Thermodynamic Formalism and Iterated Function Systems
in
Transcendental Dynamics

Resonances of Complex Dynamics

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My Bergweiler Number is the the same as my Erdös Number and

is equal to 2.

Happy Birthday Walter!

Resonances:

- Functional Analysis
- Probability Theory
- Thermodynamic Formalism
- Conformal Iterated Function Systems
- Fractal Geometry

Thermodynamic Formalism

The founders: D. Ruelle, O. E. Lanford, Ya. Sinaj, R. Bowen, P. Walters.

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Key concepts:

- Topological pressure; topological entropy
- Gibbs states
- Equilibrium states/measures
- Variational Principle
- Perron–Frobenius (Ruelle, transfer) operators
- Spectral properties of Perron–Frobenius operators
- Kolmogorov–Sinaj metric entropy
- Stochastic Laws

Thermodynamic Formalism; (very) General Scheme

- $T : X \longrightarrow X$ – a (nearly) continuous map
- $\phi : X \longrightarrow \mathbb{R} \cup \{-\infty\}$ – a continuous (usually much better) function
- $\mathcal{L}_\phi : C_b(X) \to C_b(X)$ – the associated Perron–Frobenius operator.

$$\mathcal{L}_\phi g(x) := \sum_{y \in T^{-1}(x)} g(y) e^{\phi(y)}.$$

- The dual operator: $\mathcal{L}_\phi^* : C_b^*(X) \longrightarrow C_b^*(X)$

$$\mathcal{L}_\phi^* \nu(g) = \nu(\mathcal{L}_\phi g).$$

- Eigenvalues and eigenmeasures (Gibbs states) of $\mathcal{L}_\phi^*$:

$$\mathcal{L}_\phi^* m_\phi = \lambda m_\phi.$$

$$\nu(T(A)) = \lambda \int_A e^{-\phi}\, d\nu,$$

where $A \subset X$ is Borel and $T|_A$ is 1–to–1.

- Invariant Gibbs states: $\mu_\phi = \rho_\phi m_\phi$, where $\rho_\phi \in C_b(X)$, $\rho_\phi \geq 0$, and

$$\mathcal{L}_\phi \rho_\phi = \lambda \rho_\phi$$

Expanding Rational Functions

$f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ - a rational function

$J(f)$ - the Julia set

f is expanding if $\exists (k \geq 1)$ s. t.

$$|(f^k)'(z)| \geq 2 \ \ \forall z \in J(f).$$

Equivaletntly

$$J(f) \cap \overline{\bigcup_{n=0}^{\infty} f^n(\mathrm{Crit}(f))} = \emptyset.$$

Geometric Thermodynamic Formalism for Expanding Rational Functions

$t \geq 0$: the topological pressure of the potential $-t \log |f'|$:

$$\mathrm{P}(t) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{w \in f^{-n}(z)} |(f^n)'(w)|^{-t}, \quad z \in J(f).$$

The Perron–Frobenius, transfer, operator:

$$\mathcal{L}_t : C(J(f)) \to C(J(f))$$

$$\mathcal{L}_t(g) := \sum_{w \in f^{-1}(z)} g(w)|f'(w)|^{-t}$$

$$\mathcal{L}_t(\mathrm{H}_\alpha) \subset \mathrm{H}_\alpha.$$

Geometric Thermodynamic Formalism for Expanding Rational Functions

$$\mathrm{P}(t) = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{L}_t^n \mathbb{1}(z).$$

Theorem

$\exp(\mathrm{P}(t)) =$ *the spectral radius of* $\mathcal{L}_t$.

Conformal Measures

A Borel probability measure m on $J(f))$ is called t-conformal if it is an eigenvector of the dual operator $\mathcal{L}_t^*$:

$$\mathcal{L}_t^* m = \lambda m$$

where

$$\mathcal{L}_t^* \nu(g) = \nu(\mathcal{L}_t g).$$

Equivalently:

$$m(f(A)) = \lambda \int_A |f'|^t \, dm$$

whenever $B \subset J(f)$ is Borel and $f|_A$ is 1–to–1.

We call it also $\lambda|f'|^t$–conformal or a Gibbs state for the potential $-t \log |f'|$.

Particularly important case if $\lambda = 1$.

Geometric Thermodynamic Formalism for Expanding Rational Functions

Theorem

If $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is expanding and $t \geq 0$, then the following are true.

(1) *The topological pressure $\mathrm{P}(t) = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{L}_t^n \mathbb{1}(w)$ exists and is independent of $w \in J(f)$.*

(2) *The function $[0, +\infty) \ni t \longmapsto \mathrm{P}(t) \in \mathbb{R}$ is strictly decreasing, convex, thus continuous, in fact real–analytic, and $\lim_{t\to+\infty} \mathrm{P}(t) = -\infty$.*

(3) *There exists a unique $\lambda|f'|^t$–conformal measure m_t and necessarily $\lambda = e^{\mathrm{P}(t)}$. Also, there exists a unique f-invariant Gibbs state μ_t, the latter meaning that μ_t is and equivalent to m_t and*

(4) *the Radon–Nikodym derivative $\rho_t := d\mu_t/dm_t$ is log bounded. In fact it is Lipschitz continuous and has a real analytic extension to an open neghborhood of $J(f)$.*

(5) *Both measures m_t and μ_t are ergodic, metrically exact, and more ... .*

Quasi–Compactness and Spectrum Gap

Theorem

(a) *The number* 1 *is a simple isolated eigenvalue of the operator*

$$\hat{\mathcal{L}}_t := e^{-\mathrm{P}(t)}\mathcal{L}_t : \mathrm{H}_\beta \to \mathrm{H}_\beta$$

and the rest of the spetrum is contained in a disk of radius strictly smaller than 1 *(more than quasi–compactness). More precisely:*

(b) *There exists a bounded linear operator* $S : \mathrm{H}_\beta \to \mathrm{H}_\beta$ *such that*

$$\hat{\mathcal{L}}_t = Q_1 + S,$$

where the projector $Q_1 : \mathrm{H}_\beta \to \mathbb{C}\rho_f$*, the eigenspace of* 1*, is:*

$$Q_1(g) = \left(\int g\, dm_t\right)\rho_t,$$

$Q_1 \circ S = S \circ Q_1 = 0$ *and for all* $n \geq 1$*:*

$$||S^n||_\beta \leq C\xi^n, \qquad \xi \in (0,1).$$

Stochastic/Random Laws and Behavior

Corollary

$\forall n \geq 1$,

$$\hat{\mathcal{L}}^n = Q_1 + S^n$$

and $\hat{\mathcal{L}}^n(g)$ *converges to* $\left(\int g\, dm_\phi\right) \rho_t$ *exponentially fast when* $n \to \infty$.

Stochastic/Random Laws and Behavior

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and $\hat{\mathcal{L}}^n(g)$ *converges to* $\left(\int g\, dm_\phi\right)\rho_t$ *exponentially fast when* $n \to \infty$. *More precisely:*

$$\left\| \hat{\mathcal{L}}^n(g) - \left(\int g\, dm_\phi\right)\rho \right\|_\beta = \|S^n(g)\|_\beta \leq C\|g\|_\beta \xi^n, \quad g \in H_\beta.$$

Corollary (Exponential Decay of Correlations)

For all $\psi \in \mathrm{H}_\beta$, *all* $\psi_2 \in L^1(\mu_t)$ *and all integers* $n \geq 1$*:*

$$\left| \int (\psi_1 \circ f^n \cdot \psi_2)\, d\mu_t - \int \psi_1\, d\mu_t \int \psi_2\, d\mu_t \right| \leq C\|\psi_1\|_{\mathrm{H}_\beta}\|\psi_2\|_{L^1(\mu_t)}\xi^n,$$

Stochastic/Random Laws and Behavior

Corollary (Central Limit Theorem)

For every $\psi \in \mathrm{H}_\beta$ not cohomological to a constant, the sequence of random variables

$$\frac{\sum_{j=0}^{n-1} \psi \circ f^j - n \int \psi \, d\mu_t}{\sqrt{n}}$$

converges in distribution with respect to the measure μ_t to the Gauss (normal) distribution $\mathcal{N}(0, \sigma^2)$ with some $\sigma > 0$.

Stochastic/Random Laws and Behavior

Corollary (Central Limit Theorem)

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converges in distribution with respect to the measure μ_t to the Gauss (normal) distribution $\mathcal{N}(0, \sigma^2)$ with some $\sigma > 0$. Precisely, for every $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \mu_t\left(\left\{z \in J(f) : \frac{\sum_{j=0}^{n-1} \psi \circ f^j(z) - n \int \psi \, d\mu_t}{\sqrt{n}} \le t\right\}\right) =$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left(-\frac{u^2}{2\sigma^2}\right) du.$$

Stochastic/Random Laws and Behavior

Corollary (Law of Iterated Logarithm)

For every $\psi \in \mathrm{H}_\beta$ not cohomological to a constant and for μ_t–a.e. $z \in J(f)$:

$$\overline{\lim_{n\to\infty}} \frac{\sum_{j=0}^{n-1} \psi(z) \circ f^j - n \int \psi \, d\mu_t}{\sqrt{n \log \log n}} = \sqrt{2}\sigma.$$

In fact the Almost Sure Invariance Principle holds meaning that the sequence of random variables

$$J(f) \ni z \longmapsto \sum_{j=0}^{n-1} \psi \circ f^j(z) - n \int \psi \, d\mu_t \in \mathbb{R}$$

can be approximated sufficiently well by a Brownian motion.

Bowen's formula for Expanding Rational Functions

Theorem (Bowen's formula)

If a rational function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is expanding, then

$$h := \mathrm{HD}(J(f)) = \textit{the unique zero of the pressure function}$$

$$[0, +\infty) \ni t \longmapsto \mathrm{P}(t) \in \mathbb{R}.$$

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all of them are h-Ahlfors measures, meaning that

$$C^{-1} \le \frac{m_h(B(z,r))}{r^h} \le C$$

for all $z \in J(f)$ and all $r \in (0, 1]$.

Barański's Meromorphic Functions

In 1995 Krzysztof Barański considered meromorphic functions $f : \mathbb{C} \to \hat{\mathbb{C}}$ satisfying the following conditions:

- There exist $T \in \mathbb{C} \setminus \{0\}$ and a non-polynomial rational function $h : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ with poles in $\mathbb{C} \setminus \{0\}$ such that
$$f(z) = h\left(\exp\left(\frac{2\pi i}{T} z\right)\right), \quad z \in \hat{\mathbb{C}}$$
- $$J(f) \cup \overline{\bigcup_{n=0}^{\infty} f^n\big(\mathrm{Sing}(f^{-1})\big)} = \emptyset$$
$$\tilde{f}(z) := \exp\left(\frac{2\pi i}{T} h(z)\right).$$

Then

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{f} & \mathbb{C} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ Q & \xrightarrow{\tilde{f}} & \mathbb{C} \end{array}$$

Barański's Meromorphic Functions

$$J(\tilde{f}) = \exp(J(f) \cap \mathbb{C}).$$

$$J(\tilde{f}) \cap \overline{\bigcup_{n=0}^{\infty} f^n\big(\mathrm{Sing}(\tilde{f}^{-1})\big)} = \emptyset$$

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$$J(\tilde{f}) = \exp(J(f) \cap \mathbb{C}).$$

$$J(\tilde{f}) \cap \overline{\bigcup_{n=0}^{\infty} f^n(\mathrm{Sing}(\tilde{f}^{-1}))} = \emptyset$$

So, all holomorphic inverse branches of all iterates of $\tilde{f}$ are well defined on all balls centered at points of $J(\tilde{f})$ with sufficiently small radius.
Then

$$\mathcal{L}_t(g)(z) = \sum_{w \in \tilde{f}(z)} g(w)|\tilde{f}'(w)|^{-t}$$

is well defined (and finite) iff

$$t > \frac{q}{q+1}$$

where $q \geq 1$ is the largest order of a pole of h.

Barański's Meromorphic Functions

Theorem (Barański, 1995)

- *Bowen's Formula holds, i.e.* $h := \mathrm{HD}(J(f)) =$ *the unique zero of the pressure function.*

(a) *If* $h < 1$*, then* $0 < \mathrm{P}_h(J(\tilde{f})) < \infty$ *and* $\mathrm{H}_h(J(\tilde{f})) = 0$.

(b) *If* $h = 1$*, then* $0 < \mathrm{P}_h(J(\tilde{f})),\ \mathrm{H}_h(J(\tilde{f})) < \infty$.

(c) *If* $h > 1$*, then* $0 < \mathrm{H}_h(J(\tilde{f})) < \infty$ *and* $\mathrm{P}_h(J(\tilde{f})) = \infty$,

Examples:

$$f(z) = \lambda \tan z; \quad h(z) = \lambda i \frac{z-1}{z+1}; \quad |\lambda| > 0.$$

$$f(z) = (\lambda \tan z)^p; \quad h(z) = \left(-\lambda i \frac{z-1}{z+1}\right)^p; \quad 0 < |\lambda| < 1, p \in \mathbb{N}.$$

In 2002 J. Kotus and M. U. extended Barański's case to maps of the form

$$H \circ \exp \circ Q : \mathbb{C} \longrightarrow \hat{\mathbb{C}},$$

where H and Q are rational functions.

Problems with Transcendental Functions

$$f_\lambda(z) = \lambda e^z$$

$\lambda \in \mathbb{C} \setminus \{0\}$ chosen so that f_λ has an attracting periodic orbit.

$$\mathcal{L}_t(\mathbb{1})(z) = \sum_{w \in f_\lambda^{-1}(z)} |f_\lambda'(w)|^{-t} = \sum_{w \in f_\lambda^{-1}(z)} |z|^{-t} = +\infty,$$

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always

Problems with Transcendental Functions: First Remedies

1. Projection onto the infinite cylinder

$$Q := \mathbb{C}/2\pi i\mathbb{Z}, \quad \pi : \mathbb{C} \to Q$$

(Anna Zdunik, M.U.; 2003, 2004).

$$F_\lambda : Q \to Q$$

$$F_\lambda(z) := \pi(f_\lambda(\pi^{-1}(z))).$$

In particular:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{f_\lambda} & \mathbb{C} \\ \pi \downarrow & & \downarrow \pi \\ Q & \xrightarrow{F_\lambda} & Q \end{array}$$

The geometric thermodynamic formalism fully works:
Spectral gap; Exponential decay of Correlatins; Central Limit Theorem; Law of Iterated Logarithm.

Problems with Transcendental Functions: First Remedies

But: (A. Zdunik, M. U.)

The function

$$(1, +\infty) \ni t \longmapsto \mathrm{P}(t) \in \mathbb{R}$$

is strictly decreasing, convex, thus continuous, in fact real–analytic, and

$$\lim_{t \to +\infty} \mathrm{P}(t) = -\infty \quad \text{while} \quad \lim_{t \searrow 1} \mathrm{P}(t) = +\infty.$$

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Bowen's Formula holds but:

$$J_r(F_\lambda) := \left\{ z \in J(F_\lambda) : \varliminf_{n\to\infty} \left| F_\lambda^n(z) \right| < +\infty \right\}$$

is the radial (conical) Julia set of f [Misha Lyubich (1983)].

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Radial becuse this is the set of all points $z \in \mathbb{C}$ whose forward iterates $f^n(z)$ have holomorphic pullbacks

$$F_{\lambda,z}^{-n} : B_s(F_\lambda^n(z), \delta) \longrightarrow \mathbb{C}, \quad F_{\lambda,z}^{-n}(F_\lambda^n(z)) = z$$

for infinitely many ns.

Problems with Transcendental Functions: First Remedies

Theorem (Anna Zdunik, M. U.)

$$h := \mathrm{HD}(J_r(F_\lambda)) = \textit{the unique zero of the pressure function}$$

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$$1 < \mathrm{HD}(J_r(F_\lambda)) < \mathrm{HD}(J(F_\lambda)) = 2,$$

where $\mathrm{HD}(J_r(f_\lambda)) > 1$ proved earlier for $\lambda \in (0, 1/e)$ by Bogusia Karpińska while $\mathrm{HD}(J(f_\lambda)) = 2$ is due to Curtis McMullen (1989).

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$\mathrm{HD}(J_r(F_\lambda))$ known as dynamical dimension or hyperbolic dimension, equal also to

(a)

$$\sup \left\{ \mathrm{HD}(\mu) : \mu \circ F_\lambda^{-1} = \mu \text{ (ergodic)} \right\}$$

(b) the supremum of Hausdorff dimensions of all F_λ–invariant conformal repellers in $\mathbb{C}$.

Problems with Transcendental Functions: First Remedies

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In particular neither m_λ (the h_λ–conformal measure) nor H_{h_λ} are Ahlfors.

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Corollary

Hausdorff measure $\mathrm{H}_{h_\lambda}|_{J_r(f_\lambda))}$ *is positive and* σ*–finite.*

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This approach works only for periodic functions.

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Problems with Transcendental Functions: Further Remedies

Change of Riemannian metric on $\mathbb{C}$ (Volker Mayer, M.U.; 2008, 2010):

$$|dz|/|z|.$$

Then

$$|f'(z)|_1 = |f'(z)|\frac{|z|}{|f(z)|}.$$

So,

$$|f_\lambda'(z)|_1 = |z|$$

Therefore

$$\mathcal{L}_t\mathbb{1}(w) = \sum_{z\in f_\lambda^{-1}(w)} |f'(z)|_1^{-t} = \sum_{z\in f_\lambda^{-1}(w)} |z|^{-t} = \sum_{n\in\mathbb{Z}} |\log(w/\lambda) + 2\pi i n|^{-t}$$

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chance for $t > 1$

Problems with Transcendental Functions: Dynamically Regular Meromorphic Functions

Definition

A meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ is called expanding if $\exists (k \geq 1)$ s. t.

$$|(f^k)'(z)| \geq 2 \quad \forall z \in J(f).$$

It is called topologically hyperbolic [Gwyneth Stallard] if

$$\mathrm{dist}_e\left(J(f), \overline{\bigcup_{n=0}^{\infty} f^n(\mathrm{Crit}(f)))}\right) > 0.$$

f is called hyperbolic if it is both expanding and topologically hyperbolic.

Problems with Transcendental Functions: Dynamically Regular Meromorphic Functions

Definition (V. Mayer, M.U. 2008, 2010)

A hyperbolic meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ is called dynamically regular if f is of finite order and if

$$C^{-1}(1+|z|)^{\alpha_1}(1+|f(z)|^{\alpha_2}) \leq |f'(z)| \leq C(1+|z|)^{\alpha_1}(1+|f(z)|^{\alpha_2})$$

for all $z \in J(f) \setminus f^{-1}(\infty)$, where

$$\alpha_2 > \max\{-\alpha_1, 0\}.$$

$\alpha_2 = 1$ if f is entire.
f is called dynamically semi–regular if only the LHS is assumed.

Change of Riemannian metric:

$$\frac{|dz|}{|z|^{\alpha_2}}.$$

Problems with Transcendental Functions: Dynamically Regular Meromorphic Functions

For dynamically semi–regular meromorphic functions

1. The full thermodynamic formalism holds (V. Mayer, M.U.; 2008, 2010) for all $t > \rho/\alpha$.
2. If in addition f is dynamically regular and of divergence type, then $h := \mathrm{HD}(J_r(F_\lambda)) =$ the unique zero of the pressure function

Divergence type means that

$$\int_1^\infty \frac{T(r)}{r^{\rho+1}} dr = +\infty$$

if f is not entire, and, with some $A, B > 0$:

$$\int_{\log R}^R \frac{T(r)}{r^{\rho+1}} dr - B(\log R)^{1-\rho} \geq A$$

if f is entire.

Problems with Transcendental Functions: Dynamically Regular Meromorphic Functions

This method uses Borel series

$$\sum_{z \in f^{-1}(w)} |z|^{-t},$$

shown to be comparable to

$$\mathcal{L}_t \mathbb{1},$$

and Nevanlina's Theory, needed to gain some uniformity of the Borel series.

Examples of Dynamically Regular Meromorphic Functions

1. Entire functions:
   - Classical families like $f_\lambda(z) = \lambda e^z$ or $f(z) = sin(az + b)$. $\rho = 1$, $\alpha_1 = 0$ and $\alpha_2 = 1$.
   - $f(z) = cos(\sqrt{az + b})$ $\rho = \frac{1}{2}$, $\alpha_1 = -\frac{1}{2}$ and $\alpha_2 = 1$.
   - $g = P\ f \circ Q$ with f one of the above functions and P, Q polynomials.
   - $f(z) = \int_0^z P(\xi) \exp(Q(\xi)) d\xi + c$. Always $\alpha_2 = 1$!

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   - $f(z) = \int_0^z P(\xi) \exp(Q(\xi)) d\xi + c$. Always $\alpha_2 = 1$!

2. Meromorphic functions: If f has a pole b of multiplicity m then, near b, α_2 does depend on m. $(|f'(z)| \asymp |f(z)|^{1+1/m})$.
   - Elliptic functions + compositions with polynomials.
   - Certain solutions of Ricatti differential equations: $f(z) = \frac{Ae^{2z^k}+B}{Ce^{2z^k}+D}$, $AD - BC \neq 0$.

     $\rho = k$, $\alpha_1 = k - 1$, $\alpha_2 = 2$.
   - Functions having polynomial Schwarzian derivative.
     Nice class containing e^z, $tan(z)$, the Airy functions, $\int_0^z \exp(Q(\xi)) d\xi + c$ ... and it is invariant under Möbius transformations.

A New Class of Meromorphic Functions; Asymptotic Tracts

(Volker Mayer, M. U., 2017) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Let

$S(f)$ be the singular set of f^{-1}.

Eremenko–Lyubich class $\mathcal{B}$: $S(f)$ is bounded.
Speiser class $\mathcal{S}$: $S(f)$ is finite.

$$\mathcal{S} \subset \mathcal{B}.$$

We consider only entire functions in class $\mathcal{B}$. WLOG

$$S(f) \subset \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}.$$

$$\mathbb{D}^* := \mathbb{C} \setminus \overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| > 1\}$$

By Eremenko–Lyubich,

$$f : f^{-1}(\mathbb{D}^*) \to \mathbb{D}^*$$

is a covering map.

A New Class of Meromorphic Functions; Asymptotic Tracts

The connected components of $f^{-1}(\mathbb{D}^*)$ are called asymptotic tracts and the restriction of f to any of these tracts, call them Ω, has the special form

$$f_{|\Omega} = \exp \circ \tau$$

where

$$\varphi = \tau^{-1} : \mathcal{H} := \{z \in \mathbb{C} : \Re(z) > 0\} \to \Omega$$

is a conformal homeomorphism.

We always assume that f has only finitely many asymptotic tracts:

$$f^{-1}(\mathbb{D}^*) = \bigcup_{j=1}^{N} \Omega_j$$

This is for example the case if f has finite order.

A New Class of Meromorphic Functions; Asymptotic Tracts; Disjoint Type

If

$$f^{-1}(\overline{\mathbb{D}^*}) = \overline{f^{-1}(\mathbb{D}^*)} \subset \mathbb{D}^*,$$

then f is called a function of disjoint type [Krzysztof Barański, Lasse Rempe] .

Equivalently:

$$\bigcup_{j=1}^{N} \overline{\Omega}_j \subset \mathbb{D}^*.$$

Equivalently:

$$\bigcup_{j=1}^{N} \overline{\Omega}_j \cap \overline{\mathbb{D}} = \emptyset.$$

If $f \in \mathcal{B}$ and $\lambda \in \mathbb{C} \setminus \{0\}$ has sufficiently small modulus, the function λf is of disjoint type

Asymptotic Tracts; Class $\mathcal{D}$

Put

$$Q_T := \{\xi \in \mathbb{C} : 0 < \Re\xi < 4T \ \text{ and } \ -4T < \Im\xi < 4T\}$$

and

$$\Omega_T := \varphi(Q_T).$$

Definition

An entire function $f : \mathbb{C} \to \mathbb{C}$ in $\mathcal{B}$ belongs to the class $\mathcal{D}$ if the following hold:

1. *f has only finitely many tracts.*
2. *f is of disjoint type.*
3. *The corresponding function $\varphi : \mathcal{H} \to \Omega$ of f satisfies the following geometric condition: there exists a constant $M \in (0, +\infty)$ such that for every $T \geq 1$ large enough,*

$$|\varphi(z)| \leq M|\varphi(w)|, \quad z, w \in Q_T \setminus Q_{T/8}.$$

Thermodynamic Formalism

Riemannian metric:

$$|dz|/|z|$$

The corresponding derivative:

$$|h'(z)|_1 = |h'(z)|\frac{|z|}{|h(z)|}.$$

The Perron–Frobenius operator:

$$\mathcal{L}_t g(w) := \sum_{f(z)=w} |f'(z)|_1^{-t} g(z) \quad \text{for every} \quad w \in \overline{\Omega}\,.$$

Thermodynamic Formalism

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In the meromorphic context, the whole thermodynamic formalism can be established provided that

- The Perron–Frobenius operator $\mathcal{L}_t$ is well-defined and bounded,
- $$\lim_{S\to\infty} \|\mathcal{L}_t \mathbb{1}_{\mathbb{D}_S^*}\|_\infty = 0.$$

Integral Means

Let

$$h : Q_2 \to U$$

be a conformal map onto a bounded domain $U \subset \mathbb{C}$.
Define:

$$\beta_h(r,t) := \frac{\log \int_I |h'(r+iy)|^t dy}{\log 1/r} \quad , \quad r \in (0,1) \ \text{ and } \ t \geq 0\,.$$

The integral is taken over $I = [-2,-1] \cup [1,2]$ since this will correspond to the part of the boundary of U that is important for our purposes.

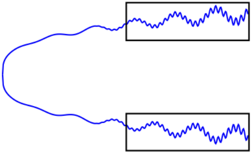

Figure: The part of the boundary relevant for integration.

Rescallings

Model:

$$\tau : \Omega \to \mathcal{H}, \quad f = e^{\tau}, \quad \varphi : \mathcal{H} \to \Omega.$$

$T > 0$ yields

$$\varphi_T := \frac{1}{|\varphi(T)|} \varphi \circ T : \mathcal{H} \to \mathbb{C}.$$

In particular

$$|\varphi_T(1)| = 1.$$

Frequently, we consider

$$\varphi_T = \frac{1}{|\varphi(T)|} \varphi \circ T : Q_2 \longrightarrow \frac{1}{|\varphi(T)|} \Omega_T, \quad T \geq \gamma.$$

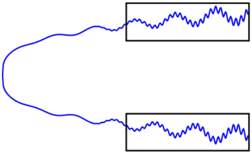

Figure: After rescaling as $T \to \infty$.

Integral Means

$$\beta_\infty(t) := \limsup_{r\to 0} \beta_{\varphi_{1/r}}(r,t) = \limsup_{T\to+\infty} \beta_{\varphi_T}(1/T,t)$$

Proposition

The function $[0,+\infty) \ni t \mapsto \beta_\infty(t)$ *is convex, thus continuous, and*

$$\beta_\infty(0) = 0 \quad \text{and} \quad \beta_\infty(2) \le 1.$$

Integral Means and Negative Spectrum

$$b_\infty(t) := \beta_\infty(t) - t + 1, \quad t \ge 0\,.$$

As an immediate consequence of the previous proposition:

Proposition

The function b_∞ is also convex, thus continuous, with

$$b_\infty(0) = 1 \text{ and } b_\infty(2) \le 0.$$

Consequently, the function b_∞ has at least one zero in $(0,2]$ and we can introduce a number $\Theta_f \in (0,2]$ by

$$\Theta_f := \inf\{t > 0 : b_\infty(t) = 0\} = \inf\{t > 0 : b_\infty(t) \le 0\}\,.$$

Definition

A function $f \in \mathcal{D}$ has negative spectrum if

$$b_\infty(t) < 0 \quad \textit{for all} \quad t > \Theta_f\,.$$

Integral Means and Transfer Operator

Proposition (Volker Mayer, M. U., 2017)

If $f \in \mathcal{D}$ and $t \geq 0$, then

$$\mathcal{L}_t \mathbb{1}(w) \asymp (\log |w|)^{1-t} \left\{ \int_{-1}^{1} \left| \varphi'_{\log |w|}(1+iy) \right|^t dy + \right.$$
$$\left. + \sum_{n \geq 1} 2^{n\left(1-t+\beta_{\varphi_{2^n \log |w|}}(2^{-n}, t)\right)} \right\}$$

for every $w \in \Omega$ with the above series being possibly divergent.

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for every $w \in \Omega$ with the above series being possibly divergent.

Negative Spectrum and Transfer Operator

Theorem (V. Mayer, M. U., 2017)

If $f \in \mathcal{D}$ has negative spectrum, then

- *If $t > \Theta_f$, then $\|\mathcal{L}_t \mathbb{1}\|_\infty < +\infty$.*
- *If $t < \Theta_f$, then $\mathcal{L}_t \mathbb{1}$ is divergent at every point of its domain of definition.*

Proposition (V. Mayer, M. U., 2017)

If $f \in \mathcal{D}$ has negative spectrum and $t > \Theta_f$, then

$$\lim_{S \to \infty} \|\mathcal{L}_t \mathbb{1}_{\mathbb{D}_S^*}\|_\infty = 0\,.$$

Thus, the whole thermodynamic formalism holds

Strongly Regular Functions and Bowen's Formula

A function f in $\mathcal{D}$ with negative spectrum is called strongly regular if there exists $t > \Theta_f$ such that

$$\mathrm{P}(t) > 0.$$

Theorem (V. Mayer, M. U., 2017; Bowen's Formula)

If $f \in \mathcal{D}$ has negative spectrum, then the following are equivalent.

- *The function f is strongly regular.*
- *The function $(\Theta_f, +\infty) \ni t \mapsto \mathrm{P}(t)$ has a (unique) zero $h > \Theta_f$.*
- $\mathrm{HypD}(J(f)) > \Theta_f$.

If one of these holds, then

$$\mathrm{HypD}(J(f)) = h.$$

Strongly Regular Functions and Bowen's Formula

Theorem (K. Barański, B. Karpińska and A. Zdunik, 2009)

If $f \in \mathcal{D}$, then

$$\mathrm{HypD}(J(f)) > 1.$$

Theorem

If $f \in \mathcal{D}$ has negative spectrum and $\Theta_f \leq 1$, then f is strongly regular; whence

$$\mathrm{HypD}(J(f)) = h.$$

Examples of Functions with Negative Spectrum

Definition

An entire function $f:\mathbb{C}\to\mathbb{C}$ is said to be of balanced growth if it has finite order $\rho=\rho(f)$, and if

$$|f'(z)| \asymp |f(z)|\,|z|^{\rho-1}, \quad z\in J(f).$$

Proposition

If $f\in\mathcal{D}$ is of balanced growth, then f is elementary in the sense that

$$b_\infty(t)=\beta_\infty(t)-t+1=1-t, \quad t\geq 0.$$

In particular, f has negative spectrum with $\Theta_f=1$.

Poincaré's functions of TCE polynomials.

Thus, the whole thermodynamic formalism holds

Barański, Karpińska, Zdunik

Theorem (BKZ, 2012)

If $f \in \mathcal{S}$ and $t \geq 0$, then

$$\mathrm{P}(t) = \mathrm{P}_z(t) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{w \in f^{-n}(z)} |(f^n)'(w)|_s^{-t}, \quad z \in \mathbb{C}.$$

exists and is independent of all $z \in \mathbb{C}$ outside a set of Hausdorff dimension zero. Furthermore,

$$\mathrm{P}(t) = \mathrm{P}_{\mathrm{hyp}}(t),$$

where $\mathrm{P}_{\mathrm{hyp}}(t)$ is the supremum of the pressures $\mathrm{P}(f|_X, t)$ taken over all transitive conformal invariant repellers $X \subset J(f)$.
Bowen's formula holds:

$$\mathrm{HD}(J_r(f)) = \mathrm{HypD}(\mathrm{J(f)}) = \mathrm{h} := \inf\{\mathrm{t} \geq 0 : \mathrm{P(t)} \leq 0\}.$$

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The first equality is due to [Lasse Rempe, 2008].

Barański, Karpińska, Zdunik

Theorem (BKZ, 2012)

If $f \in \mathcal{B}$ is tame, i.e.

$$J(f) \setminus \overline{\bigcup_{n=0}^{\infty} f^n\big(Sing(\tilde{f}^{-1})\big)} \neq \emptyset,$$

then the same holds for all z in this diffference of sets.

Barański, Karpińska, Zdunik

Motivated by a question of Dan Mauldin:

Theorem (BKZ, 2018)

If $t > 0$ and a topologically hyperbolic meromorphic function $f : \mathbb{C} \longrightarrow \hat{\mathbb{C}}$ admits a t–conformal measure m_t on $J(f)$ with respect to spherical metric, then

$$\mathrm{P}(t) \leq 0.$$

In addition, if $m_t(J(f) \setminus I_\infty(f)) > 0$, then

$$\mathrm{P}(t) = 0.$$

Barański, Karpińska, Zdunik

Conversly:

Theorem (BKZ, 2018)

Fix $t > 0$. Assume that either $f \in \mathcal{S}$ or $f \in \mathcal{B}$ is a non–exceptional tame function. If $\mathrm{P}(t) = 0$, then

there exists a t–conformal measure m_t on $J(f)$, with respect to the spherical metric.

Barański, Karpińska, Zdunik

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$$m_t\big(\mathbb{C} \setminus B(0,r)\big) = o\left(\frac{(\log r)^{3t}}{r^t}\right) \quad \text{as} \quad r \to +\infty.$$

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f is called exceptional, if there exists a (Picard) exceptional value $\xi \in J(f)$ of f and f has a non–logarithmic singularity over ξ.

Conformal Iterated Function Systems

Let (X, ρ) be a compact metric spces. Let E be a countable (either finite or infinite) set. A collection

$$\mathcal{S} = \{\phi_e : X \longrightarrow X\}_{e \in E}$$

is called an Iterated Function System (or IFS) if all maps ϕ_e are one–to–one contractions with Lipschitz constants $\kappa \in (0,1)$.

For every word $\omega \in E^*$, say $\omega \in E_A^n$, $n \geq 0$, put

$$\phi_\omega := \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_n} : X \longrightarrow X.$$

For any $\omega \in E^{\mathbb{N}}$, the sets

$$\{\phi_{\omega|_n}(X)\}_{n \geq 1}$$

form a descending sequence of nonempty compact sets. So

$$\bigcap_{n \geq 1} \phi_{\omega|_n}(X) \neq \emptyset.$$

For every $n \geq 1$,

$$\mathrm{diam}\left(\phi_{\omega|_n}(X)\right) \leq \kappa^n \mathrm{diam}(X).$$

Conformal Iterated Function Systems

We thus conclude that the intersection

$$\bigcap_{n\in\mathbb{N}} \phi_{\omega|_n}\left(X_{t(\omega_n)}\right)$$

is a singleton and we denote its only element by $\pi_{\mathcal{S}}(\omega)$ or simpler, by $\pi(\omega)$. In this way we have defined a map

$$\pi_{\mathcal{S}} := \pi : E_A^\infty \longrightarrow X.$$

The map π is called the *coding map*, and the set

$$J = J_{\mathcal{S}} := \pi(E_A^\infty)$$

is called the limit set of the IFS $\mathcal{S}$.

Conformal Iterated Function Systems

The IFS $\mathcal{S}$ is called conformal if for some $d \in \mathbb{N}$, the following are satisfied:

(a) X is a compact connected subset of $\mathbb{R}^d$, and $X = \overline{\mathrm{Int}(X_v)}$.

(b) (Open Set Condition) For all $a, b \in E$ such that $a \neq b$,

$$\phi_a(\mathrm{Int}(X_{t(a)})) \cap \phi_b(\mathrm{Int}(X_{t(b)})) = \emptyset.$$

(c) (Conformality) There exists an open connected sets $W \supset X$, such that for every $e \in E$, the map ϕ_e extends to a C^1 conformal diffeomorphism from W into W with Lipschitz constant $\leq \kappa$.

(d) (Bounded Distortion Property (BDP)) There are two constants $L \geq 1$ and $\alpha > 0$ such that for every $e \in E$ and every pair of points $x, y \in X$,

$$\left| \frac{|\phi_e'(y)|}{|\phi_e'(x)|} - 1 \right| \leq L \|y - x\|^\alpha,$$

where $|\phi_\omega'(x)|$ denotes the scaling factor of the derivative $\phi_\omega'(x) : \mathbb{R}^d \to \mathbb{R}^d$ which is a similarity map.

Thermodynamic Formalism for Conformal Iterated Function Systems

Perron–Frobenius Operators: For every real number $t \geq 0$, let

$$\mathcal{L}_t : C(X) \longrightarrow C(X),$$

$$\mathcal{L}_t g(x) := \sum_{e \in E} g(\phi_e(X)) |\phi_e'(x)|^t.$$

For $n \in \mathbb{N}$ define the partition function:

$$Z_n(t) := \sum_{|\omega|=n} \|\phi_\omega'\|_\infty^t$$

and the topological pressure of t:

$$\mathrm{P}(t) := \lim_{n \to +\infty} \frac{1}{n} \log Z_n(t).$$

Thermodynamic Formalism for Conformal Iterated Function Systems

The whole thermodynamic formalism holds

$$\theta_{\mathcal{S}} := \inf\{t \geq 0 : \mathrm{P}(t) < +\infty\} = \inf\{t \geq 0 : Z_1(t) < +\infty\}.$$

Theorem (D. Mauldin, M. U., 1996, 2003; Bowen's Formula)

If $\mathcal{S}$ is a conformal IFS, then

$$h = h_{\mathcal{S}} := \mathrm{HD}(J_{\mathcal{S}}) = \inf\{s \geq 0 : \mathrm{P}(s) \leq 0\} \geq \theta_{\mathcal{S}}.$$

Conformal IFSs and Elliptic Functions

$$f : \mathbb{C} \longrightarrow \hat{\mathbb{C}} \quad - \quad \text{an elliptic function}$$

q – the maximal order of poles of f; $I_\infty(f)$ – the escaping set

Theorem (J. Kotus, M. U., 2003)

If $f : \mathbb{C} \to \overline{\mathbb{C}}$ is an elliptic function, then

$$\mathrm{HD}(J(f)) \geq \mathrm{HD}(J_r(f)) > \frac{2q}{q+1} \geq 1.$$

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If $f : \mathbb{C} \to \overline{\mathbb{C}}$ is an elliptic function, then

$$\mathrm{HD}(I_\infty(f)) = \frac{2q}{q+1}.$$

Further Estimates of Hausdorff Dimension using IFSs

Theorem (V. Mayer, 2009; J. Kotus, M. U., 2008)

Let $f : \mathbb{C} \longrightarrow \hat{\mathbb{C}}$ *be a transcendental meromorphic function with finite order* ρ.

- *Suppose that* f *has a pole* $b \in \mathbb{C} \setminus \overline{Sing(f^{-1})}$ *with multiplicity* m.
- *Suppose also that*
$$|f'(z)| \leq K|z|^{\alpha}$$
on $f^{-1}(D)$ *where* $\alpha > -(1 + \frac{1}{m})$ *and* D *is a neighborhood of* b.

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Theorem (V. Mayer, 2009; J. Kotus, M. U., 2008)

Let $f : \mathbb{C} \longrightarrow \hat{\mathbb{C}}$ be a transcendental meromorphic function with finite order ρ.

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- *Suppose also that*

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  on $f^{-1}(D)$ where $\alpha > -(1 + \frac{1}{m})$ and D is a neighborhood of b.

Then

$$\mathrm{HD}(J(f)) \geq \frac{\rho}{\alpha + 1 + 1/M}.$$

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Then
$$\mathrm{HD}(J(f)) \geq \frac{\rho}{\alpha + 1 + 1/M}.$$

With $R \in [0, +\infty]$ Let

$$I_R(f) := \{z \in \mathbb{C} : \liminf_{n\to\infty} |f^n(z)| \geq R\}.$$

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Further Estimates of Hausdorff Dimension using IFSs

Theorem (W. Bergweiler, J. Kotus, 2012)

Let $f \in \mathcal{B}$, $\rho = \rho(f) < \infty$, ∞ is not an asymptotic value and all but finitely many poles have multiplicities bounded above by M. Then

$$\mathrm{HD}(I_\infty(f)) \leq \lim_{R\to\infty} \mathrm{HD}(I_R(f)) \leq \frac{2M\rho}{2+M\rho}$$

Theorem (W. Bergweiler, J. Kotus, 2012)

$\forall \rho \in (0,+\infty)$ $\forall M \in \mathbb{N}$ $\exists f \in \mathcal{B}$ with $\rho(f) = \rho$, all poles being of multiplicity M and for which ∞ is not an asymptotic value, such that

$$\mathrm{HD}(I_\infty(f)) = \frac{2M\rho}{2+M\rho} \quad \text{while} \quad \mathrm{HD}(I_R(f)) > \frac{2M\rho}{2+M\rho}$$

for all $R > 0$.

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Theorem (W. Bergweiler, J. Kotus, 2012)

$\forall \rho \in (0,+\infty)$ $\forall M \in \mathbb{N}$ $\exists f \in \mathcal{B}$ *with* $\rho(f) = \rho$, *all poles being of multiplicity* M *and for which* ∞ *is not an asymptotic value, such that*

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for all $R > 0$.

Proving the last inequality Walter and Janina used the above mentioned result of Volker Mayer (which uses IFSs).

Hyperbolic Dimension

By building an appropriate conformal IFS, Lasse Rempe proved the following.

Theorem (L. Rempe, 2008)

If $f : \mathbb{C} \longrightarrow \hat{\mathbb{C}}$ is a non–constant, non–linear meromorphic function, then

$$(\mathrm{DD}(J(f)) =)\mathrm{HypD}(\mathrm{J}(\mathrm{f})) = \mathrm{HD}(\mathrm{J}_\mathrm{r}(\mathrm{f})).$$

Nice Sets

Defined by Juan Rivera–Letelier in 2007.

Theorem (N. Dobbs, 2011)

Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a tame meromorphic function. Fix $z \in \mathcal{J}(f) \setminus \mathcal{P}(f)$, $\kappa > 1$, and $K > 1$.
Then $\exists L > 1$ and $\forall r > 0$ sufficiently small $\exists$ an open connected set

$$U = U(z,r) \subset \mathbb{C} \setminus \mathcal{P}(f), \text{ called a Nice Set, such that}$$

(a) *If $V \in Comp(f^{-n}(U))$, then either*

$$V \cap U = \emptyset \quad \text{or} \quad V \subset U.$$

(b) *If $V \in Comp(f^{-n}(U))$ and $V \subset U$, then, for all $w, w' \in V$,*

$$|(f^n)'(w)| \geq L \;\; and \;\; \frac{|(f^n)'(w)|}{|(f^n)'(w')|} \leq K.$$

(c) $\overline{B(z,r)} \subset U \subset B(z, \kappa r) \subset \mathbb{C} \setminus \mathcal{P}(f)$.

Nice Sets

If

$$V \in \mathrm{Comp}(f^{-n}(U)) \quad \text{and} \quad V \subset U$$

then there exists a unique holomorphic inverse branch

$$f_V^{-n} : B(z, \kappa r) \longrightarrow \mathbb{C}$$

such that

$$f_V^{-n}(U) = V.$$

Assume in addition that

$$f^k(V) \cap U = \emptyset$$

for all integers $k = 1, 2, \ldots, n-1$.

The collection $\mathcal{S}_U$ of all such inverse branches forms a conformal iterated function system.

Nice Sets Techniques

Theorem (B. Skorulski, M. U., 2014)

if $f : \mathbb{C} \to \bar{\mathbb{C}}$ is a tame meromorphic function, then

(a) $h = \mathrm{HypD}(\mathrm{J}(\mathrm{f})) = \mathsf{HD}(\mathcal{J}_{\mathrm{r}}(\mathrm{f}))) = \mathsf{HD}(\mathcal{J}_{\mathrm{U}})$ *for every nice set U.*

(b) *The h-dimensional Hausdorff measure H_h restricted to each nice limit set $\mathcal{J}_U$, $U \in \mathcal{U}$, is finite.*

(c) *The h-dimensional Hausdorff measure H_h restricted to $\mathcal{J}_r(f)$ is σ-finite.*

Nice Sets Techniques

Theorem (B. Skorulski, M. U., 2014)

Assume that a tame meromorphic function $f : \mathbb{C} \longrightarrow \hat{\mathbb{C}}$ is strongly N–regular. Let $\Lambda \subset \mathbb{C}^d$ be an open set and let $\{f_\lambda\}_{\lambda\in\Lambda}$ be an analytic family ($\Lambda \ni \lambda \mapsto f_\lambda(z) \in \mathbb{C}$ is anallytic for all $z \in \mathbb{C}$) of meromorphic functions with the following properties:

1. $f_{\lambda_0} = f$ *for some* $\lambda_0 \in \Lambda$,
2. *there exists a holomorphic motion* $H : \Lambda \times \overline{\mathcal{J}_{\lambda_0}} \to \mathbb{C}$ *such that each map* H_λ *is a topological conjugacy between* f_{λ_0} *and* f_λ *on* $\mathcal{J}_{\lambda_0}$.

Then the map

$$\Lambda \ni \lambda \longmapsto \mathrm{HD}\big(\mathcal{J}_r(f_\lambda)\big)$$

is real–analytic on some neighborhood of λ_0.

Nice Sets Techniques

Give rise to Lai-Sang Young Towers approach. This was exploited in:

- [F. Przytycki, J. Rivera–Letelier, 2007] (topological Collet-Eckmann rational functions)
- [M. Szostakiewicz, M. U., A Zdunik, 2015], (all rational functions)
- [M. Pollicott, M. U., 2018] (tame topological Collet-Eckmann rational functions)
- [J. Kotus, M. U., 2019+] (elliptic functions)

Non–Autonomous Conformal IFSs

- Systematically developed in [L. Rempe-Gillen, M. U., 2016]
- Main result: Bowen's Formula.
- Application to transcendental dynamics:

  Let $f : \mathbb{C} \to \mathbb{C}$ be a non-linear, non-constant meromorphic function, and let $\mathrm{Tr(f)}$ denote the set of transitive points of f.
  Then
  $$\mathrm{HD}\big(\mathrm{Tr(f)}\big) \geq \mathrm{HypD(J(f))}.$$
- Generalized to Conformal Non–Autonomous Graph Directed Markov Systems in [Jason Atnip, 2017]
- Application to transcendental dynamics:

  Lower estimates of Hausdorff dimension of Julia sets of non–autonomous perturbations of elliptic and V. Mayer's functions.

Thank You!