Growth of some iterated monodromy groups

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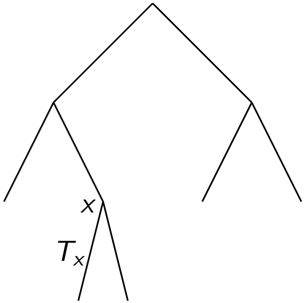
joint with Mikhail Hlushchanka

July 13th, 2018

Self-similar groups

T rooted d-ary tree.
"Groups acting on T by automorphisms" $G \curvearrowright T$, $G < \mathrm{Aut}(T)$.
For every vertex x of T consider subtree T_x.



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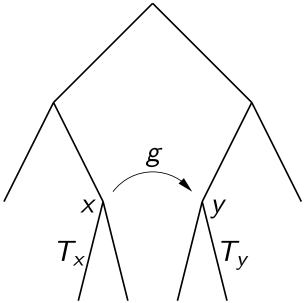

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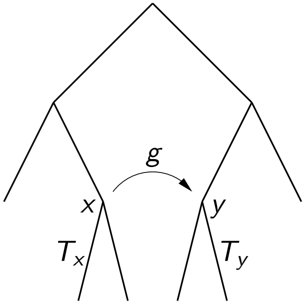

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Let $g \in \mathrm{Aut}(T)$

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Induces map $g|x \colon T \to T$, formally

$$g|x = \iota_{g(x)} \circ g \circ \iota_x^{-1},$$

Restriction of g to x.

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Definition

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Write

$$g = \langle\langle g|x_1, \dots, g|x_n \rangle\rangle h$$

here $X^1 = \{x_1, \dots, x_n\}, h \in \mathrm{Sym}(X)$, wreath recursion of g.

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Grigorchuk group solves first two problems ('84).
Other self-similar groups solve other problems.

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  Grigorchuk '84: yes, Grigorchuk group (self-similar group).

The monodromy group

Let $f\colon X \to Y$ be a covering map. Fix $t \in Y$.

Let $\gamma \subset Y$ be a loop at t. Let $s \in f^{-1}(t)$.

There is a lift $\widetilde{\gamma} \subset X$ of γ by f starting at s.
Let $s' \in f^{-1}(t)$ be endpoint of $\widetilde{\gamma}$. Obtain

$$\pi_1(X, t) \curvearrowright f^{-1}(t).$$

Formally, there is group homomorphism $\varphi\colon \pi_1(Y, t) \to \mathrm{Sym}(f^{-1}(t))$.
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Definition

$\mathrm{mon}(f) = \pi_1(Y, t)/\ker\varphi \simeq \varphi(\pi_1(Y, t))$.

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f^n unramified over $\widehat{\mathbb{C}} \setminus \operatorname{post}(f)$ for all $n \in \mathbb{N}$,
$f^n\colon \widehat{\mathbb{C}} \setminus f^{-n}(\operatorname{post}(f)) \to \widehat{\mathbb{C}} \setminus \operatorname{post}(f)$ is a covering map.
Can define $\operatorname{mon}(f^n)$ for all $n \in \mathbb{N}$.

The preimage tree

Fix $t \in \widehat{\mathbb{C}} \setminus \mathrm{post}(f)$. Let

$$T = \bigsqcup_{n \geq 0} f^{-n}(t).$$

Also $x \sim f(x)$ $\forall x \in f^{-n}(t), n \geq 1$.
d-ary tree, $d = \deg(f)$.

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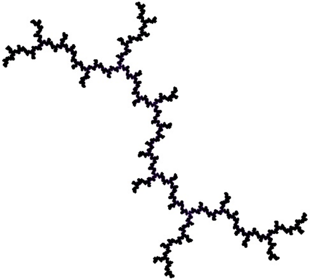
Definition

$$\mathrm{img}(f) = G / \ker \varphi \simeq \varphi(G).$$

Self-similar, defined by Kameyama, Nekrashevych '03.

Growth of IMGs



$\mathrm{img}(z^2 + i)$ intermediate growth (Bux-Pérez '06).
pcf $0 \longrightarrow i \longrightarrow -1 + i \rightleftarrows -i$
Julia set tree or dendrite
postcritical points leaves
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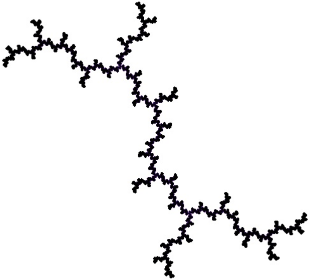

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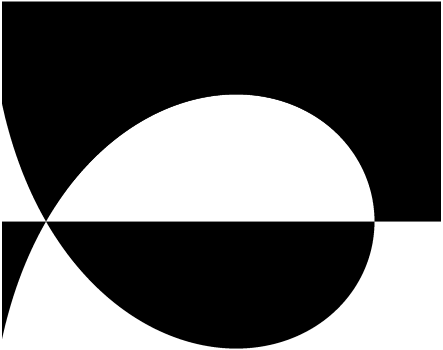
$\operatorname{img}(z^2 - 1)$ exponential growth.
growth img(airplane)?
Conjecture: maps "similar to" $z^2 + i$ have img of intermediate growth.

Tiles

Let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a postcritically finite rational map (or Thurston map).
$\mathcal{C} \subset \widehat{\mathbb{C}}$ Jordan curve with $\operatorname{post}(f) \subset \mathcal{C}$.
n-tile X: closure of component of $\widehat{\mathbb{C}} \setminus f^{-n}(\mathcal{C})$. Colored b/w.



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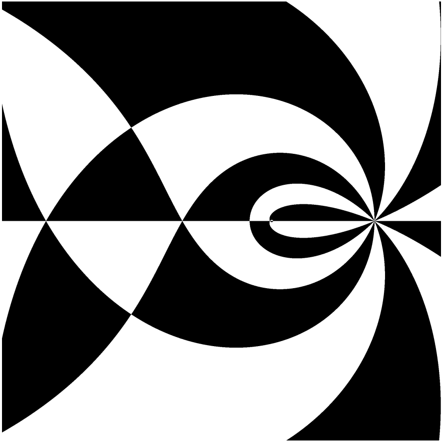

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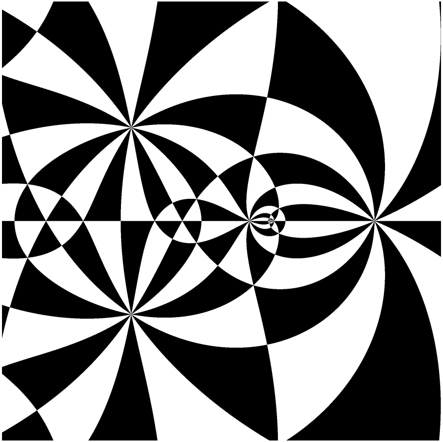

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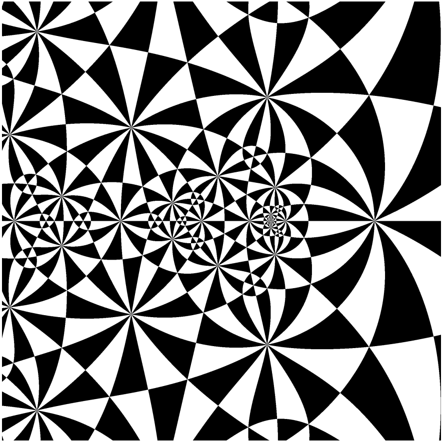

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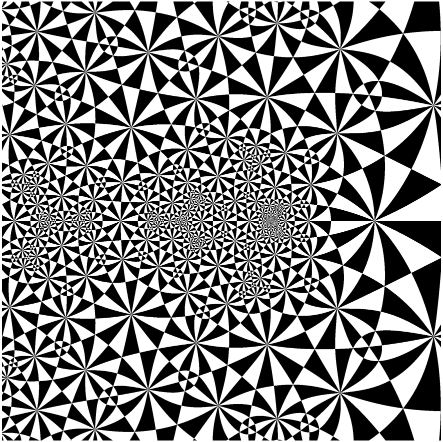

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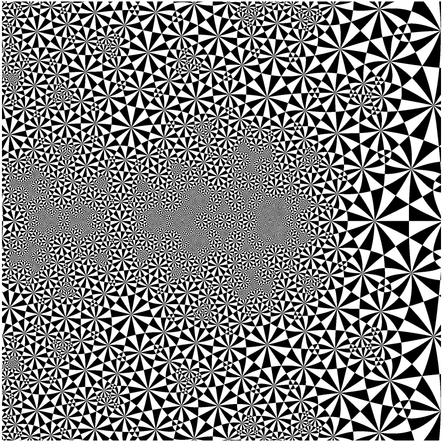

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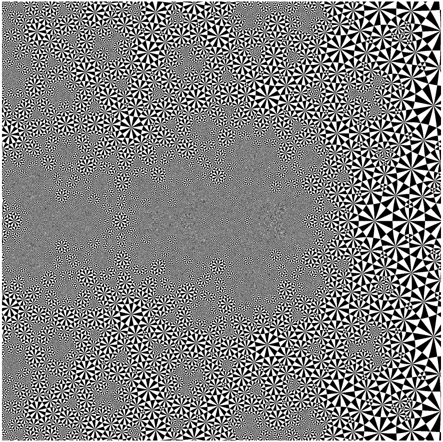
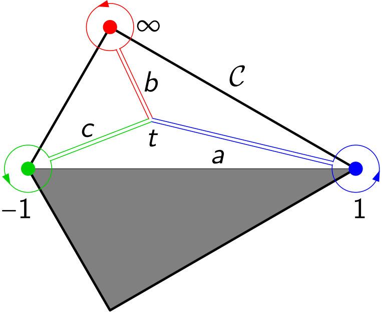


The iterated monodromy group

Fix basepoint $t \in \widehat{\mathbb{C}} \setminus \mathcal{C}$ ($t \notin \operatorname{post}(f)$).
Fix loops at t around postcritical points,
these generate $G = \pi_1(\widehat{\mathbb{C}} \setminus \operatorname{post}(f), t)$.



Let $g = [\gamma] \in G$ and $s \in f^{-n}(t)$. We may lift γ by f^n to $\widetilde{\gamma}$ starting at s.
$\widetilde{\gamma}$ ends in $s' \in f^{-n}(t)$, depends only on g, not on homotopy of γ.
$G \curvearrowright f^{-n}(t)$

IMG and flowers

Each white n-tile contains point from $f^{-n}(t)$,
each point $s \in f^{-n}(t)$ contained a white n-tile.
Can identify $f^{-n}(t) = \{\text{white } n\text{-tiles}\}$.

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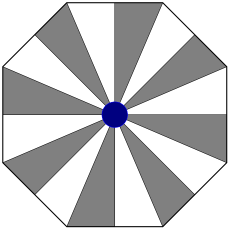

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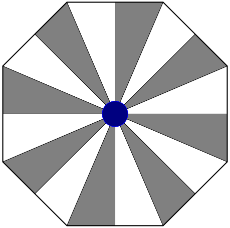

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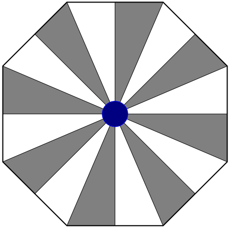

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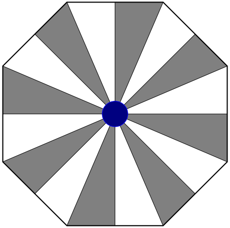

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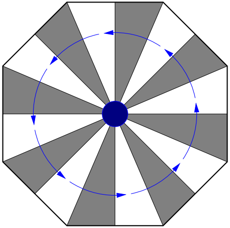
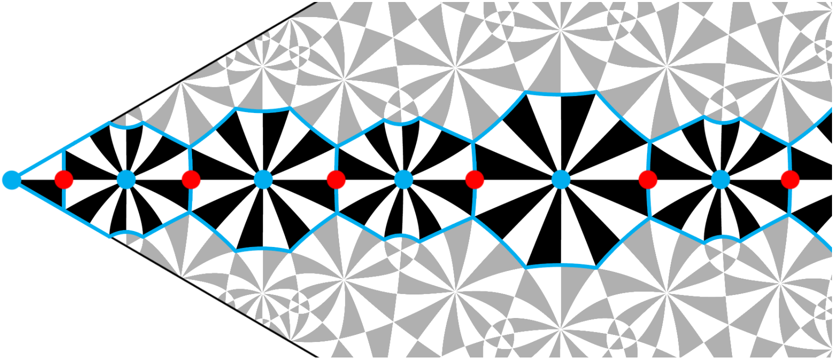


a acts by rotating tiles around center

IMG

Careful: generator a rotates all flowers at same time.
img acts on sequence of n-tiles.
Alternatively, img acts effectively on (any) weak tangent of snowball.



A polynomial

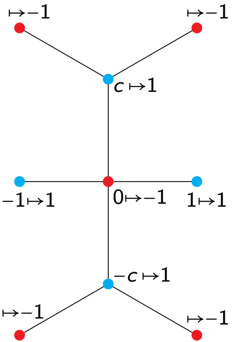
$$P(z) = \frac{2}{27}(z^2+3)^3(z^2-1)+1 = \frac{2}{27}z^8 + \frac{16}{27}z^6 + \frac{4}{3}z^4 - 1.$$

critical points $\pm\sqrt{3}i, 0$, are mapped as follows

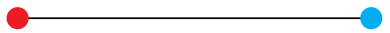
$$\begin{tikzcd} \pm\sqrt{3}i \arrow[r, "3:1"] & 1 \arrow[loop above] & -1 \arrow[l] & 0. \arrow[l, "4:1"'] \end{tikzcd}$$

Thus $\mathrm{post}(P) = \{-1, 1, \infty\}$, pcf.
P is a Shabat polynomial.
Julia set $\mathcal{J}$ of P dendrite, $-1, 1$ leaves of $\mathcal{J}$.
Have $P([-1,1]) = [-1,1]$, this is the Hubbard tree of P.

A polynomial



(a) The pre-Hubbard tree/ dessin d'enfant of P.



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P not renormalizable.

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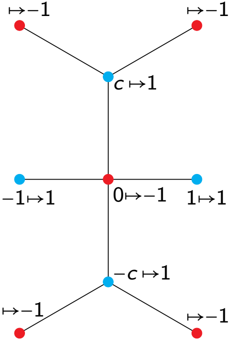

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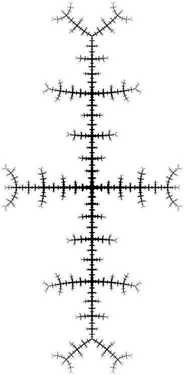

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Theorem (Hlushchanka-M '18)

$\mathrm{img}(P)$ *is of exponential growth.*

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Proof: consider preimages of Hubbard tree.

Let $a \in \mathrm{img}(P)$ be loop around 1, b loop around -1.
$a^2 b^2$ acts by translation around $[-1, 1]$.
$a^{j_1} b^2 \dots a^{j_n} b^2$ acts as n translations, for $j_1, \dots, j_n \in \{2, 6, 10\}$.

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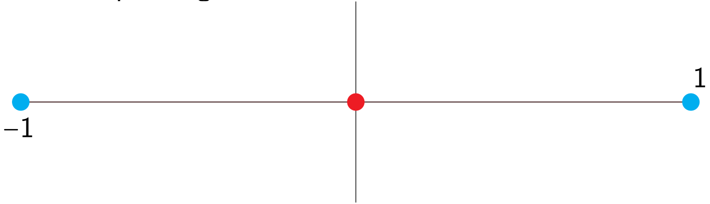

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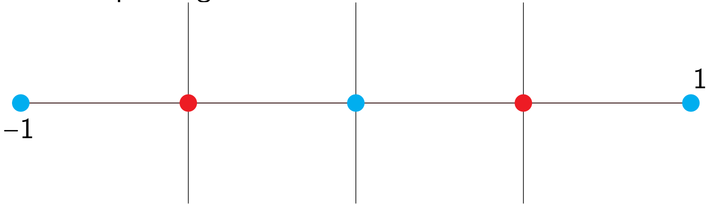

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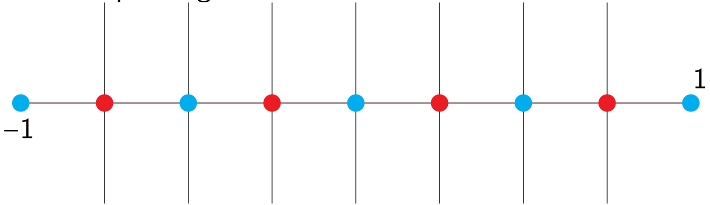

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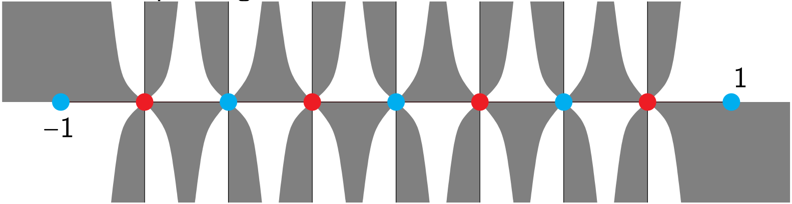

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Need to show: distinct words

$$w_1 \coloneqq a^{j_1} b^2 \dots a^{j_n} b^2 \qquad w_2 \coloneqq a^{k_1} b^2 \dots a^{k_m} b^2$$

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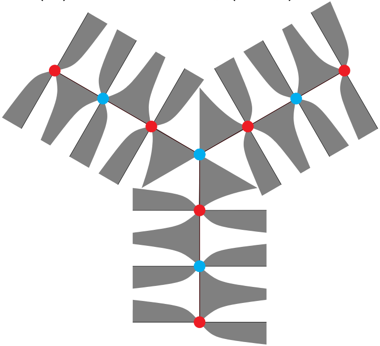

Can assume $j_1 \neq k_1$.
3 and 4 relatively prime.
w_1 and w_2 translate along distinct branches.
So $w_1 \neq w_2$ in $\mathrm{img}(P)$.
$\mathrm{img}(P)$ exponential growth.

Outlook

Get criterion for exponential growth.

Examples for rational with with Julia set $\widehat{\mathbb{C}}$, Sierpiński carpet, as well as obstructed maps.

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Examples for rational with with Julia set $\widehat{\mathbb{C}}$, Sierpiński carpet, as well as obstructed maps.

Open questions:

Are there non-polynomial rational maps with img of intermediate growth?
Are there rational maps with img that are torsion groups?