Multiply connected wandering domains of entire functions

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The Open University

Edinburgh
July 2018

Walter explains something that should have been obvious!

Photo taken at UCL by Matt Buck

Basic definitions

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The **Julia set** (or chaotic set) is

$$J(f) = \mathbb{C} \setminus F(f).$$

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The **fast escaping set** is $A(f) = \bigcup_{L \in \mathbb{N}} f^{-L}(A_R(f))$ where:

$$A_R(f) = \{z \in \mathbb{C} : |f^n(z)| \geq M^n(R) \text{ for } n \in \mathbb{N}\},$$

assuming $R > 0$ is such that $M^n(R) \to \infty$ as $n \to \infty$.

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3. **Siegel disc** There is a conformal mapping $\phi : U \to \mathbb{D}$, where $\mathbb{D}$ is the unit disc, such that $\phi(f^p(\phi^{-1}(z))) = e^{2\pi i\theta} z$, where θ is irrational.

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4. **Baker domain** For all $z \in U$, we have $f^{np}(z) \to \infty$ as $n \to \infty$.

Wandering domains

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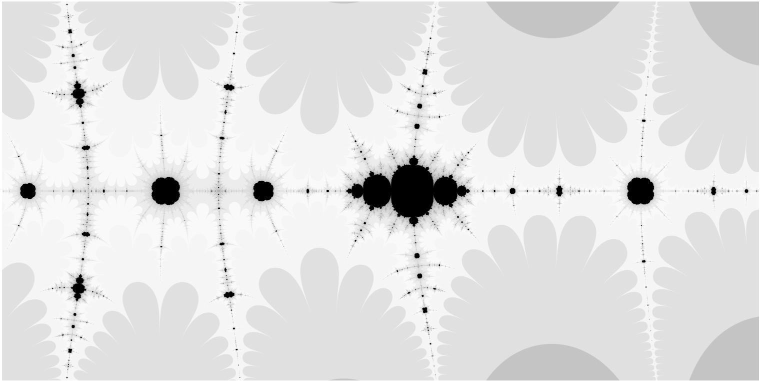
Question Can we classify wandering domains?

A wandering cauliflower



$$f(z) = z\cos z + 2\pi$$

Picture courtesy of David Marti-Pete

Multiply connected Fatou components – basic properties

Baker (first two properties), R+S

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- for large n, each component of the boundary of U_n is a component of $A_R(f) \cap J(f)$
- $A_R(f)$ is a spider's web.

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Theorem

Let U be a multiply connected wandering domain and $D \subset U$ be an open neighborhood of z_0. Then there exists $\alpha > 0$ such that

$$f^n(D) \supset A(|f^n(z_0)|^{1-\alpha}, |f^n(z_0)|^{1+\alpha}), \quad \textit{for large } n \in \mathbb{N}.$$

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Let U be a multiply connected wandering domain, $z_0 \in U$ and $r_n = |f^n(z_0)|$. For large $n \in \mathbb{N}$, the maximal annulus

$$B_n = A(r_n^{a_n}, r_n^{b_n}) \subset U_n, \quad \text{satisfies } 0 < a_n < 1 - \alpha < 1 + \alpha < b_n.$$

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Also, for every compact set $C \subset U$, $f^n(C) \subset B_n$ for $n \geq N(C)$.

Application to a question about commuting functions

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If f, g are analytic with $f \circ g = g \circ f$, does $J(f) = J(g)$?

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Julia's proof

Based on repelling periodic points (in $J(f)$!).

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Corollary

If $f \circ g = g \circ f$ and f, g have no fast escaping Fatou components, then $J(f) = J(g)$.

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There are only two known examples of functions with *simply connected* fast escaping wandering domains:

one due to Bergweiler and one due to Sixsmith.

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Using the two theorems above about multiply connected wandering domains, we obtain the following.

Theorem (Benini+R+S, 2015)

If $f \circ g = g \circ f$ and U is a multiply connected wandering domain of f, then $g(U) \subset F(f)$.

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If $f \circ g = g \circ f$ and U is a multiply connected wandering domain of f, then $g(U) \subset F(f)$.

Corollary

If $f \circ g = g \circ f$ and f, g have no simply connected fast escaping wandering domains, then $J(f) = J(g)$.

Happy Birthday Walter (Bergweiler)!