

I - Groupes algébriques - variétés de drapeaux

Def (1) Un groupe algébrique linéaire G est sous-groupe fermé (Zariski) de $GL_n(\mathbb{C})$.

(2) Un G -module rationnel est un $\mathbb{C}[G]$ -comodule, i.e., une application $V \xrightarrow{\Delta_V} V \otimes \mathbb{C}[G]$ satisfaisant des axiomes naturels.

NB (1) Si $\dim V < \infty$ et $\Delta_V(v_i) = \sum_j r_{ji} v_j \otimes f_{ji}$, (v_i) base de V , alors $\rho_V: G \rightarrow GL(V)$, $g \mapsto (f_{ji}(g))_{i,j}$ est un morphisme de groupes algébriques.

(2) $\forall V$ G -module rationnel, $V = \bigcup_{\substack{W \subset V \\ \dim W < \infty}} W$.

EX $\mathbb{C}[G]$ G -module rationnel tel que $(g \cdot f)(g') = f(g'g)$, note ρ_V .

Théorème

(1) $\forall g \in GL_n(\mathbb{C}) \exists! (g_u, g_s)$ tel que $g = g_u g_s$, $\rho(g_u)$ unipotent, $\rho(g_s)$ semi-simple, $g_u g_s = g_s g_u$.

(2) Si $f: G \rightarrow G'$ morph de grp alg alors $f(g_u) = f(g)_u$, $f(g_s) = f(g)_s$.

NB En particulier, si $i: G \hookrightarrow GL_n(\mathbb{C})$ alors $i(g_u) = i(g)_u$, $i(g_s) = i(g)_s$.

Def G gp lin

(1) $G_u = \{g_u; g = g_u\} = \{ \text{elements unipotents} \}$

(2) $G_s = \{g; g = g_s\} = \{ \text{elements semisimples} \}$

} sous-variétés
fermées de G .

(3) G unipotent si $G = G_u$

(4) G diagonalisable si G commutatif et $G = G_s$.

Théorème G gp lin.

(1) G commutatif $\Leftrightarrow G_u, G_s \subset G$ sg fermés et $G_s \times G_u \xrightarrow{m} G$ isom de groupes algébriques.

(2) G diagonalisable $\Leftrightarrow G \subseteq (\mathbb{C}^*)^N$ sg fermé.
 $\Leftrightarrow G \simeq \Pi_0(G) \times G^\circ$, G° un torse.

Def (1) Un torse est un gp alg $\simeq (\mathbb{C}^*)^N$.

(2) $\{ \text{caractères} \} = X^*(G) = \text{Hom}_{\text{gp alg}}(G, \mathbb{C}^*)$.

NB $X^*(G) \subset \mathbb{C}[G]$ Abelian sg

Def $\forall V \in \text{Rep } G, \forall \chi \in X^*(G), V_\chi = \{ v \in V; gv = \chi(g)v \forall g \in G \}$.

Théorème

(1) $X^*(G)$ consists of lin. indpt vectors of $\mathbb{C}[G]$.

(2) $\forall V \in \text{Rep } G, \sum_\chi V_\chi$ is direct and equal to V if V is rational.

(3) G diagonalisable $\Leftrightarrow \mathbb{C}[G] = \mathbb{C} X^*(G)$

G torse $\Leftrightarrow \mathbb{C}[G] = \mathbb{C} X^*(G)$ et $X^*(G)$ torsion free.

Théorème (rigidité) G gp lin, $H \subset G$ sg fermé diagonalisable

Alors $N_G(H)^\circ = \sum_G(H)^\circ$, donc $N_G(H)/Z_G(H)$ fini.

Def $(\forall \forall H, H' \subset G$ groupes, $(H, H') \subset G$ sq engendré par les commutateurs $ab^{-1}b^{-1}$, $a \in H, b \in H'$.

(2) $D^1(G) = [G, G]$, $D^{i+1}(G) = (D^i(G), D^i(G))$, G résoluble si $D^{\gg 0}(G) = \{1\}$.

(3) $C^1(G) = [G, G]$, $C^{i+1}(G) = (G, C^i(G))$, G nilpotent si $C^{\gg 0}(G) = \{1\}$.

Lemma

(1) $1 \rightarrow H' \rightarrow H \rightarrow H'' \rightarrow 1$ exact.

H rés $\Leftrightarrow H', H''$ résolubles

H nilpotent $\Rightarrow H', H''$ nilpotents

~~résoluble~~

(2) G gp linéaire $\Rightarrow D^i(G), C^i(G) \subset G$ sq fermés.

Ex $G = GL_n(\mathbb{C})$, $U_n = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ nilpotent, $B_n = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ résoluble,

$U_n = (B_n)_n$.

Théorème (Lie-Kolchin) G gp lin, $\rho: G \rightarrow GL_n(\mathbb{C})$ représentation

(1) G résoluble connexe $\Rightarrow \rho(G)$ conjugué à un sq fermé de B_n .

(2) G nilpotent $\Rightarrow \rho(G)$ conjugué à un sq fermé de U_n , donc est nilpotent.

Corollaire G résoluble connexe $\Rightarrow D(G)$ connexe nilpotent, et G_n sq fermé, connexe normal de G .

Théorème G gp linéaire résoluble connexe

(1) $\forall T \subset G$ tore max, $T \times G_n \xrightarrow{\text{mult}} G$ isom de variétés.

(2) Les tores max de G sont conjugués.

Def (1) Un \mathfrak{B} de Borel de G est un \mathfrak{B} fermé résoluble connexe maximal
 (2) $\mathfrak{B} = \{ \text{sg de Borel de } G \}$.

Théorème G gp linéaire connexe.

(1) Tous les \mathfrak{B} de Borel de G sont conjugués.

(2) $\forall B \in \mathfrak{B}, \exists$ quotient géométrique $G \rightarrow G/B$ tq.

G/B est une G -variété projective.

Ex si $G = GL_n(\mathbb{C})$ on a la variété des drapeaux.

NB

(1) G -variété $X \Leftrightarrow \sigma : G \times X \rightarrow X$ morphisme tel que

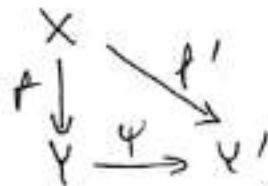
(a) $\sigma_g \circ \sigma = \sigma(g, -) \in \text{Aut}(X)$

(b) $g \mapsto \sigma_g$ gp homomorphisme.

(2) Un quotient catégorique de X par G est une paire (Y, f) tq

(a) $f : X \rightarrow Y$ G -invt

(b) $\forall f' : X \rightarrow Y'$ G -invt $\exists ! \psi$



(3) Un bon quotient de X par G est une paire (Y, f) tq

(a) $f : X \rightarrow Y$ G -invt

(b) f affine surjectif

(c) $\forall F \subset X$ G -invt fermé, $f(F) \subset Y$ fermé ($\Leftrightarrow Y$ muni de la topo quotient)

(d) $\forall F_1, F_2 \subset X$ " " " " , $F_1 \cap F_2 = \emptyset$, $f(F_1) \cap f(F_2) = \emptyset$

(e) $\forall U \subset Y$, $O_Y(U) = O_X(f^{-1}(U))^G$, ie, $(f_* O_X)^G = O_Y$.

(4) Un bon quotient est un quotient catégorique

(5) Un " " est géométrique si toutes les G -orbites dans X sont fermées.

Covollaire

- (1) Tout tore max est contenu dans un sg de Borel
- (2) Toutes les paires (B, T) , $B \in \mathcal{B}$, T tore max $\subset B$, sont conjuguées
- (3) les sg fermés connexes unipotents max de G sont tous conjugués et sont les $B_u, B \in \mathcal{B}$.

Théorème (Chevalley) G gp lin. connexe.

$$\forall B \in \mathcal{B} \text{ on a } N_G(B) = N_G(B_u) = B.$$

Covollaire $G/B \simeq \mathcal{B}$ as a set.

Def. G gp linéaire.

- (1) $\text{Rad}(G) = \left(\bigcap_{B \in \mathcal{B}} B \right)^0$ est le + grand sg fermé connexe résoluble normal de G ; c'est le radical de G .
- (2) $\text{Rad}(G)_u = \left(\bigcap_{B \in \mathcal{B}} B_u \right)^0$ est le + grand sg fermé connexe unipotent normal de G ; c'est le radical unipotent de G .
- (3) G réductif si $\text{Rad}(G)_u = \{1\}$

(4) G semi-simple si $\text{Rad}(G) = \{1\}$

NB $\left\{ \begin{array}{l} \text{Rad}(G) \text{ fermé car } G \text{ est } \text{Noetherien.} \\ \text{La normalité de } \text{Rad}(G) \text{ découle du fait qu'il est préservé par tout} \\ \text{automorphisme de } G, \text{ d'après:} \end{array} \right.$

Théorème Soit $f: G \rightarrow G'$ morphisme surjectif de gp linéaires. Alors $\mathcal{B}(G') = \{ f(B); B \in \mathcal{B}(G) \}$ et idem pour les tores max et les sg fermés unipotents connexes maximaux.

Proposition G gp lin

$G/\text{Rad}(G)$ st \approx et $G/\text{Rad}(G)_u$ est réductif.

Nb Un sg de Lévi de G , lin, connexe, est un sg connexe $L \subset G$

tp. $G = L \ltimes \text{Rad}(G)_u \Rightarrow L \simeq G/\text{Rad}(G)_u$ réductif.

$\text{car } 0 \Rightarrow L$ exist and are conjugate.

$\text{car } > 0 \Rightarrow$ parabolic groups.

Théorème G gp lin connexe, $S \subset G$ Tore

(1) $Z_G(S)$ connexe

(2) $\{ B \subset Z_G(S) \text{ sg Borel contenant } S \} \xrightarrow{\sim} \{ B \subset Z_G(S) \text{ Borel} \}$
 $B \longmapsto B \cap Z_G(S)$

Def $\forall G$ gp lin connexe, un sg de Cartan est un sg de la forme $Z_G(T)$, T tore max. 2 sg de Cartan sont conjugués.

Corollaire G gp lin connexe, $T \subset G$ tore max, $C = Z_G(T)$ sg Cartan

$\forall B \in \mathcal{B}, T \subset B \Rightarrow C \subset B$.

Dém $C \cap B$ sg Borel de $Z_G(T)$, donc

C nilpotent (non dém) $\Rightarrow C \cap B = C$, ie, $C \subset B$.

□

Def. G gp lin connexe, T tore max,

$W = W(T, G) = N_G(T)/Z_G(T)$ gp fini (indépendant des choix de T) appelé groupe de Weyl.

Théorème G gp lin connexe

- (1) $\forall S \subset G$, B^S sous-ensemble fermé de B égale à $\{B \in B; S \subset B\}$
 (2) $\forall T \subset G$ tore max, $W(T, G)$ agit simplement transitivement sur $B^T \Rightarrow |B^T| = |W(T, G)| < \infty$.

Dém

(i) $B \in B^S \Leftrightarrow s \in N_G(B) = B$

(ii) (a) $\begin{cases} n \in N_G(T) \\ B \in B^T \end{cases} \Rightarrow T = n T n^{-1} \subset n B n^{-1}$
 $\Rightarrow N_G(T) \subset B^T$

argument utilisé
 Not given because
 used in previous case.

(ii) $T \subset B \Rightarrow Z_G(T) \subset B$
 $\Rightarrow Z_G(T)$ fixe B
 $\Rightarrow W(T, G) \not\subset B^T$

(iii) $B, B' = g B g^{-1} \in B^T$
 $\Rightarrow T, g^{-1} T g \subset B$
 $\Rightarrow \exists b \in B / g^{-1} T g = b^{-1} T b$
 $\Rightarrow n := g b^{-1} \in N_G(T)$
 $\Rightarrow B' = n B n^{-1} \in W(T, G) \cdot B$

(iii) $n \in N_G(T), n B n^{-1} = B, B \in B^T \Rightarrow n \in B \cap N_G(T) = N_B(T)$
 Or $N_B(T) = Z_B(T)$ car $\forall g \in N_B(T), \forall t \in T, (g + g^{-1}) t^{-1} \in T \cap D(B) = T \cap B = \{1\}$
 $\Rightarrow n \in Z_B(T)$

□

Théorème (Chevalley) G gp réductif complexe.

$$\left(\bigcap_{B \in \mathcal{B}^T} B \right)^0 = T$$

Corollaire G gp réductif complexe

(1) T Tore max $\Rightarrow Z_G(T) = T$ (ie, T sg de Cartan)

(2) $Z(G) = \bigcap_{T \text{ Tore max}} T$

(3) $S \subset G$ tore $\Rightarrow Z_G(S)$ gp réduct. complexe.

Théorème G gp lin complexe

(1) tout elt semi simple appartient à un Tore max.

(2) " " unipotent " " sg unipotent complexe max.

(3) " " appartient à un sg de Borel.

II - Groupes réductifs

T torse, $X_{\mathbb{Z}}(\mathbb{G}) = \text{Mor}_{\text{grop}}(\mathbb{G}, T) = \text{gp Abélien des caractères.}$

La composition induit un couplage

$$X^*(T) \times X_*(T) \longrightarrow \mathbb{Z}$$

Proposition Ce couplage est parfait, i.e., il identifie $X^*(T)$ et

$\text{Hom}_{\mathbb{Z}}(X_*(T), \mathbb{Z})$ et réciproquement.

Def Une donnée radicielle est un quadruplet $(X, R, X^{\vee}, R^{\vee})$ où

(a) X^{\vee}, X 2 \mathbb{Z} -modules libres de n fini en dualité parfaite par

un couplage $\langle \cdot, \cdot \rangle$

(b) $R \subset X, R^{\vee} \subset X^{\vee}$ en bijection $\alpha \mapsto \alpha^{\vee}$ tels que

$$\langle \alpha, \alpha^{\vee} \rangle = 2 \quad \forall \alpha, \quad s_{\alpha}(R) = R, \quad s_{\alpha^{\vee}}(R^{\vee}) = R^{\vee}$$

où $s_{\alpha}, s_{\alpha^{\vee}}$ sont les réflexions de $V = X \otimes_{\mathbb{Z}} \mathbb{R}, V^{\vee} = X^{\vee} \otimes_{\mathbb{Z}} \mathbb{R}$

$$s_{\alpha}(x) = x - \langle x, \alpha^{\vee} \rangle \alpha$$

$$s_{\alpha^{\vee}}(x^{\vee}) = x^{\vee} - \langle \alpha, x^{\vee} \rangle \alpha^{\vee}$$

On suppose la donnée radicielle réduite, i.e.,

$$\forall \alpha, \beta \in R, \quad \alpha \in \mathbb{Z}\beta \Leftrightarrow \alpha = \pm \beta.$$

Nb R est un système de racines dans $V = X \otimes_{\mathbb{Z}} \mathbb{R}$.

Def (1) $Q = \mathbb{Z}R =$ réseau des racines (dans V)

(2) $P = \{x \in V; \langle x, \alpha^{\vee} \rangle \in \mathbb{Z}, \forall \alpha^{\vee} \in R^{\vee}\}$
 = réseau des poids (dans V)

(3) Q^{\vee}, P^{\vee}

(4) $W = \langle s_{\alpha} \rangle \subset GL(V)$

\mathfrak{G} gp réductif connexe

$T \subset \mathfrak{G}$ Torse max

$W(T, \mathfrak{G}) = \text{gp de Weyl}$

$\mathfrak{g} = \text{Lie}(\mathfrak{G})$, $\mathfrak{t} = \text{Lie}(T)$

$R = \text{ens des poids } \neq 0 \text{ de } T \text{ dans } \mathfrak{g}, \text{ ie,}$

$$\mathfrak{g} = \mathfrak{g}^T \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_{\alpha}, \quad \mathfrak{g}^T = \mathfrak{t}$$

$\forall \alpha \in R, \mathfrak{G}_{\alpha} = \sum_{\mathfrak{G}} ((\ker \alpha)^{\circ})$, $(\ker \alpha)^{\circ} \subset T$ subtorus.

\mathfrak{G}_{α} est un sg réductif connexe de \mathfrak{G} .

$\forall \alpha \in R \exists ! \alpha^{\vee}$ unique poids tel que $W(T, \mathfrak{G}_{\alpha}) = \{ \pm 1, \pm \alpha^{\vee} \}$, Se détermine via α^{\vee} contrappos

Proposition (1) $(X^{\vee}(T), X_{\alpha}(T), R, P^{\vee})$ est une donnée radicielle et $W(T, \mathfrak{G})$ est \cong gp de Weyl W de R .

(2) $\forall \alpha \in R \exists$ isomorphisme $u_{\alpha}: \mathfrak{G}_{\alpha} \xrightarrow{\sim} U_{\alpha}$, U_{α} closed sg of \mathfrak{G} , tel que $t u_{\alpha}(x) t^{-1} = u_{\alpha}(\alpha(t)x) \quad \forall t \in T, x \in \mathfrak{G}_{\alpha}$.

On a $\text{Im}(du_{\alpha}) = \mathfrak{g}_{\alpha}$ (donc $\dim \mathfrak{g}_{\alpha} = 1$)

(3) T et les U_{α} ($\alpha \in R$) engendrent \mathfrak{G} .

Nb les U_{α} sont des sg unipotents de \mathfrak{G} normalisés par T .

Proposition

(1) $\mathfrak{G} = \text{Rad}(\mathfrak{G}) \cdot D(\mathfrak{G})$

(2) $D(\mathfrak{G})$ est semi-simple.

(3) $Z(\mathfrak{G}) = \bigcap_{\alpha \in R} \ker \alpha$

Proposition $\alpha \in R$

(1) G_α \mathfrak{g} réductif complexe de rang 1, i.e., $D/G_\alpha \simeq SL_2(\mathbb{C}), PSL_2(\mathbb{C})$

$$\text{Lie}(G_\alpha) = \mathfrak{g} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$$

$$Q_\alpha \cap R = \pm \alpha$$

(2) $W(G_\alpha, T) = \{ \vee s\alpha \} \subset W$

(3) $U_\alpha, T \subset G_\alpha, B_\alpha = U_\alpha T \subset G_\alpha$ \mathfrak{g} de Borel.

Def $\forall B \subset G$ Borel $\supset T$, $\mathfrak{b} = \text{Lie}(B)$

$$R^+ = R^+(\mathfrak{b}) = \{ \alpha \in R; B_\alpha \subseteq B \}$$
$$= \{ \alpha \in R; \mathfrak{g}_\alpha \subseteq \mathfrak{b} \}$$

$$R^+ = \{ \text{racines positives} \}$$

Théorème $B \neq B^T$

(1) $R^+(\mathfrak{b})$ est l'ensemble des racines > 0 associées à une base $\Delta(\mathfrak{b})$ de R

(2) $B \mapsto \Delta(\mathfrak{b})$ est injective.

(3) $R = R^+(\mathfrak{b}) \cup -R^+(\mathfrak{b})$

NB (1) $w \in W$ représenté par $n_w \in N_G(T) \Rightarrow n_w U_\alpha n_w^{-1} = U_{w(\alpha)}$

(2) $B^T \simeq W$, les systèmes de racines > 0 sont les $\{w(R^+); w \in W\}$.

Théorème G semi-simple, $X = X^*(T)$. $R^+(n_w B n_w^{-1})$

(1) Le groupe P/Q est fini et $Q \subset X \subset P$.

\Rightarrow si R est fixé il y a un nombre fini de choix pour X (et donc pour G).

(2) G est adjoint si $X = Q$ (alors $Z(G) = \{1\} \Rightarrow G \subset \text{Aut}(\mathfrak{g})$)

(3) G est sc si $X = P$ (alors $Z(G) = P/Q$).

NB $\Delta^+(\mathfrak{b}) = \{\alpha_i\} \Rightarrow Q = \mathbb{Z}\alpha_i, \check{Q} = \mathbb{Z}\check{\alpha}_i$ les (co)-racines spé

$\{\omega_i\} \in P / \langle \omega_i, \check{\alpha}_j \rangle = \delta_{ij}$ les pb fondamentaux

$D = \mathbb{Z}\omega_i$

Proposition

$\forall M \subset G$ sg fermé convexe normalisé par T ou a :

M engendré par $T \cap M$ et les U_α tq $\mathfrak{g}_\alpha \subset \text{Lie}(M)$.

Théorème $B \in \mathcal{B}$, $U = B \cap U$, $(\alpha_1, \alpha_2, \dots, \alpha_m)$ une numérotation de R^+

(1) Le morphisme $\mathbb{R}^m \longrightarrow U$, $(x_i) \longmapsto U_{\alpha_1}(x_1) \dots U_{\alpha_m}(x_m)$
est un isomorphisme de variétés.

(2) " " $T \times U \longrightarrow B$, $(t, u) \longmapsto tu$
est un isomorphisme de variétés.

Nb En particulier $\left\{ \begin{array}{l} G \text{ est engendré par } T \text{ et les } U_{\pm\alpha}, \alpha \in R^+ \\ B \text{ " " " } U_\alpha, \alpha \in R^+ \\ U \text{ " " les } U_\alpha, \alpha \in R^+ \end{array} \right.$

Soit $w_0 \in W$ l'élément de plus grande longueur.

C'est l'unique élément tel que $w_0(R^+) = -R^+$.

$\forall w \in W$ soit $R(w) = \{ \alpha \in R^+ ; w(\alpha) \in -R^+ \}$.

Lemme N les $U_\alpha, \alpha \in R(w)$, engendrent un sg fermé convexe

U_w de U tq. $U_w = \prod_{\alpha \in R(w)} U_\alpha$ (ordre quelconque).

(2) $U_w \times \underbrace{U_{w_0 w}}_{U_w} \xrightarrow{\text{mult}} U$ est un isomorphisme de variétés.

Théorème (Bruhat) $B \in \mathcal{B}^T$, $U = B \cap U$.

(1) $G/B = \bigsqcup_{w \in W} U_w w B/B$, $G = \bigsqcup_{w \in W} U_w w B$

décomposition en B (ou U)-orbites.

L'orbite ouverte de G/B est $U_{w_0} w_0 B/B$.

$$(2) \forall w, U_w B/B \simeq \mathbb{C}^{\langle w \rangle}, \ell(w) = \text{longueur de } w.$$

(3) $G = \bigcup_w U_w B$ recouvrement de G par des ouverts affines. Idem pour G/B .

$$\text{Def } P_+ = \{x \in P; \langle x, \alpha \rangle \geq 0 \forall \alpha \in P^+\} \\ = \{ \text{poids dominants} \} \\ = \bigoplus_i \mathbb{Z}_{\geq 0} \omega_i.$$

Théorème

(1) les G -modules rationnels sont semi-simples.

(2) $\forall V$ G -module simple (de dim $< \infty$)

$$\exists! x \in P_+ \cap X \text{ tq } V_x \neq \{0\}.$$

Alors $\dim(V_x) = 1$, x est le + haut poids de V .

(3) Tous les G -modules simples s'obtiennent ainsi.

Def. X variété avec action de G .

Un faisceau $\tilde{\mathcal{F}}$ G -mod(\mathcal{O}_X) est G -équivariant ssi

$$\exists \mathcal{I}: a^*(\tilde{\mathcal{F}}) \xrightarrow{\sim} p_2^*(\tilde{\mathcal{F}}), \quad G \times X \xrightarrow[p_2]{a} X, \text{ tel que}$$

$$(a) p_{23}^*(\mathcal{I}) \circ (\text{id}_{G \times a})^*(\mathcal{I}) = (m \times \text{id}_X)^*(\mathcal{I}), \quad \begin{array}{ccc} G \times G \times X & \xrightarrow{p_{23}} & G \times X \\ G \times G & \xrightarrow{m} & G \end{array}$$

$$(b) \mathcal{I}_{1 \times X} = \text{id}.$$

Nb Si $\tilde{\mathcal{F}}$ est libre, c'est la m^{me} chose que la donnée d'un

$$\mathcal{I}_{g,x}: \tilde{\mathcal{F}}_{g^{-1}x} \rightarrow \tilde{\mathcal{F}}_x \quad \forall g, x$$

satisfaisant les axiomes évidents.

Soit $\chi \in X = X^*(T)$

Soit \mathbb{C}_χ le B -module de dimension 1 tel que

$$t \cdot u \cdot v = \underbrace{\chi(t)}_{\text{noté } \chi(t)} \cdot v \quad \forall t \in T, \forall u \in U, \forall v \in \mathbb{C}_\chi$$

Posons $L(\chi) = \mathbb{G}_m \times_B \mathbb{C}_\chi$ (quotient géométrique)

(C'est une \mathbb{G}/B -variété $\rightsquigarrow \mathcal{L}(\chi) = \mathcal{O}(L(\chi))$ le faisceau des germes de sections de $L(\chi) \rightarrow \mathbb{G}/B$.

Lemme (i) $\mathcal{L}(\chi)$ est un faisceau inversible \mathbb{G} -équivariant.

$$(2) \quad H^0(\mathbb{G}/B, \mathcal{L}(\chi)) = \{ f \in \mathbb{C}[B] ; f(gb) = \chi(b)^{-\langle \chi, \lambda \rangle} f(g) \}$$

Théorème (i) $H^0(\mathbb{G}/B, \mathcal{L}(\chi)) \neq \{0\} \iff \chi \in X_{NP^+}$

(2) Si $\chi \in X_{NP^+}$ alors le \mathbb{G} -module $H^0(\mathbb{G}/B, \mathcal{L}(\chi))$ est simple de plus haut poids χ .

EX.

$$(1) \quad \mathbb{G} = SL_2(\mathbb{C}), \quad \mathfrak{g} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right\}$$

$$T = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} = D(z) ; z \in \mathbb{C}^* \right\}$$

$$X = \mathbb{Z}\omega_1, \quad \omega_1(D(z)) = z$$

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{-\alpha_1}$$

$b=c=0 \quad a=c=0 \quad b=c=0$

$$\alpha_1(D(z)) = z^2$$

$$Q = \mathbb{Z}\alpha_1 \not\subset X = P = \mathbb{Z}\omega_1$$

$$\mathbb{G}/B = \mathbb{P}^1, \quad \mathcal{L}(n\omega_1) = \mathcal{O}_{\mathbb{P}^1}(n)$$

$$H^0(\mathbb{G}/B, \mathcal{L}(n\omega_1)) = S^n \mathbb{C}^2$$

$$Q^\vee = X^\vee$$

$$(2) \quad \mathbb{G} = PSL_2(\mathbb{C})$$

$$T = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = D(x,y) \text{ mod } \mathbb{C}^* \text{Id} \right\}$$

$$X = \mathbb{Z}\alpha_1, \quad \alpha_1(D(x,y)) = xy$$

$$Q = \mathbb{Z}\alpha_1 = X \not\subset P = \mathbb{Z}\omega_1$$

$$Q^\vee \not\subset X^\vee$$

III. Fibres de Springer, Variétés de Steinberg

G gp réductif de caractéristique p , $B \in \mathcal{B}$, $U = B_u$, \mathfrak{g} , \mathfrak{b} , \mathfrak{u} , etc.

Lemme \exists eq. catégories $\text{vb}_B^G(O_{\mathbb{A}^1}) \rightarrow \text{Rep}^+(B)$

Dém $E \mapsto E_B = \text{fibres en } B \in \mathcal{B}$

$$G \times_B V \leftarrow V$$

L'eq. est un cas particulier de la descente fidèlement plate.

□

Corollaire $T^*B \simeq G \times_B \mathfrak{u}$
 $\simeq \{ (B', x) ; x \in \text{Lie}(B_{\mathbb{A}^1}) \}$.

Dém On a $T^*B \simeq G \times_B (T_B^*B)$

De plus $T_B^*(G/B) = \mathfrak{g}/\mathfrak{b}$

\exists non dégénéré couplage $\mathfrak{g} \times \mathfrak{g} \xrightarrow{\kappa} \mathbb{C}$ $(\text{ad } G)$ -invariant
 tel que $\mathfrak{b}^\perp = \mathfrak{u}$.

Donc $\mathfrak{g}/\mathfrak{b} \simeq \mathfrak{u}$ comme B -module.

□

Nb Car $0 \Rightarrow \mathfrak{g}_{\text{nil}} \simeq \mathfrak{G}_{\mathbb{A}^1}$ via l'exponentielle
 $\Rightarrow T^*B \simeq \{ (B', g) ; g \in \mathfrak{G}_{\mathbb{A}^1} \}$.

Def On note $\mathbb{C}P = \mathfrak{g}_{\text{nil}}$, $\rho: T^*B \rightarrow \mathbb{C}P$, $(B', x) \mapsto x$.

ρ est la résolution de Springer de $\mathbb{C}P$.

On note $B_x = \rho^{-1}(x)$.

Théorème Il y a un nombre fini de classes de conjugaisons dans

Def $\mathcal{Z} = \underset{d^2}{T^*B \times T^*B} = \{ (B', B'', \kappa) ; \kappa \in \text{Lie}(B_u) \cap \text{Lie}(B_u^\vee) \}$
 est la variété de Steinberg.

Proposition (1) \mathcal{Z} est la réunion des fibres connexes aux G -orbites dans $B \times B$.

(2) \mathcal{Z} est de dimension pure, les composantes irréductibles sont paramétrées par W .

Dém

(1) (a) soit $(B, x) \in T^*B \subset B \times \mathfrak{g}$.

$$\forall y \in \mathfrak{g}, y \cdot B := \frac{d}{dt} (\text{ad}^{t y})(B) \Big|_{t=0} = \begin{cases} 0 & \text{si } y \in \mathfrak{b} \\ y & \text{si } y \in \mathfrak{u}^- \end{cases}$$

(via l'identification $T_B B \cong \mathfrak{g}$ donnée par la \mathcal{G} -de Bruntan

On en déduit que

$$\begin{array}{ccc} (B, x), y \cdot B = \kappa(x, y) \\ \uparrow & & \uparrow \\ \text{couplage } T^*B/TB & & \text{couplage } \mathfrak{g}/\mathfrak{g} \end{array}$$

(x*) Si $\alpha = (B_1, x_1; B_2, x_2) \in T^*_{B_1} B \times T^*_{B_2} B$ alors

$$\alpha \left(T_{(B_1, B_2)} G \cdot (B_1, B_2) \right) = 0 \Leftrightarrow \kappa(x_1, y) + \kappa(x_2, y) = 0 \quad \forall y \in \mathfrak{g}$$

$$\Leftrightarrow x_1 + x_2 = 0$$

Donc $\alpha \in \mathcal{Z}$ modulo l'identification évidente.

(2) \Leftarrow (1) car $G \backslash B \times B = G \backslash (G/B \times G/B) = B \backslash G/B$.

□

Lemme

- (1) X G -variété \Rightarrow les G -orbites dans X sont lisses et localement fermées.
- (2) G connexe \Rightarrow il préserve les composantes irréductibles de X

Dém. du Théorème

$$\mathcal{O} = \bigsqcup_0 \mathcal{O}, \quad \mathcal{O} \text{ } G\text{-orbites}$$

$$\mathcal{Z}_0 = \rho^{-1}(0) \times_0 \rho^{-1}(0)$$

$$\mathcal{Z} = \bigsqcup_0 \mathcal{Z}_0$$

$$\mathcal{Z}_0 = \bigsqcup_{G(x)} G_x \times (B_x \times B_x)$$

$G(x)$ = sy d'isotropie

$$A(x) = G(x) / G(x)^0$$

$A(x)$ paramètre les comp irr de B_x et les comp irr de \mathcal{Z}_0 sont les $\mathcal{Z}_0^i = \bigsqcup_{G(x)} G \times \mathcal{Z}_x^i$, \mathcal{Z}_x^i une $A(x)$ -orbite

de comp irr de $B_x \times B_x$.

\mathcal{Z}_0 loc^t fermée et on a:

Lemme (Spaltenstein) les composantes irréd. de B_x sont de dimension $\dim B_x = \dim B - \frac{1}{2} \dim O$ with $O = G \cdot x$.

$$\Rightarrow \dim \mathcal{Z}_0^i = \dim \mathcal{Z}_0 = 2 \dim B - \dim O + \dim O = \dim \mathcal{Z}.$$

Donc \mathcal{Z} est union des \mathcal{Z}_0 et chaque \mathcal{Z}_0 est de dim pure égale à celle de \mathcal{Z} .

$$\Rightarrow \#\{\mathcal{Z}_0\} < \infty.$$

Theorem (Richardson)

G' linear group, X smooth G' -variety.

$G \subset G'$ closed connected normal subgroup.

$S \subset G'$ reductive closed subgroup.

Then G acts transitively on $X \Rightarrow G^{S=0}$ acts transitively on the connected components of X^S

NB S reductive, X smooth $\Rightarrow X^S$ smooth.

Since G -orbits are smooth, it yields

Corollary $\mathcal{C}P \subset \mathcal{C}g$ nil cone

The $G^{S=0}$ orbits in $\mathcal{C}P^S$ are the con. comp. of the faces in $\mathcal{C}P^S$ of the G -orbits in $\mathcal{C}P$.

$\Rightarrow \exists$ finite number of them.

Def. A finite partition $X = \bigsqcup X_i$ of an alg. variety X is called an algebraic stratification if

(a) Each X_i is smooth locally closed

(b) Each $\overline{X_i}$ is a union of X_j 's

(c) $\forall i, \forall x \in X_i \exists$ a stratified slice to X_i at x , i.e.,

$\exists U$ open neighborhood of x (analytic topology) in X

\exists analytic isom. $(X_i \cap U) \times S \xrightarrow{f} U$, with

S analytic containing x such that:

f yields $\{x\} \times S \xrightarrow{\sim} S, (X_i \cap U) \times \{x\} \xrightarrow{\sim} X_i \cap U$

$(X_i \cap U) \times (X_j \cap S) \xrightarrow{\sim} X_j \cap U$.

Lemma The partition of $\mathcal{C}P$ into orbits is an algebraic

stratification in $\mathcal{C}P$. The nil cone is a union of orbits.

Def. $\pi: F \rightarrow X$ morphism of G -varieties.

It is a cellular fibration if $\exists F = F^n \supset F^{n-1} \supset \dots \supset F_0 = \emptyset$ such that

- (a) F^i G -stable, closed, $\pi|_{F^i}$ G -equivariant locally trivial
- (b) $\pi|_{F^i \setminus F^{i-1}}$ G equiv. locally trivial with affine linear fibers and affine linear transition functions.

EX.

(1) Bruhat decomposition $B = \bigsqcup_w B_w$ over point.

(2) $B \times B = \bigsqcup_w Y_w$, $Y_w =$ diagonal G -orbit of $(B, w(B))$ with $w(B) = m_w B m_w^{-1}$.

(a) $P_2|_{Y_w} \rightarrow B$ affine bundle with fibers $\cong B_w$.

because $G \cdot (B, w(B)) \cong G \times_B B / B \cap w(B)$
 $\cong G \times_B U / U_w$, $U_w = U \cap w(U)$
 $\cong G \times_B U_w$, $U_w = U \cap w(U)$
 \Rightarrow c.f. over B

(b) $Y_w = G / T U_w$ (isot. sy of $(B, w(B))$)

$G = \bigsqcup_w U_w \cdot m_w \cdot T U$
 $\Rightarrow Y_w \cong \bigsqcup_w \underbrace{U_w \cdot m_w \cdot U_w}_{\text{a cell}} \Rightarrow$ c.f. over pt.

(3) $\mathcal{Z} = \bigsqcup_w T_{Y_w}^*(B \times B) \subset T^*(B \times B) \xrightarrow{P_2} T^*B \xrightarrow{\pi} B$

$\Rightarrow T_{Y_w}^*(B \times B)$ affine bundle with fibers $\cong T_{B_w}^*B$

NB: $\overline{B_w} = \bigsqcup B_w$ $\overline{Y} = \bigsqcup Y_w$

IV - Cohomology

* X complex variety.

$\text{Sh}(X)$ = Abelian category of sheaves of \mathbb{C} -vector spaces

$C^b \text{Sh}(X)$ = category of bounded complexes in $\text{Sh}(X)$ with morphisms of complexes.

$D^b(\text{Sh}(X))$ = Derived category of $\text{Sh}(X)$

Objects = Objects of $C^b \text{Sh}(X)$

Morphisms = complicated, quasi-isomorphisms in $C^b \text{Sh}(X)$ yield isomorphisms in $D^b(\text{Sh}(X))$.

A sheaf \mathcal{E} in $\text{Sh}(X)$ is constructible if \exists algebraic stratification $X = \bigsqcup_i X_i$ such that $\forall i$, if $\gamma_i: X_i \hookrightarrow X$ the inclusion, then $\gamma_i^{-1}(\mathcal{E})$ is locally constant with finite dim. stalks.

Locally constant sheaves in $\text{Sh}(X)$ are called local systems.

$D_c^b(X)$ = full subcategory of $D^b(\text{Sh}(X))$ formed by complexes with constructible cohomology sheaves

NB $D^b(\text{Sh}(X))$, $D_c^b(X)$ are not Abelian categories.

Def. $\forall \mathcal{E}, \mathcal{F} \in D_c^b(X)$,

$$\text{Ext}_{D_c^b(X)}^k(\mathcal{E}, \mathcal{F}) = \text{Hom}_{D_c^b(X)}(\mathcal{E}, \mathcal{F}[k])$$

where $\mathcal{F}[k]^i = \mathcal{F}[k+i]$.

The composition

$$\text{Hom}_{D_c^b(X)}(\mathcal{E}, \mathcal{F}[k]) \times \text{Hom}_{D_c^b(X)}(\mathcal{F}[k], \mathcal{G}[k+l]) \longrightarrow \text{Hom}_{D_c^b(X)}(\mathcal{E}, \mathcal{G}[k+l])$$

yields an associative multiplication (Yoneda product):

$$\text{Ext}_{D_c^b(X)}^k(\mathcal{E}, \mathcal{F}) \times \text{Ext}_{D_c^b(X)}^l(\mathcal{F}, \mathcal{G}) \longrightarrow \text{Ext}_{D_c^b(X)}^{k+l}(\mathcal{E}, \mathcal{G}).$$

Nb. If $\mathcal{E} \in D_c^b(X)$ is such that $\mathcal{H}^i(\mathcal{E}) = 0 \forall i \neq i_0$, then \mathcal{E} is quasi isomorphic to $\mathcal{H}^{i_0}(\mathcal{E})[-i_0]$.

\Rightarrow Functor $\text{sh}(X) \rightarrow D_c^b(X)$ for each i_0 .

* $Y \subset X$ smooth locally closed subvariety of dimension d (over \mathbb{C})
 \mathcal{L} local system on Y .

Def. $\exists!$ intersection cohomology complex $\text{IC}(\mathcal{L})$ in $D_c^b(X)$ such that:

(a) $\mathcal{H}^i(\text{IC}(\mathcal{L})) = 0 \quad \forall i < -d$

(b) $\mathcal{H}^{-d}(\text{IC}(\mathcal{L}))|_Y = \mathcal{L}$

(c) $\dim \text{supp } \mathcal{H}^i(\text{IC}(\mathcal{L})) < -i \quad \forall i > -d$

(d) $\dim \text{supp } \mathcal{H}^i(\text{IC}(\mathcal{L})^*) < -i \quad \forall i > -d$

Here $*$ denotes Verdier duality.

Nb. Cohomology sheaves of $\text{IC}(\mathcal{L})$ are supported on \bar{Y} .

Def \exists full Abelian subcategory $\text{Perv}(X) \subset D_c^b(X)$ whose simple objects are the $\text{IC}(\mathcal{L})$'s, with $\mathcal{L} \in \{ \text{irreducible local systems on various smooth locally closed subvarieties } Y \subset X \}$.

The composition

$$\text{Hom}_{D_c^b(X)}(\mathcal{E}, \mathcal{F}[k]) \times \text{Hom}_{D_c^b(X)}(\mathcal{F}[k], \mathcal{G}[k+l]) \longrightarrow \text{Hom}_{D_c^b(X)}(\mathcal{E}, \mathcal{G}[k+l])$$

yields an associative multiplication (Yoneda product):

$$\text{Ext}_{D_c^b(X)}^k(\mathcal{E}, \mathcal{F}) \times \text{Ext}_{D_c^b(X)}^l(\mathcal{F}, \mathcal{G}) \longrightarrow \text{Ext}_{D_c^b(X)}^{k+l}(\mathcal{E}, \mathcal{G}).$$

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Nb If \bar{Y} smooth ^{connected} and $\mathcal{L} = \mathcal{O}_Y$ (= constant sheaf on Y) then

$$IC(\mathcal{L}) = \mathcal{O}_{\bar{Y}}[d]$$

Theorem (Decomposition) If $\pi: X \rightarrow X'$ projective morphism
 X smooth, $Y \subset X$ smooth locally closed, then in $D_c^b(X')$
 we have:

$$\pi_* (IC(\mathcal{O}_Y)) = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{\substack{Y', \mathcal{L}' \\ \text{l.s. on } Y'}} L_{\mathcal{L}', i} \otimes IC(\mathcal{L}') [i]$$

* If X is a G -variety there are G -equivariant sheaves:

(a) \exists isomorphism $\mathbb{I}: a^{-1}(\mathcal{E}) \xrightarrow{\sim} p_2^{-1}(\mathcal{E})$

with $G \times X \begin{matrix} \xrightarrow{a} \\ \xrightarrow{p_2} \end{matrix} X$

(b) $p_{23}^{-1}(\mathbb{I}) \circ (id_G \times a)^{-1}(\mathbb{I}) = (m \times id_X)^{-1}(\mathbb{I})$, $\begin{matrix} G \times G \times X \xrightarrow{p_{23}} G \\ G \times G \xrightarrow{m} G \end{matrix}$

(c) $\mathbb{I}_{1 \times X} = id$

Yields an Abelian category $Sh^G(X)$.

There are also equivariant perverse sheaves, and an equivariant derived category.

* Categories and K-theory

A category is formed of Objects and Morphisms such that:

(a) $\forall A \in Obj \quad \exists id_A \in Hom(A, A)$

(b) \exists composition of morphisms which is associative

(c) Composition with id_A is the identity.

A category is additive if :

(a) Hom's are Abelian groups and composition is additive.

(b) $\forall A, B, \exists A \sqcup B$ such that $\text{Hom}(A \sqcup B, -) = \text{Hom}(A, -) \times \text{Hom}(B, -)$,

$\exists A \sqcap B$ such that $\text{Hom}(-, A \sqcap B) = \text{Hom}(-, A) \times \text{Hom}(-, B)$,

$A \sqcap B \simeq A \sqcup B$ (denoted $A \oplus B$)

(c) \exists object 0 such that $\text{Hom}(0, 0) = 0$.

An additive category is Abelian if :

(a) Any morphism has a kernel and cokernel

(b) The canonical morphism $\text{Coim} f \rightarrow \text{Im} f$ is invertible.

A full additive subcategory \mathcal{A} of an Abelian category \mathcal{B} is exact if is closed under extensions, i.e.,

$$\begin{cases} 0 \rightarrow P' \rightarrow P \rightarrow P'' \rightarrow 0 \text{ exact in } \mathcal{B} \\ P', P'' \in \text{Obj}(\mathcal{A}) \end{cases} \Rightarrow P \in \text{Obj}(\mathcal{A}).$$

Then an exact sequence in \mathcal{A} is an exact sequence in \mathcal{B} whose objects belong to \mathcal{A} .

Ex R ring, Noetherian, commutative.

(a) $\text{mod}^{\text{fg}}(R)$ is Abelian

(b) $\text{proj}(R) \subseteq \text{mod}^{\text{fg}}(R)$ is exact

(c) $\text{perf}(R) \subseteq \text{mod}^{\text{fg}}(R)$ the category of perfect modules, modules with a finite resolution :

$$0 \rightarrow P_n \rightarrow \dots \rightarrow P_0 \rightarrow M \rightarrow 0$$

with $P_n \in \text{Obj}(\text{proj}(R))$ is exact. (horseshoe Lemma)

NB, R regular $\iff \text{perf}(R) = \text{mod}^{\text{fg}}(R)$.

Def. If \mathcal{A} is an exact category then $K_0(\mathcal{A})$ is the free

Abelian group on $\text{Obj}(\mathcal{A})$ modulo:

(a) $[P] = [P']$ if $P \cong P'$

(b) $[P] = [P'] + [P'']$ if $\exists 0 \rightarrow P' \rightarrow P \rightarrow P'' \rightarrow 0$ exact.

Theorem (Resolution) $\mathcal{D} \subset \mathcal{E} \subset \mathcal{A}$ full subcategories

\mathcal{A} Abelian, \mathcal{D}, \mathcal{E} exact. If

(a) Each object of \mathcal{E} has finite \mathcal{D} -resolution.

(b) $\forall 0 \rightarrow P' \rightarrow P \rightarrow P'' \rightarrow 0$ exact in \mathcal{A} we have:

$P, P'' \in \text{Obj}(\mathcal{E}) \Rightarrow P' \in \text{Obj}(\mathcal{E})$

$P, P'' \in \text{Obj}(\mathcal{D}) \Rightarrow P' \in \text{Obj}(\mathcal{D})$

Then the inclusion $\mathcal{D} \subset \mathcal{E}$ yields an isomorphism $K_0(\mathcal{D}) \cong K_0(\mathcal{E})$

Theorem (Devissage) $\mathcal{A} \subset \mathcal{B}$ Abelian categories such that

(a) \mathcal{A} closed under subobjects and quotients,

(b) $\forall B \in \text{Obj}(\mathcal{B}), \exists$ finite filtration $B = B_0 \supset B_1 \supset \dots \supset B_n = 0$
with $B_i/B_{i+1} \in \text{Obj}(\mathcal{A}) \forall i$.

Then the inclusion $\mathcal{A} \subset \mathcal{B}$ yields an isomorphism $K_0(\mathcal{A}) \cong K_0(\mathcal{B})$

* Equivariant K-theory

X variety.

$\text{Mod}(\mathcal{O}_X) \subset \text{Sh}(X)$

$\text{Coh}(\mathcal{O}_X) \subset \text{Mod}(\mathcal{O}_X)$

$\text{vb}(\mathcal{O}_X) \subset \text{Coh}(\mathcal{O}_X)$

category of sheaves of \mathcal{O}_X -modules.

category of coherent sheaves.

category of loc. free sheaves of finite rank.

$f: X \rightarrow Y$ morphism

$f_*: \text{Mod}(O_X) \rightarrow \text{Mod}(O_Y)$ such that $(f_*E)(U) = E(f^{-1}(U))$

$f^*: \text{Mod}(O_Y) \rightarrow \text{Mod}(O_X)$ such that $f^*E = f'_*E \otimes_{f'^*O_Y} O_X$

If X G -variety the categories $\text{Mod}^G(O_X)$, $\text{Coh}^G(O_X)$, $\text{vb}^G(O_X)$ are defined similarly.

Def $K_0^G(X) = K_0(\text{Coh}^G(O_X))$

Lemma $\forall X \subset Y$ closed embedding of G -varieties we have

$$K_0^G(X) = K_0^G(\text{Coh}_X^G(O_Y))$$

Proof: Devissage.

□

Thus to construct functoriality of K_0^G we will assume that

(a) $X \subset M$ closed G -subvariety.

(b) M smooth quasi projective G -variety.

Theorem (Sumihiro)

$\forall X$ smooth quasi projective G -variety

$\exists \rho \in \text{Hom}_{\text{gp. act.}}(G, GL_{n+1})$ such that X

embeds as a G -subvariety of \mathbb{P}^n (with action induced by ρ)

Corollary If X is smooth quasi-projective G -variety
 then any $\mathcal{E} \in \text{Coh}^G(X)$ has a finite $\text{ob}^G(\mathcal{O}_X)$ -resolution.

Functoriality and tensor product

$f: X \rightarrow Y$ morphism of G -varieties.

(a) If f flat there is $f^*: k_0^G(Y) \rightarrow k_0^G(X)$,

(b) If X, Y smooth quasi-projective and f is closed embedding,
 there is $Lf^*: k_0^G(Y) \rightarrow k_0^G(X)$,

(c) If f is proper there is: $Rf_*: k_0^G(X) \rightarrow k_0^G(Y)$

(d) M smooth quasi-projective G -variety.

$X, Y \subset M$ closed G -subvarieties.

$$\exists \otimes^{\mathbb{Z}}: k_0^G(X) \times k_0^G(Y) \rightarrow k_0^G(X \cap Y)$$

the Tor-product relative to M .

NB (1) $k_0^G(\text{point}) = R(G) = \text{representation ring of } G$.

If G is reductive the character yields an algebra isomorphism

$$R(G) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} \mathbb{C}[G]^G$$

If G is unipotent, by Lie-Kolchin, G has only one simple module (the trivial one). So, by Jordan-Hölder, we have:

$$R(G) = \mathbb{Z}.$$

In general, $R(G) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} \mathbb{C}[G_{\text{red}}]^G$ with $G_{\text{red}} = G/\text{Rad}(G)$.

(2) In the setting of (d) take $Y = M$, and consider $R(\mathbb{A}^1) = k_0^G(\text{point}) \rightarrow k_0^G(M)$. We get a $R(G)$ -module structure on $k_0^G(X)$. Further (a), (b), (c), (d) are $R(G)$ -linear.

Localization Let A be an Abelian reductive group (= diagonalisable group).

Fix $a \in A$, $i: X^a \hookrightarrow X$ closed embedding, X a quasi-projective A -variety.

We have $R(A) \otimes_{\mathbb{Z}} \mathbb{C} \simeq \mathbb{C}[A]$.

Thus functions vanishing at a yield a maximal ideal

$$(a) \subset R(A)_{\mathbb{C}} := R(A) \otimes_{\mathbb{Z}} \mathbb{C}$$

$\forall M$ a $R(A)$ -module set

$$\begin{cases} M_{(a)} = M \otimes_{R(A)} R(A)_{\mathbb{C}, (a)} \\ R(A)_{\mathbb{C}, (a)} = \{f; f(a) \neq 0\}^{-1} R(A)_{\mathbb{C}} \end{cases}$$

Theorem The direct image $i_{*}: K_0^A(X^a)_{\mathbb{C}} \xrightarrow{\sim} K_0^A(X)_{(a)}$ is an isomorphism.

Chern character

X quasi projective variety.

Def Given a closed embedding $X \subseteq M$, M smooth quasi-projective,

the Borel Moore homology is (not grading preserving):

$$H_{*}(X, \mathbb{C}) = H^{*}(M, M \setminus X, \mathbb{C}).$$

It has same functoriality as $K_0(X)_{\mathbb{C}} := K_0(X)$

Theorem $X \subset M$ closed, M smooth.

(1) \exists \mathbb{C} -linear ~~isomorphism~~ morphism $ch: K_0(X)_{\mathbb{C}} \rightarrow H_*(X, \mathbb{C})$
It is invertible if X has a cellular decomposition.

(2) $ch(O_X) = [X] =$ fundamental class.

(3) If $X, Y \subset M$ closed, M smooth, the following diagram commutes:

$$\begin{array}{ccc} K_0(X)_{\mathbb{C}} \times K_0(Y)_{\mathbb{C}} & \xrightarrow{\otimes} & K_0(X \cap Y)_{\mathbb{C}} \\ \downarrow ch & & \downarrow ch \\ H_*(X, \mathbb{C}) \times H_*(Y, \mathbb{C}) & \xrightarrow{\cap} & H_*(X \cap Y, \mathbb{C}) \end{array}$$

(4) ch commutes with pull-back.

(5) (Riemann-Roch) Given a proper map

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \cap & & \cap \\ M & \xrightarrow{F} & N \end{array}$$

with M, N smooth, the maps $K_0(X)_{\mathbb{C}} \rightarrow H_*(Y, \mathbb{C})$

$$Td_N \cap (f_* \circ ch) = f_* (Td_M \otimes ch)$$

are equal.

Idea of construction:

Take $\mathcal{E} \in \text{Coh}(O_X)$.

Fix a finite $\text{vb}(O_M)$ -resolution of $i_X^* \mathcal{E}$, $i: X \hookrightarrow M$.

Apply the alternate sum of Chern characters of each term of the resolution to $[M]$.

V - Convolution algebras

* Convolution in homology

M smooth complex variety, $\pi: M \rightarrow N$ proper map,
 N complex variety.

$$Z = \underset{N}{M \times M} \subset M \times M$$

$$M \times M \times M \xrightarrow{P_{ij}} M \times M \quad 1 \leq i, j \leq 3.$$

We have $P_{13} \left(P_{12}^{-1}(Z) \cap P_{23}^{-1}(Z) \right) \subset Z$ (indeed =).

In Borel-Moore homology we get

$$H_* (Z, \mathbb{C}) \times H_* (Z, \mathbb{C}) \xrightarrow{\otimes} H_* (Z, \mathbb{C})$$

$$(c_{12}, c_{13}) \longmapsto (P_{13})_* \left((P_{12})^*(c_{12}) \cap (P_{23})^*(c_{13}) \right)$$

Proposition $(H_* (Z, \mathbb{C}), \otimes)$ is a \mathbb{C} -algebra.

* Convolution in k -theory

M smooth quasi projective G -variety, G linear group

N quasi projective G -variety, $\pi: M \rightarrow N$ proper
 G -equivariant.

Z as above is closed G -subvariety.

Proposition $(k_0^G(Z), \otimes)$ is a $R(G)$ -algebra.

* Simple modules of homological convolution algebra

Set $\mathcal{L} = \pi_* (\mathbb{C}_M) \in D_0^b(N) =$ derived category
of bounded complexes of \mathbb{C} -sheaves
whose cohomology sheaves are
constructible.

Proposition there is an isomorphism of algebras

$$(H_*(Z, \mathbb{C}), \otimes) \xrightarrow{\sim} \text{Ext}_{\mathcal{D}_c^b(N)}^*(\mathcal{L}, \mathcal{L}).$$

It is not graded.

Decomposition theorem $\Rightarrow \mathcal{L} = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{\substack{S \in \text{Per}v(N) \\ \text{simple}}} L_{S,i} \otimes S[i]$

$L_{S,i}$ f. dim. \mathbb{C} -vector space.

$$\begin{aligned} \text{Thus } H_*(Z, \mathbb{C}) &\simeq \bigoplus_{k \in \mathbb{Z}} \bigoplus_{i,j \in \mathbb{Z}} \bigoplus_{S, S'} \text{Hom}_{\mathbb{C}}(L_{S,i}, L_{S',j}) \otimes \\ &\quad \otimes \text{Ext}_{\mathcal{D}_c^b(N)}^{k-2} (S[i], S'[j]) \\ &\simeq \bigoplus_{i,j,k} \bigoplus_{S, S'} \text{Hom}_{\mathbb{C}}(L_{S,i}, L_{S',j}) \otimes \text{Ext}_{\mathcal{D}_c^b(N)}^{k-j-i} (S, S') \end{aligned}$$

Since $\text{Per}v(N)$ is Abelian category, we get:

$$H_*(Z, \mathbb{C}) \simeq \bigoplus_{k \geq 0} \bigoplus_{S, S'} \text{Hom}_{\mathbb{C}}(L_S, L_{S'}) \otimes \text{Ext}_{\mathcal{D}_c^b(N)}^k(S, S')$$

where $L_S = \bigoplus_i L_{S,i}$, $L_{S'} = \bigoplus_j L_{S',j}$.

Lemma If S, S' are simple perverse sheaves then

$$\text{Hom}_{\mathcal{D}_c^b(N)}(S, S') = \begin{cases} 0 & \text{if } S \not\cong S' \\ \mathbb{C} & \text{if } S \cong S'. \end{cases}$$

$$\text{Thus } H_*(Z, \mathbb{C}) = \bigoplus_S \text{End}_{\mathbb{C}}(L_S) \oplus \underbrace{\left(\bigoplus_{S, S', k > 0} \text{Hom}_{\mathbb{C}}(L_S, L_{S'}) \otimes \text{Ext}_{\mathcal{D}_c^b(N)}^k(S, S') \right)}$$

Theorem $\{L_S \neq 0\}$ is a complete set of isomorphism classes of simple $H_*(Z, \mathbb{C})$ -modules.

* Localization of K-theoretic convolution algebras

A diagonalisable group.

M smooth quasi projective A -variety

N quasi projective A -variety

$\pi: M \rightarrow N$ A -equivariant proper map.

$$Z = M \times_M M$$

Proposition (1) $\exists \mathbb{C}$ -algebra isomorphism

$$\left(K_0^A(Z) \otimes_{R(A)} \mathbb{C}_a, \otimes \right) \xrightarrow{\sim} \left(K_0^A(Z^a) \otimes_{R(A)} \mathbb{C}_a, \otimes \right)$$

where $\mathbb{C}_a = R(A)_{\mathbb{C}} / (a)$.

(2) If A is topologically generated by a then

$$K_0^A(Z^a) = K_0(Z^a) \otimes_{\mathbb{Z}} R(A)$$

Nb Part (2) is a general fact: If \mathcal{G} acts trivially on X

$$\text{Then } K_0^{\mathcal{G}}(X) = R(\mathcal{G}) \otimes_{\mathbb{Z}} K_0(X).$$

* Chern character and convolution algebras

M, N, Z, π as above, with cellular decompositions.

Proposition The map $K_0(Z)_{\mathbb{C}} \rightarrow H_*(Z, \mathbb{C}), \epsilon \mapsto (1 \otimes \text{td}_M) \text{Nch}$ is an isomorphism of convolution algebras.

VII - Affine Hecke algebras

- G connected reductive group with simply connected derived subgroup, $T \subset G$ max. torus, $W = \text{Weyl group}$
- The AHA of G is the $\mathbb{C}[[t^{\pm 1}]]$ -algebra H generated by

(a) α_λ , $\lambda \in X = X^*(T) = \text{weight lattice}$.

(b) $t_i = t_{s_i}$, $\{s_i; i \in I\} = \text{simple reflections}$.

with relations:

(c) $\alpha_\lambda \alpha_\mu = \alpha_{\lambda + \mu}$

(d) $t_{s_i} t_{s_j} t_{s_i} \dots = t_{s_j} t_{s_i} t_{s_j} \dots$ (braid relations)

(e) $(t_{s_i} - t)(t_{s_i} + t^{-1}) = 0$

(f) $\alpha_\lambda t_{s_i} - t_{s_i} \alpha_{\lambda - r\alpha_i} = (t - t^{-1}) \alpha_\lambda (1 + \alpha_{-i} + \dots + \alpha_{-i}^{r-1})$ if $\check{\alpha}_i \cdot \lambda = r \geq 0$

Theorem (1) H is free of rank $|W|^2$ over its center

(1) $Z(H) = \mathbb{C}[\alpha_\lambda]^W \mathbb{C}[[t^{\pm 1}]]$

(2) $R(\mathfrak{G})_{\mathbb{C}} = R(T)_{\mathbb{C}}^W = \mathbb{C}[\alpha_\lambda]^W$

(3) $\forall w = s_{i_1} s_{i_2} \dots s_{i_r}$ reduced expression, set $t_w = t_{s_{i_1}} \dots t_{s_{i_r}}$.

Then H is a free $\mathbb{C}[[t^{\pm 1}]]$ -module with basis $\{\alpha_\lambda t_w\}$.

Nb.

(1) The restriction yields a map $\text{res}: R(\mathfrak{G})_{\mathbb{C}} = R(\mathfrak{G})^{\mathbb{C}} \rightarrow \mathbb{C}[[T]]^W$.

It is injective because $\{(odg)/t\}; g \in G, t \in T_{\text{reg}}\}$ is dense open set in G , where

$$T_{\text{reg}} = \{t; \alpha(t) \neq 1, \forall \alpha \in R\}.$$

Surjectivity follows from M. Weyl's formula.

(2) Due to Bernstein.

The inclusion $Z' = \mathbb{C}[\alpha, \lambda]^w[t^{\pm 1}] \subset Z(H)$ is computationally

Surjectivity:

$$(a) \quad H/(t-1) = \mathbb{C}[W \rtimes X]$$

$$Z(H) \cap (t-1)H = (t-1)Z(H) \quad \left(\begin{array}{l} h \in H \text{ s.t. } h \in Z(H) \\ \Rightarrow h \in Z(H) \text{ because } H \text{ torsion-free} \\ \Rightarrow H/Z(H) \text{ torsion-free} \\ \text{Then use short ex. seq + } \text{Tor}^1 = 0 \end{array} \right)$$

Thus we have an exact sequence

$$0 \rightarrow (t-1)Z(H) \rightarrow Z(H) \xrightarrow{t-1} Z'/(t-1) \rightarrow 0.$$

(b) We have

$$Z' \subset Z(H)$$

$Z(H)$ Z' -finite (because so is H)

$$\xRightarrow{\text{Nakayama}} Z(H)_{(t-1)} = Z'_{(t-1)}$$

(c) We have

$$Z(H) \subset Z(H)_{(t-1)} \cap H = Z'_{(t-1)} \cap H = Z'.$$

Theorem $\exists \mathfrak{A}$ -algebra homomorphism

$$K_0^{\text{Gxd}^r}(\mathfrak{A})_{\mathfrak{A}} \xleftarrow[\Theta]{N} H.$$

Idea of proof

(1) We define Θ on generators.

We check relation by embedding $K_0^{\text{Gxd}^r}(Z)_{\mathfrak{A}} \subset \text{End}(K_0^{\text{Gxd}^r}(\mathbb{B}_{\mathfrak{A}}))$

Embedding given by (faithful) convolution action.

(2) We prove Θ is invertible by filtering

$$k_0^{G \times G^x}(\mathbb{Z})_0 \text{ as in III p19.}$$

Proposition We have an identification of $R(G)$ -algebras:

$$\begin{array}{ccc} H & \xrightarrow[\Theta]{\sim} & k_0^{G \times G^x}(\mathbb{Z})_0 \\ \cup & & \cup \\ \mathbb{Z}(H) & \xrightarrow{\sim} & R(G \times G^x) \end{array}$$

Idea of proof

$$(a) \tilde{\mathbb{Z}}_{\text{diag}} = T_{\Delta(B)}^{\mathbb{Z}}(B \times B) \xrightarrow{i} \mathbb{Z} \quad \text{closed embedding.}$$

$\pi \downarrow$ projection

B

$$\text{yields } k_0^{G \times G^x}(B) \xrightarrow{\text{ind}} k_0^{G \times G^x}(\tilde{\mathbb{Z}}_{\text{diag}}) \xrightarrow{i_*} k_0^{G \times G^x}(\mathbb{Z})$$

ind || induction

$\uparrow \Theta$

$$R(G \times G^x) \longleftrightarrow R(T \times G^x) \xleftarrow{\text{obvious inclusion}} H$$

commutative diagram by definition of Θ .

(b) $\forall V \in \text{Rep}^{\text{fd}}(G \times G^x)$, $\text{ind}(V) = V \otimes_{\mathbb{C}} G_B$ by definition of induction.

$$(c) \forall V \in \text{Rep}^{\text{fd}}(G \times G^x), \quad [V \otimes_{\mathbb{C}} G_B] \otimes [\mathcal{F}] = [V] \otimes [\mathcal{F}]$$

$$\forall \mathcal{F} \in \text{Coh}^{G \times G^x}(\mathbb{Z})$$

$$\cap$$

$$R(G \times G^x).$$

Part (c) follows from

Lemma Given $M, 2$ as in p29 we have:

$$\forall \varepsilon \in \text{Coh}^0(M \times M), \quad \forall \mathcal{F} \in \text{Coh}^0(H) \simeq \text{Coh}^0(\Delta(M))$$

$$[\Delta_* (\mathcal{F})] \otimes [\varepsilon] = [p_1^* (\mathcal{F})] \otimes_{M^2}^{\mathbb{L}} [\varepsilon]$$

Proof

$$\begin{array}{ccccc} M \times M & \xleftarrow{p_{12}} & M \times M \times M & \xrightarrow{p_{13}} & M \times M \\ \Delta \cup & & \Delta \times \text{id} \cup & & \parallel \\ M & \xleftarrow{p_1} & M \times M & = & M \times M \end{array}$$

$$[\Delta_* (\mathcal{F})] \otimes [\varepsilon] = p_{13,*} \left([p_{12}^* \Delta_* (\mathcal{F})] \otimes_{M^3}^{\mathbb{L}} [p_{23}^* (\varepsilon)] \right)$$

$$\stackrel{\text{base chg}}{=} p_{13,*} \left((\Delta \times \text{id})_* [p_1^* (\mathcal{F})] \otimes_{M^3}^{\mathbb{L}} [p_{23}^* (\varepsilon)] \right)$$

$$\stackrel{\text{pr. form.}}{=} p_{13,*} (\Delta \times \text{id})_* \left([p_1^* (\mathcal{F})] \otimes_{M^2}^{\mathbb{L}} [(\Delta \times \text{id})^* p_{23}^* (\varepsilon)] \right)$$

$$= [p_1^* (\mathcal{F})] \otimes_{M^2}^{\mathbb{L}} [\varepsilon]$$

because $p_{13} \circ (\Delta \times \text{id})_M = p_{23} \circ (\Delta \times \text{id})_M = \text{id}_{M^2}$.

Theorem Assume $\zeta \in \mathbb{C}^\times$ not root of unity.

The simple $H|_{T=\zeta}$ -modules are parametrized by G -conjugacy classes of triples (s, α, χ) where:

(a) $s \in G$ is semi-simple.

(b) $\alpha \in \mathcal{A}^D$ is nilpotent and $s\alpha s^{-1} = \zeta\alpha$

(c) $\chi \in \text{Irr } A(s, \alpha)$, $A(s, \alpha) = Z_G(s, \alpha) / Z_G(s, \alpha)^0$,

such that χ is a J.H. factor of $H_\chi(B_{\alpha}^D, \mathbb{C})$.

Idea of proof: write H for $H|_{T=\zeta}$.

By Schur's Lemma, for any simple H -module L we have $Z(H)$ acts by a scalar on L .

$\Rightarrow \exists s \in T$ s.t. $W(s) \in T/W$ is the central char of L

Put $a = (s, \zeta)$.

We must classify $H \otimes_{Z(H)} Z(H)/(a) \cong K_0^{G \times \mathbb{C}^\times}(\mathbb{Z}) \otimes_{R(G \times \mathbb{C}^\times)} \mathbb{C}_a$.

$$\stackrel{\text{La. Hkr.}}{=} K_0(\mathbb{Z}^a)_{\mathbb{C}}$$

$$\stackrel{\text{ch.}}{=} H_\chi(\mathbb{Z}^a, \mathbb{C})$$

Two conjugate s, s' yield isomorphic algebras $H_\chi(\mathbb{Z}^a, \mathbb{C})$.
 \Rightarrow by [I p 31] simple H mod. with central char. are classified

by direct summands of $\mathcal{L}^a = \Pi_\chi(\mathbb{C}_{(T^x B)^a})$, where Π^a is

with of $\Pi: T^x B \rightarrow \mathcal{A}^D$ to $T^x B$.

Richardson Theorem (III, p10)

$\Rightarrow \exists$ finite number of $ZG(\mathfrak{g})$ -orbits in

$$\mathcal{O}^{\mathfrak{g}} = \{x \in \mathcal{O}; \sigma \times \sigma' = \mathfrak{z} \times \mathfrak{z}\}.$$

Thus direct summands of $\mathcal{Z}^{\mathfrak{g}}$ are labelled by \mathbb{V} ^{irreducible} equivariant local systems on orbits in $\mathcal{O}^{\mathfrak{g}}$, hence by $\text{Irr}(A(\mathfrak{g}, \alpha))$.

□

Nb For the proof to work it is important that $G(\mathfrak{g})$ acts trivially on $H_x(\mathbb{Z}^{\mathfrak{g}}, \mathbb{C})$. This is a consequence of a theorem of Steinberg which implies that $G(\mathfrak{g})$ is connected. Further, since $G(\mathfrak{g})$ acts on $(T^*B)^{\mathfrak{g}}$ it implies that direct summands of $\mathcal{Z}^{\mathfrak{g}}$ are labelled by $G(\mathfrak{g})$ equivariant local systems on $\mathcal{O}^{\mathfrak{g}}$. Since we have

$$\forall x \in \mathcal{O}^{\mathfrak{g}}, \quad \mathcal{O} := (\text{ad } G(\mathfrak{g}))(x) \subset \mathcal{O}^{\mathfrak{g}}$$

$$\Pi_2(G(\mathfrak{g})) \rightarrow \Pi_2(\mathcal{O}) \rightarrow A(\mathfrak{g}, \alpha) \rightarrow \Pi_0(G(\mathfrak{g})) = \mathbb{Z}$$

exact, we get that such local systems (on \mathcal{O}) are labelled by $A(\mathfrak{g}, \alpha)$ -modules.

VII - Double Affine Hecke algebras

* G_0 simple, simply connected, connected group, $\mathfrak{g}_0 = \text{Lie}(G_0)$

$T_0 \subset B_0 \subset G_0$ max torus, Borel subgroup, $W_0 = \text{Weyl group}$

$$k = \mathbb{C}((\epsilon)), \quad A = \mathbb{C}[[\epsilon]]$$

$$G = G_0(k), \quad \mathfrak{g} = \mathfrak{g}_0 \otimes k$$

$$B_{\pm} = \sigma^{\pm 1}(B_0), \quad w_0^{\pm}: G_0(A) \longrightarrow G_0, \quad \epsilon \mapsto 0.$$

Def. An Iwahori subgroup of G is a sg G -conjugate to B_{\pm}

$$B := \{ \text{Iwahori subgroups} \}$$

= Affine flag "manifold".

Proposition (1) $B^{T_0} = \{ B^{\star} \in B; T_0 \subset B^{\star} \}$

(2) It is a W -torsor, with $W = N_G(T_0)/T_0(A)$.

(3) $W = \text{Affine Weyl group} = X_{\ast}(T_0) \rtimes W_0$.

Nb Since G_0 is simply connected we have

$$\left\{ \begin{array}{l} X_{\ast}(T_0) = \varphi^{\vee} \\ \quad = \text{concoils lattice} \\ \\ X^{\ast}(T_0) = P \\ \quad = \text{weights lattice} \end{array} \right.$$

W is the Weyl group of an affine root system.

* $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C} \cdot d \oplus \mathbb{C} \cdot c$ affine Kac-Moody Lie algebra.

$$[d, x \otimes \varepsilon^m] = m \cdot x \otimes \varepsilon^m,$$

$$[x \otimes \varepsilon^m, y \otimes \varepsilon^n] = [x, y] \otimes \varepsilon^{m+n} + (x, y) \mathcal{D}_{m+n, 0}^c,$$

where $(,)$ is a \mathfrak{G}_0 -invariant bilinear form on \mathfrak{g}_0 .

Set $\tilde{\mathfrak{t}} = \mathfrak{t}_0 \oplus \mathbb{C}d \oplus \mathbb{C}c$, and decompose $\tilde{\mathfrak{g}}$ by weights for adj. $\tilde{\mathfrak{t}}$ -action. Got the set of affine roots:

$$\Delta = \underbrace{(\Delta_0 \times \mathbb{Z})}_{\Delta_{re}} \cup \underbrace{\{0\} \times \mathbb{Z}}_{\Delta_{im}}.$$

$\forall \alpha \in \Delta_{re}$, \exists isomorphism $u_\alpha: \mathfrak{G}_\alpha \xrightarrow{\sim} U_\alpha \subset \mathfrak{G}$
 \parallel
 (α, m)

similar to II, p10.

* We have $B_1 = \left(\prod_{\alpha \in \Delta_{re}^+} U_\alpha \right) \times T_0(A)$.

We set $U = \left(\prod_{\alpha \in \Delta_{re}^+} U_\alpha \right) \times T_0(1 + \varepsilon A) =$ pro-unipotent radical

of B_1 .

~~Central extensions of \mathfrak{G} are constructed from 2-cocycles $\mathfrak{G} \times \mathfrak{G} \rightarrow \mathfrak{G}^1$~~

Central extensions of \mathfrak{G} are constructed from 2-cocycles $\mathfrak{G} \times \mathfrak{G} \rightarrow \mathfrak{G}^1$
 $\Rightarrow \exists$ a maximal central extension (non-split)

$$1 \rightarrow \mathbb{C}^x \rightarrow \tilde{\mathfrak{G}} \rightarrow \mathfrak{G} \times \mathbb{C}_d^x \rightarrow 1$$

