Construction of transcendental entire functions of arbitrarily slow growth with prescribed polynomial dynamics

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Resonances in Complex Dynamics

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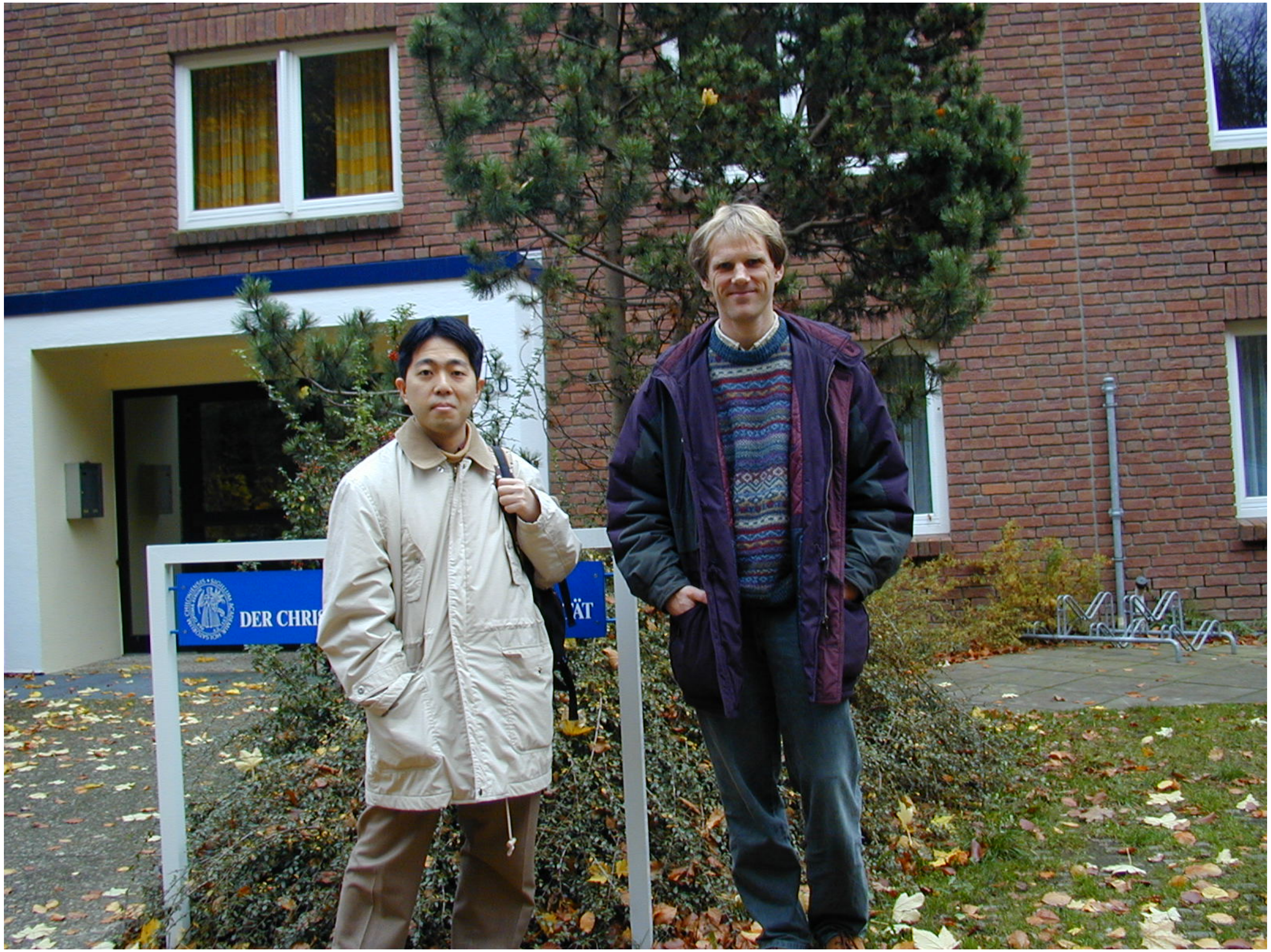


1999.11.10 12:19, in Kiel (Germany)

2003.9.17 16:58, in Kochi (Japan)

§1 Introduction

Definition 1 :

f **: a transcendental entire function,** f^n **: the** n**-th iterate of** f

- $F(f) := \{z \in \mathbb{C} \mid {}^{\exists}U : \textbf{nbd. of } z,\ \{f^n|_U\}_{n=1}^{\infty} \textbf{ is normal}\}$: **Fatou set**

  ($\{f^n|_U\}_{n=1}^{\infty}$ **is normal**

  $\Longleftrightarrow$ ${}^{\forall}$**subseq. of** $\{f^n|_U\}_{n=1}^{\infty}$ **contains a local unif. convergent subseq.**)

- $J(f) := \mathbb{C} \setminus F(f)$: **Julia set** $= \overline{\{\textbf{repelling periodic points}\}}$

- $\mathrm{sing}(f^{-1}) := \{$**all crit. & asympt. values and their accumulation pts**$\}$

- $P(f) := \overline{\bigcup_{n=0}^{\infty} f^n(\mathrm{sing}(f^{-1}))}$: **post-singular set**

- $S := \{f \mid f : \textbf{transcendental entire}, {}^{\sharp}\mathrm{sing}(f^{-1}) < \infty\}$

- $B := \{f \mid f : \textbf{transcendental entire},\ \mathrm{sing}(f^{-1}) : \textbf{bounded}\}$

  (trivially $S \subset B$**)**

There are two directions for research on dynamics of transcendental entire functions:

(I) Research on f with similar behavior as polynomials,

(II) Research on phenomena which never occur for polynomials.

★ **For (I):**

Example 2:

$$e^z,\ ze^z,\ \int^z P(z)e^{Q(z)}dz\ (P,\ Q : \textbf{polyn.}),\ \sin z,\ \cos z \in S, \qquad \frac{\sin z}{z} \in B \setminus S$$

Theorem 3 (Goldberg-Keen, Eremenko-Lyubich):

$f \in S \implies f$ **has no wandering domains and Baker domains.**

Theorem 4 (Eremenko-Lyubich, 1992):

$f \in B \implies {}^{\forall} z \in F(f)$, $f^n(z) \not\to \infty\ (n \to \infty)$. **In particular, f has no Baker domains.**

★ **For (II):**

Theorem 5 (Baker, 1963+1976)：

$^{\exists}f(z) = cz^2 \prod_{n=1}^{\infty}\left(1+\frac{z}{r_n}\right), \quad c > 1,\ 1 < r_1 < r_2 < \cdots$ **(r_n satisfies some recursive formula) has a multiply connected wandering domain.**

Theorem 6 (Herman, 1984)：

$f(z) := z + 1 - e^z + 2\pi i$ **has a simply connected wandering domain.**

Definition 7 ：

There exists an f of arbitrarily slow growth with property P

$\Longleftrightarrow$ **For any monotone increasing function $\varphi(r) > 0$ $(r > 0)$ with $\lim_{r\to\infty}\varphi(r) = +\infty$, there exists f with the property P and satisfies**

$$\log M(r,f) < \varphi(r)\log r, \quad {}^{\forall}r > r_0 \quad \left(\ M(r,f) := \max_{|z|=r}|f(z)|\ \right)$$

(Note that if $\varphi(r) \equiv$ const, then f is a polynomial.)

f with sufficiently slow growth has similar properties with polynomials.

Theorem 8 (Hayman, 1960) :

$\log M(r,f) < A(\log r)^2 \implies |f(z)| > {}^{\exists}K$ **outside small neighborhoods of zeros of f**

On the other hand,

Theorem 9 (Baker, 1984) :

There exists a transcendental entire function with arbitrarily slow growth which has a multipy-connected wandering domain.

Theorem 10 (Bergweiler-Eremenko, 2000) :

There exists a transcendental entire function with arbitrarily slow growth which satisfies $J(f) = \mathbb{C}$.

§2 Main Result

Theorem A :

For a given polynomial P with $\deg P \geq 2$, There exists a transcendental entire function with arbitrarily slow growth which satisfies the following:

(1) There exists a topological disk U such that $(f|U, U, f(U))$ is polynomial-like and conjugate to P.

(2) Periodic Fatou components of $(f|U, U, f(U))$, (which come from P) are the only periodic Fatou components of f and any Fatou component of f is eventually mapped to one of these components. In particular,

(i) f has no wandering domains.

(ii) If $J(P) = K(P) := \{z \mid P^n(z) \not\to \infty \ (n \to \infty)\}$, then $J(f) = \mathbb{C}$

(3) f has no asymptotic values and all the critical points of f escape to ∞ under the iterate of f, except for the ones which correspond to the non-escaping critical points of P.

(Outline of Proof):

Proposition B :

Suppose a given polynomial P **with** $d = \deg P \geq 2$ **and** $z_1,\ z_2, \cdots, z_{k-1} \in \mathbb{C}$ **satisfy the following:**

(a) $P(0) = 0,\ P(1) = 1$**,**

(b) $z_1,\ z_2, \cdots, z_{k-1} \in K(P)$

(c) Let $c_1,\ c_2, \cdots, c_l$ **be the distinct critical points of** P**, then** $c_1, \cdots, c_m \in K(P),\ c_{m+1}, \cdots, c_l \in \mathbb{C} \setminus K(P)$**.**

Then for any given $z_k \in \mathbb{C} \setminus K(P),\ \varepsilon > 0$ **and** $R > 0$**, there exist a polynomial** Q **and** $z_1',\ z_2', \cdots, z_k'$ **which satisfy the following:**

(1) $\deg Q = d + 1$

(2) $Q(0) = 0,\ Q(1) = 1$

(3) There exists a quasiconformal map φ **and a topological disk** U **such that** $K(P) \subset \varphi(U)$ **and** $Q|U \sim_\varphi P$**.**

(4) $z'_j := \varphi^{-1}(z_j)$ $(1 \leq j \leq k)$ **satisfy**

$|z_j - z'_j| < \varepsilon$ $(1 \leq j \leq k)$, $\quad z'_1,\ z'_2, \cdots, z'_{k-1},\ z'_k \in K(Q)$ (**in fact** ${}^{\exists}m,\ Q^m(z'_k) = 0$)

(5) $|P(z) - Q(z)| < \varepsilon$ **for** $|z| < R$

(6) $c'_j := \varphi^{-1}(c_j)$ $(1 \leq j \leq l)$ **and** ${}^{\exists}c'_{l+1}$ **are the distinct critical points of** Q **and satisfy**

$$|c_j - c'_j| < \varepsilon, \quad c'_1 \sim c'_m \in K(Q), \quad c'_{m+1} \sim c'_l,\ c'_{l+1} \in \mathbb{C} \setminus K(Q)$$

(7) Let $a_1, \cdots, a_d$ **be the zeros of** P**, then the zeros of** Q **are** $a'_1, \cdots a'_d,\ a'_{d+1}$ **and satisfy**

$$|a_j - a'_j| < \varepsilon\ (1 \leq j \leq d), \quad |a'_{d+1}| > R$$

(Outline of Proof of Proposition B):

(Construction of $Q(z)$) :

[1] Define the quasiregular map $Q_1(z)$ as follows:

$$Q_1(z) = \begin{cases} P(z) & |z| \le r \\ \psi(z) & r < |z| < 2r \\ \widetilde{P}(z) := z^d(a - az/P^n(z_k)) & |z| \ge 2r \end{cases}$$

$$P(z) = z^d(a + h_1(z)), \quad \widetilde{P}(z) = z^d(a + h_2(z)) \quad (\textbf{i.e. } h_2(z) = -az/P^n(z_k))$$

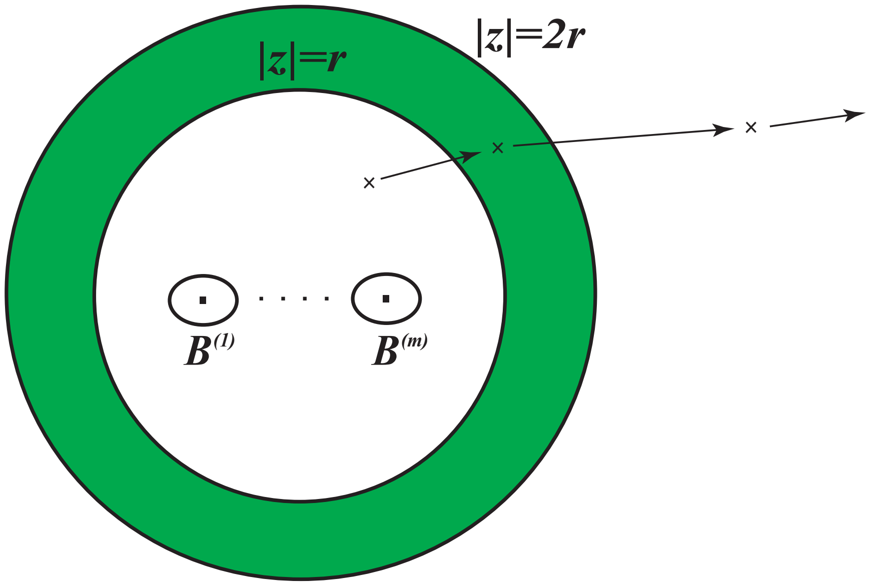
$$\psi(z) := z^d(a + h(z)), \quad h(z) := \left(2 - \frac{|z|}{r}\right) h_1(z) + \left(\frac{|z|}{r} - 1\right) h_2(z)$$

[2] Construction of Q_1-invariant ellipse field :

Since $^\forall z \in \{r \le |z| \le 2r\}$ **satisfies**

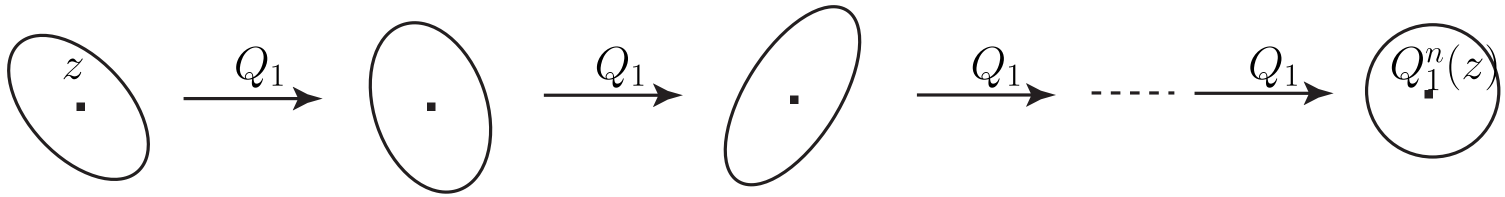
$$|Q_1(z)| > 2r, \; Q_1^n(z) \to \infty \; (n \to \infty),$$

the orbit of $^\forall z \in \mathbb{C}$ **passes the region** $\{r \le |z| \le 2r\}$**, where** Q_1 **is not holomorphic, at most once. Then define**



$$X_{\mu_n} := (Q_1^n)^*(X_0), \quad X_0 : \textbf{circle field}$$

(i.e. X_{μ_n} = **pull back of** X_0 **by** Q_1^n).



It follows that $X_{\mu_n} \to {}^{\exists}X_{\mu} \quad (n \to \infty)$ **and** X_μ **is a** Q_1**-invariant ellipse field by the constrcution.**

[3] By measurable Riemann mapping theorem, ${}^{\exists}\varphi$ **with** $\mu_\varphi(z) = \mu(z)$.

[4] $Q = \varphi \circ Q_1 \circ \varphi^{-1}$ **is holomorphic on** $\mathbb{C}$ **and since it is finite to one,** Q **is a polynomial.** $\square$

Take a dense subset $\{z_j\} \subset \mathbb{C}$ **with** $z_1 = 0$. **Also take a sequence** $\{R_n\}$ **with** $R_n \nearrow \infty$, $R_1 >> 1$ **and** $\sum \frac{1}{R_n} < \infty$. **Take** $\{\varepsilon_n\}$ **with** $\varepsilon_n > 0$ **and** $\sum \varepsilon_n < {}^{\exists}\varepsilon_\infty$ **(small enough). By starting with** $P_0(z) := P(z)$, z_1, $\varepsilon_1 > 0$, $R_1 > 0$ **and applying Proposition B over and over again, we get** $\{P_n(z)\}_{n=0}^{\infty}$ **with** $\deg P_n = d + n$ **which converges to an** $f(z)$.

Since $P_n(0) = 0$ **and** $P_n(1) = 1$**, we have**

$$P_n(z+1) = \prod_{k=1}^{n} \left(1 - \frac{z}{c_{n,k}}\right), \ c_{n,k} := a_{n,k} - 1$$

Then $^{\exists}c_k := \lim_{n\to\infty} c_{n,k}$ **and** $|c_k| > R_k - 2$ **and we have**

$$f(z+1) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{c_k}\right).$$

This shows that f **has genus 0 and if we choose** $\{R_k\}$ **so that it increases rapidly, then** f **has arbitrarily slow growth (by a standard theory of entire functions). Also it is shown by the construction that preimages of** $\mathrm{int}K(f|_U)$ **(if any) are dense in** $\mathbb{C}$**. So there are no wandering domains.** $\square$

§3 Applications

Corollary 11 = Theorem 10 (Bergweiler-Eremenko, 2000) :

There exists a transcendental entire function with arbitrarily slow growth which satisfies $J(f) = \mathbb{C}$.

($\because$ $P(z) := 4z^2 - 3z$ **(or in general,** P **with** $K(P) = J(P)$**))**

Corollary 12 (Baker (2001), Boyd (2002)) :

There exists a transcendental entire function with arbitrarily slow growth which satisfies the following:

(1) 0 **is an attracting fixed point.**

(2) $F(f) =$ **the attractive basin of** 0**.**

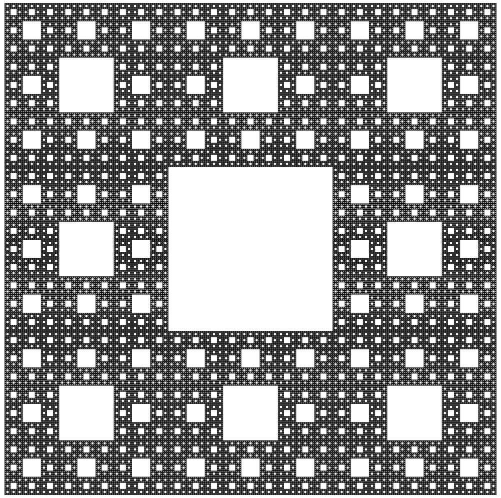
($\because$ $P(z) := z^2$ **(or in general,** P **with only one attractive fixed point))**

Corollary C (K, 2013) :

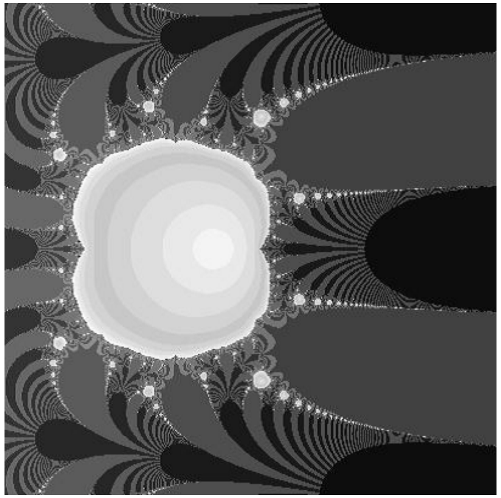
There exists a transcendental entire function with arbitrarily slow growth such that $J(f)$ is a Sierpiński carpet.

($\because$ $P(z) =$ polynomial with m attractive fixed points and their immediate basins have no common boundary points.)

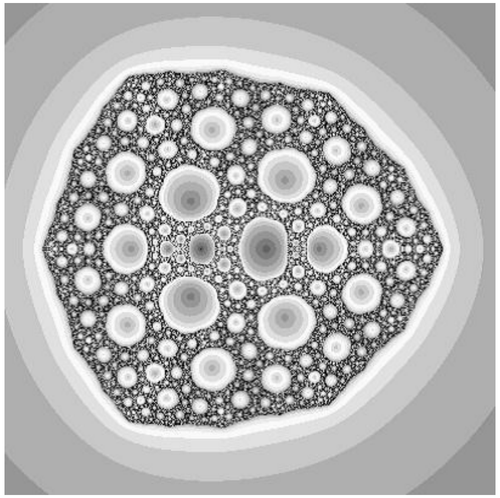
(pictures : by S. Morosawa)



typical Sierpiński carpet



$g_a(z) := ae^a\{z-(1-a)\}e^z \;\; (a=1.2)$



$f(z) := 27z^2(z-1)/\{(3z-2)^2(3z+1)\}$

Remark 13 :

Since $P_n \to f$ locally uniformly on $\mathbb{C}$ and $F(f)$ consists only of attractive basins, it follows that $J(P_n) \to J(f)$ wrt Hausdorff metric (K, 1996). Note that $J(P_n)$ is disconnected and therefore it is not locally connected at any point, while $J(f)$ is locally connected.

Corollary D :

There exists a transcendental entire function with arbitrarily slow growth which has prescribed finite number of attracting, parabolic, Siegel and Cremer cycles.

($\because$ $P(z)$ = polynomial with prescribed finite number of attracting, parabolic, Siegel and Cremer cycles.)

Corollary E :

There exists a transcendental entire function with arbitrarily slow growth such that f has a Cremer point but $J(f)$ is locally connected.

($\because$ $P(z)$ = polynomial with a Cremer point and an attractive fixed point which satisfies some condition.)

Thank you for your attention and

Herzlichen Glückwunsch zum 60 geburtstag, Walter!!