Dynamical moduli spaces and multipliers of periodic points

Lukas Geyer (Montana State University)

joint with Adam Epstein (University of Warwick)

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Outline

Dynamical Moduli Space

Definition

For $d \geq 2$, the *dynamical moduli spaces* of rational maps and polynomials are

$$\mathcal{M}_d = \operatorname{End}_d(\hat{\mathbb{C}}) / \operatorname{M\ddot{o}b}(\hat{\mathbb{C}}) \quad \text{and} \quad \mathcal{M}_d^P = \operatorname{End}_d(\mathbb{C}) / \operatorname{M\ddot{o}b}(\mathbb{C})$$

Here $\operatorname{End}_d(\hat{\mathbb{C}})$ and $\operatorname{End}_d(\mathbb{C})$ are the rational maps and polynomials of degree d, respectively, and $\operatorname{M\ddot{o}b}(\hat{\mathbb{C}})$ and $\operatorname{M\ddot{o}b}(\mathbb{C})$ are the groups of Möbius transformations and affine transformations, respectively, acting by conjugation.

Coordinates on Polynomial Moduli Space

- Easy to see

$$\mathcal{M}_d^P \approx \{f(z) = z^d + a_{d-2}z^{d-2} + \ldots + a_0\}/C_{d-1}$$

where C_{d-1} is the group of rotations by $(d-1)$-th roots of unity ω, acting on coefficients by

$$(a_0, \ldots, a_{d-2}) \mapsto (\omega^{-1}a_0, a_1, \omega a_2, \ldots, \omega^{d-3}a_{d-2}).$$

- Simple description as a quotient of $\mathbb{C}^{d-1}$ by a finite cyclic group (not acting freely), but coefficients a_k do not have an obvious dynamical significance.
- Are there better (dynamically significant) coordinates?

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Coordinates on Rational Moduli Space

Theorem (Milnor 1993)

$\mathcal{M}_2 \approx \mathbb{C}^2$. More explicitly, quadratic rational moduli space has global coordinates given by the symmetric functions $\sigma_1 = \mu_1 + \mu_2 + \mu_3$ and $\sigma_2 = \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3$ of the three multipliers μ_1, μ_2, μ_3 of fixed points.

- Obviously, these coordinates σ_1 and σ_2 have dynamical significance.
- In higher dimensions things are more complicated and less explicit, see Silverman for an algebraic approach.

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Rigidity of Multipliers and Lattès Locus

Definition

A rational map f is a *flexible Lattès map* if there is a doubly periodic non-constant meromorphic $g : \mathbb{C} \to \hat{\mathbb{C}}$ and an integer $n \geq 2$ with $f(g(z)) = g(nz)$ for all z. The *flexible Lattès locus* Λ_d is the corresponding subset of $\mathcal{M}_d$.

Theorem (McMullen 1987)

Outside of the flexible Lattès locus, the Möbius conjugacy class of a rational map is determined up to finitely many choices by the set of all multipliers of periodic points.

Note that deformations of the torus associated to g give rise to a complex one-dimensional family in $\mathcal{M}_d$ with all multipliers constant.

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Rigidity of Multipliers, quantitatively

Corollary (McMullen 1987)

For every $d \geq 2$ there exist N_d and K_d such that the multipliers of periodic points up to period N_d determine at most K_d points in $\mathcal{M}_d \setminus \Lambda_d$.

Question

What are the minimal pairs (N_d, K_d)?

- Milnor: $N_2 = K_2 = 1$ is optimal.
- For all $\epsilon > 0$ there exists $C_\epsilon > 0$ such that $K_d \geq C_\epsilon d^{1/2-\epsilon}$. (Silverman, using non-flexible Lattès maps.)
- There are d independent multipliers of fixed points, and $\dim \mathcal{M}_d = 2d - 2 > d$ for $d \geq 3$, so $N_d \geq 2$ for $d \geq 3$.

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Generic rigidity definition

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For $d \geq 2$, $N_d, K_d \geq 1$, we say that multipliers up to period N_d *generically* determine at most K_d points in dynamical moduli space $\mathcal{M}_d$ if there exists an open dense subset $\widetilde{\mathcal{M}}_d \subset \mathcal{M}_d$ such that this holds in $\widetilde{\mathcal{M}}_d$.

Theorem (G., Epstein)

For $d \geq 3$, multipliers up to period $N_d = 2$ generically determine at most a finite number K_d of points in dynamical moduli space.

- Gorbovickis 2015: Multipliers of $2d - 2$ distinct periodic cycles, under mild additional conditions, are algebraically independent. Requires that at least one cycles has period ≥ 3, so generic rigidity with $N_d = 3$ follows.

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Generic rigidity, idea of proof

- $\mathrm{End}_d(\hat{\mathbb{C}})$ is an irreducible affine variety, and generic rigidity follows from the fact that the rank of the derivative of the multiplier map $\mu_2 : \mathrm{End}_d(\hat{\mathbb{C}}) \to \mathbb{C}^m$ is $2d - 2$ on an open dense set. (Here μ_2 maps to the multipliers of periods 1 and 2, and we have to avoid multipliers equal to 1.)
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Generic rigidity, idea of proof, continued

- Parametrizing as $f_{a,b}(z) = \frac{z^d + a_{d-1}z^{d-1} + \ldots + a_1 z}{1 + b_{d-1}z + \ldots + b_1 z^{d-1}}$ with $a, b \in \mathbb{C}^{d-1}$ we can calculate the partial derivatives of all the multipliers of period 1 and 2 with respect to a and b at $a = b = 0$.
- Interesting observation: The matrix of derivatives of fixed point multipliers with respect to a is up to constants the Vandermonde matrix of the $(d-1)$-th roots of unity.
- Computationally it helps that $1/f_{a,b}(1/z) = f_{b,a}(z)$, so that $f_{a,b}$ and $f_{b,a}$ have the same multipliers, which means that we really only need the partial derivatives with respect to a.

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Rigidity for polynomials, preliminaries

- All the same questions can be asked for the polynomial moduli space $\mathcal{M}_d^P$, where the situation is considerably nicer.
- Multipliers of fixed points determine a unique point in $\mathcal{M}_d^P$ only for $d = 2$ and $d = 3$. For $d \geq 4$, pick integers $a, b \geq 2$ with $a + b = d$ and let $f_\lambda(z) = z + (z - a\lambda)^b(z + b\lambda)^a$. Then f_λ is a one-dimensional non-constant family of monic centered polynomials of degree d with all fixed point multipliers equal to 1.

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Holomorphic index of a fixed point

Definition

The *holomorphic index* of f at a fixed point z_0 is the residue

$$\iota(f, z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{dz}{f(z) - z}$$

for sufficiently small $r > 0$.

- For a simple fixed point with $\lambda = f'(z_0) \neq 1$, we have $\iota(f, z_0) = \frac{1}{\lambda - 1}$ by simple complex analysis.
- The holomorphic index is invariant under analytic conjugation.

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Fixed point data rigidity for polynomials

Definition

The *fixed point data* of a polynomial f is the collection of all multiplicities and holomorphic indices of fixed points.

Theorem (G., Epstein)

For every $d \geq 2$, any given fixed point data determines at most $(d-2)!$ points in dynamic moduli space $\mathcal{M}_d^P$.

- Fujimura 2007: Generically, multipliers determine $(d-2)!$ points in $\mathcal{M}_d^P$.
- Sugiyama 2017: Fiber structure of the multiplier map for polynomials without multiple fixed points.
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Fixed point data proof idea

Write

$$f_\zeta(z) = z + \prod_{k=1}^{l} (z - \zeta_k)^{m_k} = z + p_\zeta(z)$$

where ζ_k are distinct with $\sum_{k=1}^{l} m_k \zeta_k = 0$. Then f_ζ is monic, centered, with fixed points ζ_k of holomorphic index

$$\eta_k(\zeta) = \frac{1}{(m_k - 1)!} \left(\frac{\partial}{\partial \zeta_k} \right)^{m_k - 1} \prod_{j \neq k} \frac{1}{(\zeta_k - \zeta_j)^{m_j}}.$$

and

$$\frac{\partial \eta_k}{\partial \zeta_j}(\zeta) = m_j \operatorname{Res}_{\zeta_k} \frac{1}{(z - \zeta_j) p_\zeta(z)} =: a_{kj}$$

By the holomorphic fixed point formula, $\sum_{k=1}^{l} a_{kj} = 0$ for all j, and the vector $(1, 1, \ldots, 1)$ is in the kernel of $D\eta_\zeta$.

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If $v = (v_1 \dots, v_l)$ is in the kernel of $D\eta_\zeta$, then

$$\mathrm{Res}_{\zeta_k} \sum_{j=1}^{l} \frac{m_j v_j}{(z-\zeta_j)p_\zeta(z)} = \mathrm{Res}_{\zeta_k} g(z) = 0.$$

- Then g is a rational function with all zero residues, vanishing at ∞ to order $\geq d+1$, so there exists a non-constant rational anti-derivative G with $G' = g$, vanishing at ∞ to order $\geq d$, so $\deg G \geq d$.
- On the other hand, multiplicity of pole at ζ_j is $\leq m_j$, so $\deg G \leq \sum_j m_j = d$.

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Fixed point data proof, conclusion

- So G is a rational function of degree d without finite zeros, and $1/G$ is a polynomial of degree d with zeros of multiplicities m_j at ζ_j, which implies $1/G(z) = cp_\zeta(z)$.
- This implies

$$g(z) = G'(z) = -\frac{1}{c}\sum_{j=1}^{l} \frac{m_j}{(z-\zeta_j)p_\zeta(z)},$$

so $v = -\frac{1}{c}(1, 1, \ldots, 1)$.

- This shows that the kernel of η is spanned by $(1, 1, \ldots, 1)$, so the rank of $D\eta_\zeta$ is $d-1$.

Fixed point data proof, conclusion

- So G is a rational function of degree d without finite zeros, and $1/G$ is a polynomial of degree d with zeros of multiplicities m_j at ζ_j, which implies $1/G(z) = cp_\zeta(z)$.
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Hutz-Tepper Conjecture

A natural question is whether one can trade off more multipliers (increase N_d) to reduce the ambiguity (decrease K_d).

Theorem (G., Epstein)

For $d \geq 4$, the multipliers up to period $N_d = 2$ generically determine a unique point in (polynomial) dynamical moduli space $\mathcal{M}_d^P$.

- Hutz and Tepper showed this for $d \leq 5$ using explicit algebraic computations of Groebner bases and conjectured that this holds generally.
- Polynomials $f \circ g$ and $g \circ f$ always have the same multipliers, so uniqueness only holds generically.
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Hutz-Tepper Conjecture proof idea

- There is exactly one point in $\mathcal{M}_d^P$ with $d-1$ multipliers zero, represented by $P_0(z) = \frac{d}{d-1}z + z^d$.
- Standard construction using quasiconformal surgery shows that a neighborhood of P_0 can be analytically parametrized by the multipliers $\lambda = (\lambda_1, \ldots, \lambda_{d-1}) \in \mathbb{D}^{d-1}$.
- Similarly, any tuple of attracting multipliers $\lambda = (\lambda_1, \ldots, \lambda_{d-1}) \in \mathbb{D}^{d-1}$ is obtained this way.
- Ambiguity comes from permuting the cyclic order of multipliers, which can be done in $(d-2)!$ ways.
- Explicitly compute the derivatives of period 2 multipliers with respect to fixed point multipliers and show non-symmetry w.r.t. (non-cyclic) permutations.

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Open questions

- What are the minimal pairs (N_d, K_d) for McMullen's original question? Some non-trivial bounds on minimal N_d?
- Silverman showed that one always has an ambiguity K_d of at least the class number of $\mathbb{Q}[\sqrt{-d}]$ (which is the number of equivalence classes of tori admitting a multiplication by $i\sqrt{d}$). Is this bound sharp?
- Using the fact that $f \circ g$ and $g \circ f$ have the same multipliers, the ambiguity K_d^P for polynomials of degree d is at least the number of prime factors of d. Is this bound sharp?
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