Random dynamics od some entire functions and random invariant measures; random non- hyperbolic exponential maps

Anna Zdunik

University of Warsaw

Edinburgh, July 2018

Plan

Plan

Plan

Plan

Plan

Plan

1. Dynamics of entire maps
2. Non-autonomous and random setting
3. Transcendental dynamics
4. Random entire and meromorphic maps
5. Random non-hyperbolic exponentials
6. Hausdorff Dimension of the radial Julia set

Julia set.

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire map. Notation: $f^n = f \circ f \circ \cdots \circ f$. The Julia set $J(f)$ is (informally) the set where the chaotic part of the dynamics is concentrated.

Fatou set and Julia set

The set

$$F(f) = \{z \in \mathbb{C} : \exists U \ni x \text{ such that } \{f^n_{|U}\} \text{ is normal}\}$$

is called the Fatou set of f, and its complement is the Julia set $J(f)$.

Clearly, $F(f)$ is open and $J(f)$ is closed.

Given a sequence $f_1, f_2 \ldots f_n \ldots$ of entire maps consider the iterations

$$f^n := f_n \circ f_{n-1} \circ \cdots \circ f_1. \tag{1}$$

Then the Julia set of the non- autonomous system is defined in an analogous way, using normality criterion.

The theme of Hausdorff dimension for entire functions was taken up, in particular, in a series of papers by G. Stallard and by Urbański –Zdunik. These latter papers concerned hyperbolic exponential functions, i.e. those of the form

$$\mathbb{C} \ni z \longmapsto f_\lambda(z) := \lambda e^z \in \mathbb{C},$$

where λ is such that the map f_λ is hyperbolic, i.e. it has an attracting periodic orbit.

Our papers used the ideas of thermodynamic formalism and, particularly, of conformal measures. In these papers the concept of a radial (called also conical) Julia set, denoted by $J_r(f)$ occured in a natural way.

Conical (radial) limit set

This is the set of points z in the Julia set $J(f)$ for which infinitely many holomorphic pullbacks from $f^n(z)$ to z are defined on balls centered at points $f^n(z)$ and having radii uniformly bounded from below in the spherical metric. For hyperbolic functions f_λ this is just the set of points that do not escape to infinity under the action of the map f_λ.

Results on $J_r(f_\lambda)$

What we have discovered in is that $HD(J_r(f_\lambda)) < 2$ for hyperbolic exponential functions f_λ defined above. This is in stark contrast with McMullen's results asserting that $HD(J(f_\lambda)) = 2$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. In addition we proved in that its Hausdorff dimension $HD(J_r(f_\lambda))$ is equal to the unique zero of the pressure function $t \mapsto P(t)$ defined absolutely independently of $J_r(f_\lambda)$. Moreover, this dimension depends (real) analytically on the parameter λ, as long as λ varies inside a *hyperbolic component* in the set of parameters.

Importance of the set $J_r(f)$

The set $J_r(f_\lambda)$ is dynamically significant as every finite Borel f_λ–invariant measure on $\mathbb{C}$ is supported on this set.

Non-hyperbolic exponential maps

In our next papers, the ergodic theory and conformal measures was provided for a large class of non-hyperbolic exponential functions f_λ, namely those for which the number 0 escapes fast to infinity; it includes all maps for wihich λ is real and larger than $1/e$.

Our next work on this subject stems from this one and provides a systematic account of ergodic theory and conformal measures for randomly iterated functions f_λ, where $\lambda > 1/e$. The theory of random dynamical systems is a large fast developing subfield of dynamical systems with a specific variety of methods, tools, and goals.

Random hyperbolic transcendental functions

In the paper
"RANDOM DYNAMICS OF TRANSCENDENTAL FUNCTIONS"
Volker Mayer and Mariusz Urbański worked with non- autonomous and random dynamics of some classes of hyperbolic meromorphic functions of finite order, satisfying, in particular, *balanced growth condition*. In particular, they provided the following tools:

The thermodynamical formalism which, in particular, produces unique fiberwise geometric and fiberwise invariant Gibbs states. Spectral gap property for the associated transfer operator.

In this random setting the (random) radial Julia set is defined analogously to the autonomous cases. Question: How does the dimension of the random radial Julia set depend on the ”range of randomness”? We deal with this question in joint paper with Volker Mayer and Mariusz Urbański:
”REAL ANALYTICITY FOR RANDOM DYNAMICS
OF TRANSCENDENTAL FUNCTIONS”
Below is the application of our results in the special case of ”random hyperbolic exponential dynamics”.

Theorem

Let $f_\eta(z) = \eta e^z$ and let $a \in (\frac{1}{3e}, \frac{2}{3e})$ and $0 < r < r_{max}$, $r_{max} > 0$. Suppose that $\eta_1, \eta_2, ..$ are i.i.d. random variables uniformly distributed in $\mathbb{D}(a, r)$ and let

$$J_r(\eta_1, \eta_2, ...) = \{z \in J_{\eta_1,\eta_2,...} ;\ \liminf_{n\to\infty} |f_{\eta_n} \circ ... \circ f_{\eta_1}(z)| < \infty\}$$

be the radial Julia set of $(f_{\eta_n} \circ ... \circ f_{\eta_1})_{n\geq 1}$. Then, the Hausdorff dimension of $J_r(\eta_1, \eta_2, ...)$ is almost surely constant and depends real-analytically on the parameters (a, r) provided r_{max} is sufficiently small.

i

Random non-hyperbolic exponential maps

In the paper
RANDOM NON-HYPERBOLIC EXPONENTIAL MAPS
(Mariusz Urbański and A.Z)
the randomness is modeled by a measure preserving invertible dynamical system $\theta : \Omega \to \Omega$, where $(\Omega, \mathcal{F}, m)$ is a complete probability measurable space, and θ is a measurable invertible map, with θ^{-1} measurable, preserving the measure m. Fix some real constants $B > A > 1/e$ and let

$$\eta : \Omega \longmapsto [A, B]$$

be measurable function. Furthermore, to each $\omega \in \Omega$ associated is the exponential map $f_\omega := f_{\eta(\omega)} : \mathbb{C} \to \mathbb{C}$; precisely

$$f_\omega(z) := \eta(\omega)e^z.$$

Consequently, for every $z \in \mathbb{C}$, the map

$$\Omega \ni \omega \longmapsto f_{\eta(\omega)}(z) \in \mathbb{C}$$

is measurable.

Change of coordinates

We consider the dynamics of random iterates of exponentials:

$$f_\omega^n := f_{\theta^{n-1}\omega} \circ \cdots \circ f_{\theta\omega} \circ f_\omega : \mathbb{C} \longrightarrow \mathbb{C}.$$

Instead of f_ω, we consider F_ω, the map on the cylinder $Q = \mathbb{C}/\sim$, where $Z \sim W$ if $Z - W = 2k\pi i$ for some $k \in \mathbb{Z}$.

Global dynamics

Put $X = \Omega \times Q$ The global map $F : X \to X$ is the skew product

$$F(\omega, x) = (\theta(\omega), F_\omega(x))$$

Put $X = \Omega \times Q$ and let $\mathcal{M}_m \subset \mathcal{M}(X)$ be the set of all non-negative probability measures on X that project onto m under the map $\pi_1 : X \to \Omega$, i.e.

$$\mathcal{M}_m = \{\mu \in \mathcal{M}(X) : \mu \circ \pi_1^{-1} = m\}.$$

The members of $\mathcal{M}_m$ are called random measures with respect to m. Their disintegration measures μ_ω, $\omega \in \Omega$, with respect to the partition of X into sets $\{\omega\} \times \mathbb{C}$, are called (fiberwise) random measures.

Geometric random measures: random conformal measures

We are interested in conformal random measures, their existence, uniqueness, and their geometrical and dynamical properties. Such measures are characterized by the property that

$$\nu_{\theta\omega}(F_\omega(A)) = \lambda_{t,\omega} \int_A |(F'_\omega)^t| d\nu_\omega$$

for m–a.e. $\omega \in \Omega$ and for every Borel set $A \subset Q$ such that $F_\omega|_A$ is 1–to–1,where $\lambda_t : \Omega \to (0, +\infty)$ is some measurable function.

Random conformal and invariant measures

Existence of conformal measures

Theorem: For every $t > 1$ there exists $\nu^{(t)}$, a random t–conformal measure, for the map $F : Q \to Q$.

Theorem.

For every $t > 1$ there exists a unique Borel probability F–invariant random measure $\mu^{(t)}$ absolutely continuous with respect to $\nu^{(t)}$, the random t–conformal measure. In fact, $\mu^{(t)}$ is equivalent with $\nu^{(t)}$ and ergodic.

In terms of fiberwise invariant measures, F–invariance of the measure $\mu^{(t)}$ means that

$$\mu_{\omega}^{(t)} \circ F_{\omega}^{-1} = \mu_{\theta\omega}^{(t)}$$

Expected pressure and Bowen's formula

For $t > 1$ put

$$\mathcal{E}P(t) := \int_\Omega \log \lambda_{t,\omega} dm(\omega).$$

Then

Results on the expected pressure

1. $\mathcal{E}P(t) < +\infty$ for all $t > 1$,
2. The function $(1, +\infty) \ni t \mapsto \mathcal{E}\,\mathrm{Pr}(t)$ is stricitly decreasing, convex, and thus continuous,
3. $\lim_{t\to 1} \mathcal{E}P(t) = +\infty$ and $\mathcal{E}P(2)) \leq 0$.
4. (Bowen's formula) Let $h > 1$ be the unique value $t > 1$ for which $\mathcal{E}P(t) = 0$. Then
$$HD(J_{r,\omega}) = h$$
for m–a.e.$\omega \in \Omega$.

Some consequences

The Hausdorff dimension $h = HD(J_{r,\omega})$ of the random radial Julia set $J_{r,\omega}$, is constant for m–a.e. $\omega \in \Omega$ and satisfies $1 < h < 2$. In particular, the 2–dimensional Lebesgue measure of m–a.e. $\omega \in \Omega$ set $J_{r,\omega}$ is equal to zero.

More consequences

Trajectory of a (Lebesgue) typical point

For m–almost every $\omega \in \Omega$ there exists a subset $Q_\omega \subset Q$ with full Lebesgue measure such that for all $z \in Q_\omega$, the set of accumulation points of the sequence

$$(F_\omega^n(z))_{n=0}^\infty$$

is contained in $[0, +\infty] \cup \{-\infty\}$

These last two properties are truly astonishing and were first time observed for the exponential map $\mathbb{C} \ni z \mapsto e^z \in \mathbb{C}$ by M. Rees and M. Lyubich. Our approach to establish these two properties is different than those of Rees and Lyubich and relies on investigation of h–dimensional packing measure Q.

Question

What is the actual dependence of the dimension of the random radial Julia set on the *range of randomness*? Do we have an analogue of the hyperbolic case?

Theorem?

Let $A, B > 1/e, f_\eta(z) = \eta e^z$ and let $a \in [A, B])$ and $0 < r < r_{max}$, $r_{max} > 0$. Suppose that $\eta_1, \eta_2, ..$ are i.i.d. random variables uniformly distributed in $(a - r, a + r)$ and let

$$J_r(\eta_1, \eta_2, ...)$$

be the radial Julia set of $(f_{\eta_n} \circ ... \circ f_{\eta_1})_{n \geq 1}$. Then, the Hausdorff dimension of $J_r(\eta_1, \eta_2, ...)$ is almost surely constant and depends how? on the parameters (a, r) provided r_{max} is sufficiently small.