Pressure and conformal measures for transcendental meromorphic maps

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This is a joint work with

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Setup

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We study iterates of **transcendental meromorphic** maps

$$f : \mathbb{C} \to \overline{\mathbb{C}}$$

and geometric properties of the **Julia set**

$$J(f) = \mathbb{C} \setminus \{z : \{f^n\}_{n>0} \text{ is defined and normal in a nbhd of } z\}$$

and its invariant subsets (**Convention: $\infty \notin J(f)$**).

Notation

Singular set $\quad \mathrm{Sing}(f) = \{z : f^{-1} \text{ has a singularity at } z\}$
$\quad = \{\text{critical and asymptotic values of } f\}$.

Post-singular set $\quad \mathcal{P}(f) = \bigcup_{n=0}^{\infty} f^n(\mathrm{Sing}(f))$.

Speiser class $\quad \mathcal{S} = \{f : \mathrm{Sing}(f) \text{ is finite}\}$.

Eremenko–Lyubich class $\quad \mathcal{B} = \{f : \mathrm{Sing}(f) \text{ is bounded}\}$.

Hyperbolicity and conformal repellers

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A set $X \subset J(f)$ is a **conformal expanding repeller**, if it is compact, forward-invariant and $|(f^n)'||_X \geq cQ^n$ for every $n > 0$, where $c > 0$, $Q > 1$.

A conformal expanding repeller X is **transitive**, if for all non-empty open subsets V, W of X we have $f^n(V) \cap W \neq \emptyset$ for some $n \geq 0$.

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Remark

If a rational map f is hyperbolic, then $J(f)$ is a transitive conformal expanding repeller. In the transcendental case, $J(f)$ is not compact in $\mathbb{C}$ and the hyperbolicity of f does not always imply that f is expanding on $J(f)$.

Conformal measures

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Definition

A Borel probability measure ν on an invariant set $X \subset J(f)$ is ***t*-conformal** for some $t > 0$, if

$$\nu(f(A)) = \int_A |f'(z)|^t d\nu(z)$$

for every Borel set $A \subset X$ on which f is injective.

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Proposition

If ν is a t-conformal measure on $X = J(f)$, then ν is either positive on non-empty open sets in $J(f)$ or it is supported on the set of (at most two) exceptional values of f.

Example

For $f(z) = ze^z$, the value 0 is the unique finite exceptional value of f, with $f^{-1}(0) = \{0\}$, $f(0) = 0$ and $f'(0) = 1$. Consequently, $0 \in J(f)$ and the Dirac measure at 0 is t-conformal for every $t > 0$.

Classical thermodynamic formalism (Bowen, Ruelle, Walters)

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Let $X \subset J(f)$ be a transitive conformal expanding repeller. Then the **topological pressure function**

$$P(f|_X, t) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\substack{w\in f^{-n}(z)\\ w\in X}} |(f^n)'(w)|^{-t},$$

is well-defined for $t > 0$ and does not depend on $z \in X$.
We have $P(f|_X, t) = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{L}_t^n(\mathbb{1})$, where $\mathbb{1} \equiv 1$ and

$$\mathcal{L}_t : C(X) \to C(X), \qquad \mathcal{L}_t(\phi)(z) = \sum_{\substack{w\in f^{-1}(z)\\ w\in X}} \phi(w)|f'(w)|^{-t}$$

is the **Perron–Frobenius (transfer) operator***. Moreover, there exist a t-conformal measure m_t (eigenmeasure of the dual operator $\mathcal{L}_t^*$) and an f-invariant Gibbs measure on X, equivalent to m_t.*

Classical Bowen's formula

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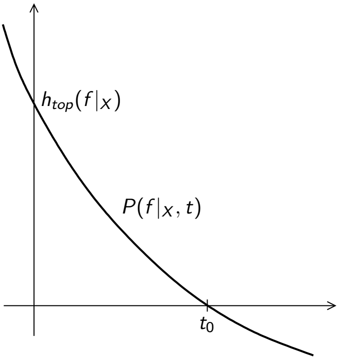
Theorem (Bowen 1979)

Let $X \subset J(f)$ be a transitive conformal expanding repeller. Then $\mathbf{dim}_{\boldsymbol{H}}\boldsymbol{(X) = t_0}$, *where t_0 is the unique zero of the pressure function $t \mapsto P(f|_X, t)$ and $\dim_H$ denotes the Hausdorff dimension.*



Thermodynamic formalism for transcendental maps

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Aim

Establish elements of thermodynamic dynamic formalism on $J(f)$ for transcendental entire or meromorphic maps f.

Difficulties compared with the finite degree case

Due to the lack of compactness, the standard Perron–Frobenius operator and the pressure can be not well-defined.

Tricks

- project the map f to a cylinder or torus (for periodic or doubly periodic maps)
- consider derivative of f in a different (non-Euclidean) metric

Example – the exponential map $E(z) = \lambda e^z$, $\lambda \in \mathbb{C} \setminus \{0\}$

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- The standard Perron–Frobenius–Ruelle operator on the constant function $\mathbb{1}$ is infinite for all $t > 0$:

$$\mathcal{L}_t(\mathbb{1})(z) = \sum_{w \in E^{-1}(z)} |E'(w)|^{-t} = \sum_{w \in E^{-1}(z)} \frac{1}{|z|^t} = \infty.$$

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- For the quotient map $\tilde{E} : \mathbb{C}/2\pi i\mathbb{Z} \to \mathbb{C}/2\pi i\mathbb{Z}$ the modified operator $\tilde{\mathcal{L}}_t$ on the function $\mathbb{1}$ is finite for $t > 1$:

$$\tilde{\mathcal{L}}_t(\mathbb{1})(z) = \sum_{w \in \tilde{E}^{-1}(z)} |\tilde{E}'(w)|^{-t} = \sum_{k \in \mathbb{Z}} \frac{1}{|z + 2\pi i k|^t} < \infty.$$

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- Alternatively, in the new metric $d\sigma = dz/|z|$, the modified operator $\mathcal{L}_{\sigma,t}$ on the function $\mathbb{1}$ is finite for $t > 1$:

$$\mathcal{L}_{\sigma,t}(\mathbb{1})(z) = \sum_{w \in E^{-1}(z)} |E'(w)|_\sigma^{-t} = \sum_{w \in E^{-1}(z)} \frac{1}{|w|^t} = \sum_{k \in \mathbb{Z}} \frac{1}{\left|\log\left|\frac{z}{\lambda}\right| + i \operatorname{Arg}\left(\frac{z}{\lambda}\right) + 2\pi i k\right|^t} < \infty.$$

Classes of transcendental maps admitting elements of thermodynamic formalism

Classes of transcendental maps admitting elements of thermodynamic formalism

B. 1995

Hyperbolic periodic maps of the form $f(z) = R(e^z)$, where R is a non-polynomial rational map, e.g. $f(z) = \lambda \tan z$.

Kotus–Urbański, Mayer–Urbański 2004–2005

Hyperbolic doubly periodic elliptic functions, e.g. $f(z) = \lambda\wp(z)$, where $\wp$ is the Weierstrass function.

Urbański–Zdunik 2003–2004

Hyperbolic exponential maps $E(z) = \lambda e^z$.

Mayer–Urbański 2005–2008

Hyperbolic maps of finite order with **rapid/balanced derivative growth** ($|f'(z)| \asymp |z|^\alpha |f(z)|^\beta$ as $|z| \to \infty$), e.g. previous examples, $f(z) = P(e^{Q(z)})$, where P, Q polynomials, $f(z) = \sin(az + b)$.

Mayer–Urbański 2017

Maps with Hölder tracts.

Escaping set and radial Julia set

Escaping set and radial Julia set

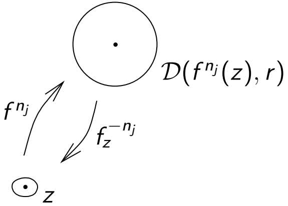
Definition

The **escaping set** $I(f)$ is defined as

$$I(f) = \{z \in \mathbb{C} : f^n(z) \text{ is defined for all } n > 0 \text{ and } \lim_{n\to\infty} f^n(z) = \infty\}.$$

Definition

The **radial Julia set** $J_r(f)$ is the set of $z \in J(f)$ for which there exists $r > 0$ and a sequence $n_j \to \infty$, such that a branch of f^{-n_j} sending $f^{n_j}(z)$ to z is well-defined on the disc $\mathcal{D}(f^{n_j}(z), r)$ with respect to the spherical metric on $\overline{\mathbb{C}}$.



Properties of the escaping set and radial Julia set

Properties of the escaping set and radial Julia set

Proposition

(a) *If f has a finite number of poles, then* $J_r(f) \subset J(f) \setminus (I(f) \cup \bigcup_{n=1}^{\infty} f^{-n}(\infty))$.
In particular, if f is entire, then

$$J_r(f) \subset J(f) \setminus I(f).$$

(b) *If f is hyperbolic, then* $J(f) \setminus (I(f) \cup \bigcup_{n=1}^{\infty} f^{-n}(\infty)) \subset J_r(f)$.
In particular, if f is hyperbolic entire, then

$$J_r(f) = J(f) \setminus I(f).$$

Proposition

(a) *If $J(f) \neq \mathbb{C}$,*
then $J_r(f)$ has 2-dimensional Lebesgue measure zero.

(b) *If f is hyperbolic,*
then $J(f) \setminus I(f)$ has 2-dimensional Lebesgue measure zero.

Further remarks

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Theorem (Schubert 2007, B. 2008, Bergweiler, Karpińska, Stallard 2009)

If $f \in \mathcal{B}$ is entire and has not too large growth rate (e.g. has finite order), then $\dim_H(J(f)) = 2$.

Theorem (Karpińska, Zdunik, B. 2009)

If $f \in \mathcal{B}$ has a logarithmic tract over ∞ (this holds, in particular, for every $f \in \mathcal{B}$, which is entire or has a finite number of poles), then $\dim_H(J_r(f)) > 1$.

Theorem (Rempe 2013)

There exists a hyperbolic entire map f of finite order with $\dim_H(J_r(f)) = 2$.

Pressure and conformal measures in spherical metric

Pressure and conformal measures in spherical metric

From now on, we consider pressure and conformal measures in the **spherical metric**

$$ds = \frac{2\,dz}{1+|z|^2},$$

i.e. we set

$$P(f,t,z_0) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{w\in f^{-n}(z_0)} |(f^n)^*(w)|^{-t},$$

where

$$f^*(z) = \frac{(1+|z|^2)f'(z)}{1+|f(z)|^2}$$

is the **spherical derivative** of f. Analogously, the condition for t-conformality of a measure ν has now the form

$$\nu(f(A)) = \int_A |f^*(z)|^t d\nu(z)$$

for every Borel set A on which f is injective.

Theorem (Karpińska, Zdunik, B. 2010)

Let f be an arbitrary map from $\mathcal{S}$ or a hyperbolic map from $\mathcal{B}$. Then for every $t > 0$ the pressure $P(f,t) = P(f,t,z_0)$ exists (possibly equal to $+\infty$) and is independent of $z_0 \in \mathbb{C}$ up to a set of Hausdorff dimension zero. The following version of Bowen's formula holds:

$$\dim_H(J_r(f)) = \dim_{hyp}(J(f)) = t_0,$$

where $t_0 = \inf\{t > 0 : P(f,t) \leq 0\}$.

Remark

In fact, the result is valid for all non-exceptional tame maps $f \in \mathcal{B}$, i.e. maps with $J(f) \setminus \overline{\mathcal{P}(f)} \neq \emptyset$ and without non-logarithmic singularity over an exceptional value of f contained in $J(f)$.

Remark

The same results (and more) were proved for rational maps by Przytycki, Rivera-Letelier and Smirnov in 2004.

Theorem (Rempe 2009)

For every transcendental meromorphic map f,

$$\dim_H(J_r(f)) = \dim_{hyp}(J(f)),$$

where $\dim_{hyp}(J(f))$ *is the* **hyperbolic dimension** *of $J(f)$, i.e. the supremum of* $\dim_H(X)$ *over all conformal expanding repellers $X \subset J(f)$.*

Question (Mauldin 2013)

Is the condition $P(f, t) = 0$ equivalent to the existence of a t-conformal measure on $J(f)$?

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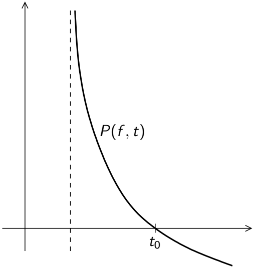
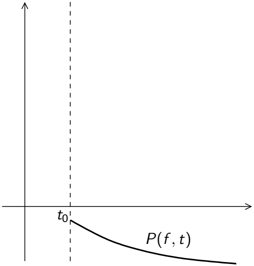
Remark

The function $t \mapsto P(f,t)$ is non-increasing and convex when it is finite and satisfies $P(f,2) \leq 0$.

Some possible situations





Existence of the zero of the pressure

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Theorem (Karpińska, Zdunik, B. 2018)

If a hyperbolic map f admits a t-conformal measure m_t for some $t > 0$, then $P(f, t) \leq 0$. Moreover, if $m_t(J(f) \setminus I(f)) > 0$, then $P(f, t) = 0$.

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Example

For $f(z) = \lambda \sin z$, $\lambda \in \mathbb{C} \setminus \{0\}$, the set $I(f)$ has positive 2-dimensional Lebesgue measure (McMullen 1987), and the normalized 2-dimensional spherical Lebesgue measure on $I(f)$ is 2-conformal. If, additionally, f is hyperbolic, then $P(f, 2) < 0$ (Coiculescu–Skorulski 2007).

Existence of the conformal measure

Existence of the conformal measure

Theorem (Karpińska, Zdunik, B. 2018)

Let f be an arbitrary map from $\mathcal{S}$ or a hyperbolic map from $\mathcal{B}$, such that f has a logarithmic tract over ∞. If $P(f,t)=0$ for some $t>0$, then there exists a t-conformal measure m_t on $J(f)$ such that

$$m_t(\mathbb{C}\setminus\mathbb{D}(r)) = o\left(\frac{(\ln r)^{3t}}{r^t}\right) \quad \text{as} \quad r\to\infty,$$

where $\mathbb{D}(r)=\{z\in\mathbb{C} : |z|<r\}$.

Remark

All maps from $\mathcal{B}$, which are entire or have a finite number of poles admit a logarithmic tract over ∞.

Remark

In fact, the result is valid for all non-exceptional tame maps $f\in\mathcal{B}$ with logarithmic tracts.

Logarithmic tracts

Logarithmic tracts

Definition

An unbounded simply connected domain $U \subset \mathbb{C}$ is called a **logarithmic tract** of F over ∞, if the following are satisfied:

- ∂U is a smooth open simple arc in $\mathbb{C}$,
- $F : \overline{U} \to \mathbb{C}$ is continuous, holomorphic on U,
- $F|_U$ is a universal covering of $V = \mathbb{C} \setminus \overline{\mathbb{D}(r)}$ for some $r > 0$,
- $F(\partial U) = \partial \mathbb{D}(r)$.

Spherical Distortion Theorem for logarithmic tracts

Theorem (Karpińska, Zdunik, B. 2010)

Let $F : U \to V = \{z : |z| > R\}$ be a logarithmic tract for some $R > 1$, $0 \notin U$ and let $z_1, z_2 \in V$ with $|z_1| \geq |z_2| \geq LR$ for some $L > 1$. If g is a branch of F^{-1} near z_1, then

$$c_1 \frac{|z_1|}{|z_2|} \left(\frac{\log |z_1|}{\log |z_2|} \right)^{-3} \leq \frac{|g^*(z_1)|}{|g^*(z_2)|} \leq c_2 \frac{|z_1|}{|z_2|} \frac{\log |z_1|}{\log |z_2|},$$

for some extension of g, where c_1, c_2 depend only on R, L (not on F).

Construction of the conformal measure m_t (following Patterson, Denker–Urbański...)

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Suppose $P(f, t) = 0$. Define

$$\mu_s = \frac{1}{\Sigma_s} \sum_{n=1}^{\infty} b_n e^{-ns} \sum_{w \in f^{-n}(z_0)} \frac{\delta_w}{|(f^n)^*(w)|^t},$$

where $s > 0$, $z_0 \in J(f)$, $b_n > 0$, δ_w is the Dirac measure at w, and

$$\Sigma_s = \sum_{n=1}^{\infty} b_n e^{-ns} \sum_{w \in f^{-n}(z_0)} |(f^n)^*(w)|^{-t} < \infty.$$

We can choose the sequence b_n so that

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = 1, \qquad \lim_{s\to 0^+} \Sigma_s = +\infty.$$

Lemma

For sufficiently large $r > 0$,

$$\mu_s(J(f) \setminus \mathbb{D}(r)) < c\frac{(\log r)^{3t}}{r^t}$$

for a constant $c > 0$ independent of s.

Corollary

The family $\{\mu_s\}_{s\in(0,1)}$ is tight. Consequently, there exists a weak limit

$$m_t = \lim_{j\to\infty} \mu_{s_j}$$

for some sequence $s_j \to 0^+$, which is a probability measure with support in $J(f)$. The measure m_t is t-conformal with respect to the spherical metric.

Thank you for your attention!

Wszystkiego najlepszego!