 19 $a \Rightarrow b$: set of strongly normalizing proofs π s.t.. if π reduces to $\lambda\alpha \pi_1$ then for all π' in a , $(\pi'/\alpha)\pi_1$ is in b

$\tilde{\forall} A$: ...

...

$\mathcal{C}^+ = \mathcal{C}$

Always $a \leq b$ (not antisymmetric, one point quotient)

Proving normalization
without knowing what a reducibility candidate is

Just prove super-consistency

No need to know what a candidate is

Slide 20 The candidates are hidden in the proof that super-consistency implies normalization

Explains why the choice among the thousand variants of candidates is immaterial

The candidates: only their algebraic structure matters

Another cut elimination proof (joint with Hermant)

Another truth values algebra

Candidates : sets of proofs

Replace proofs by their types

Slide 21 Sets of sequents

“Semantic” cut elimination proofs: just

A generalization of V -complexes

Open problems

Are there theories that strongly normalize without being super-consistent ? (For cut elimination yes (Hermant))

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L.-S. ? (2^{\aleph_0})