

Literature

I Dunkl operators for complex reflection groups

(joint with C. Dunkl;

Proc. LMS (3) 86 (2003))

II Category \mathcal{O} for rational Cherednik algebras

(joint with Ginzburg, Guay & Rouquier;

Invent. Math. ... (2003))

I gives a natural construction of the rational Cherednik algebra.

II gives the natural context for the main problem that arises as a result of the construction in **I**

III Quasi-invariants of complex reflection groups

(unpublished preprint (to appear))
of Yuri Berest & Oleg Chalykh

PLANNING

- 1) Dunkl representation of the Cherednik algebra
(Dunkl operators, Dunkl pairing, singular parameters, shift operators, Spherical algebra)
- 2) KZ-functor (localization, monodromy Hecke algebra, duality, main results KZ-functor, decomposition numbers)
- 3) Quasi-invariants (differential operators on quasi-invariants, symmetries spherical algebras, existence shift operators)
- 4) KZ-twist & category \mathcal{O} (Fake degrees)

Berest-Chalykh

I. Complex reflection groups

①

K field of char. 0

V fin. diml. vector space / K

DEF $g \in GL(V)$ is called a **pseudo reflection**

if g has finite order, and if $\text{Ker}(g - \mathbb{1})$ is a hyperplane in V

DEF $G \subset GL(V)$ finite group. G is called a **pseudo reflection group** if G is generated by pseudo reflections.

THM (Chevalley; Shephard-Todd) $G \subset GL(V)$ finite.

The following are equivalent:

(I) $S(V)^G$ is a polynomial algebra.

(II) G is a pseudo-reflection group.

(III) $S(V)$ is a free, rank one $S(V)^G[G]$ module

- If $K = \mathbb{Q}$: G is a (finite) Weyl group
- If $K \cong \mathbb{R}$: G is a real reflection group
- If $K \cong \mathbb{C}$: G is a complex refl. gp.

THM (Bessis '98) K is splitting field for G

Classification (Shephard-Todd)

$G \subset GL(V)$ irreducible CRG. G isom. to:

(I) S_{n+1} acting on $V = \{x \in \mathbb{C}^{n+1} \mid \sum x_i = 0\}$.

(II) $G(m, p, n) \subset GL_n(\mathbb{C})$ with $m, p, n \in \mathbb{N}$,
 $m > 1, p \geq 1, n \geq 1$ and $p \mid m$.

$$G(m, p, n) \subset (\mathbb{Z}/m\mathbb{Z})^m \rtimes S_n = G(m, 1, n)$$

diagonal with entries m -th roots of 1 permutation matrices

$$G(m, p, n) = \{g = D \times \sigma \mid (\det(D))^{m/p} = 1\}$$

(III) One of 34 exceptional cases; the largest is E_8 ; many rank 2 cases related to the finite subgroups $\Gamma \subset SU(2)$ by central

ext: $G = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^n \end{pmatrix} \mid \alpha^m = 1 \right\} \cdot \Gamma$ (n suitable)

Remark (Shephard-Todd) G irreducible CRG

of rank n . Then G can be generated by a set S of m or $n+1$ pseudo reflections with relations of the form (i) homogeneous relations (braid relations) and (ii) finite order rels ($s_i^{e_i} = 1$)

Ex. $G(m, p, n)$ needs $|S| = n+1 \iff n > 1, p \neq 1, m$.

Braid groups From now on $K = \mathbb{C}$

(3)

$G \subset GL(V)$ a CRG. Let \mathcal{A} denote the arrangement of reflection hyperplanes of G . Put

$$V^{reg} = V \setminus \bigcup_{H \in \mathcal{A}} H$$

G acts freely on V^{reg} (Steinberg)

Put $X^{reg} = G \backslash V^{reg} (\simeq \mathbb{C}^n \setminus \{\Delta_G = 0\})$

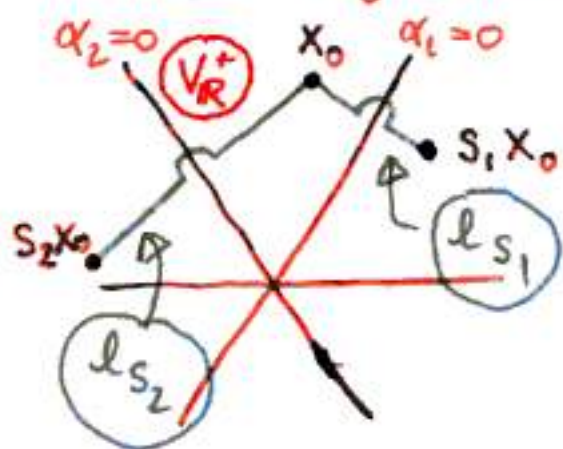
for the regular orbit space of G

DEF Choose $x_0 \in X^{reg}$. The group $B^G = \pi_1(X^{reg}, x_0)$ is called "the" topological braid group of G .

If $W \subset GL(V_{\mathbb{R}})$ is a real reflection group then W admits a Coxeter system (W, S)

DEF The braid group $B^{(W, S)}$ of (W, S) is the group with generators $\sigma_s (s \in S)$ and relations given by the braid relations only

THM (Brieskorn; Deligne) Choose closed loops l_s ($s \in S$) in X^{reg} as follows:



Then the ~~map~~ assignment

$$B^{(W, S)} \ni \sigma_s \rightarrow l_s \in B^W$$

extends to an isomorphism

$$B^{(W, S)} \xrightarrow{\sim} B^W$$

DEF G a CRG. A generator system (G, S) is called "Coxeter-like" if there exists a similar isomorphism $B^{(G, S)} \xrightarrow{\sim} B^G$.

CONJ. (Broué, Malle, Rouquier) There exist a Coxeter-like presentation (BMR proved this in all but 6 exceptional cases)

II The Dunkl-De Rham complex

(5)

$G \subset GL(V)$ a complex reflection group.

For $H \in \mathcal{A}$ we put $G_H = \{g \in G \mid g|_H = \text{Id}_H\}$.

Then $G_H \cong \mathbb{Z}/e_H\mathbb{Z}$ a cyclic group (Steinberg).

and $\hat{G}_H = \{\chi_H^{-i} \mid i=0, \dots, e_H-1\}$ ($\chi_H = \det|G_H$)

Choose $\alpha_H \in V^*$ such that $H = \ker(\alpha_H)$

Thus $\forall g \in G_H : \alpha_H^g = \chi_H^{-1}(g) \alpha_H$

Observation If $p \in \mathcal{P}(V)$ and $\forall g \in G_H :$

$p^g = \chi_H^{-i}(g) p$, then p divisible by α_H^i
($i=0, \dots, e_H-1$)

Let $E_{H,i} = \frac{1}{e_H} \sum_{g \in G_H} \chi_H^i(g) \cdot g \in \mathbb{C}[G_H]$

(projection on χ_H^{-i}). Then for $i \neq 0$ we

have a well defined linear operator

$$\alpha_H^{-1} E_{H,i} : \mathcal{P}(V) \rightarrow \mathcal{P}(V)$$

Choose constants $k_{H,1}, \dots, k_{H,e_H-1} \in \mathbb{C}$ and put:

$$a_H(k) = \sum_{i=1}^{e_H-1} e_H k_{H,i} E_{H,i} \in \mathbb{C}[G_H]$$

Then we have well defined linear operators ⑥

$$\alpha_H^{-1} a_H(k) : P(V) \rightarrow P(V).$$

Assume that $k_{H,i}$ is constant on each G orbit $C = G \cdot H \in G \setminus A := \mathcal{C}$ and write

$$k = (k_{C,i})_{\substack{C \in \mathcal{C} \\ i=1, \dots, e_C-1}} \quad (k_{C,i} = k_{H,i} \quad \forall H \in C)$$

Then the elements $a_H(k) \in \mathbb{C}[G_H]$ satisfy

$$a_{gH}(k) = g a_H(k) g^{-1}$$

DEF For $\xi \in V$ let ∂_ξ be the corresp. constant vector field on V . Let $k = (k_{C,i})$ as above. We put (Dunkel operators):

$$T_\xi(k) = \partial_\xi + \sum_{H \in A} \alpha_H(\xi) \alpha_H^{-1} a_H(k) \in \text{End}_{\mathbb{C}}(P(V)).$$

PROP (I) $T_\xi(k)$ independent choices $\alpha_H \in V^*$

(II) Equivariance: $g T_\xi(k) g^{-1} = T_{g\xi}(k)$

(III) $T_\xi(k)$ homogeneous of degree -1 .

(7)

DEF Let $K^\bullet = P \otimes \wedge^\bullet V^*$ be the algebra of polynomial differential forms on V . Let $d(k): K^\bullet \rightarrow K^\bullet$ be the map

$$d(k)(p \otimes \omega) = \sum_{\xi \in B, \text{ basis } V} T_\xi(k)(p) \wedge d\xi' \wedge \omega$$

\(\xi \in B, \text{ basis } V\)
\'dual of \(\xi\)

$d(k)$ G -equivariant!

$$= (d + \Omega(k))(p \otimes \omega)$$

where $\Omega(k) = \sum_{H \in A} a_H(k) \left(\frac{d a_H}{dH} \wedge \cdot \right): K^\bullet \rightarrow K^\bullet$
 $:= \Theta_H = d \log(a_H)$

PROP (00) Let $\partial: K^\bullet \rightarrow K^\bullet$ be the Koszul differential

$$\partial(p \otimes dx_{i_1} \wedge \dots \wedge dx_{i_\ell}) := \sum_{r=1}^{\ell} (-1)^{r+1} x_{i_r} p \otimes dx_{i_1} \wedge \dots \wedge \widehat{dx_{i_r}} \wedge \dots \wedge dx_{i_\ell}$$

Then $\partial d(k) + d(k) \partial := E(k) = E(0) + \sum_{H \in A} a_H(k)$

where $E(0)$ is the degree operator which acts

on $K_m^\ell := P_m \otimes \wedge^\ell V^*$ by mult. with $m + \ell$

We put $\tilde{z}(k) = \sum_{H \in A} a_H(k) \in \mathbb{C}[G]$

⑧

PROP (00) $\tilde{z}(k)$ satisfies:

- (i) $\tilde{z}(k) \in \tilde{Z}(\mathbb{C}[G])$ (center of $\mathbb{C}[G]$)
- (ii) The multiplier $C_\tau(k)$ of $\tilde{z}(k)$ acting in the representation space V_τ of $\tau \in \hat{G}$ is a linear function of k with non-negative integral coefficients.
- (iii) $C_\tau(k) \equiv 0 \iff \tau = \text{triv.}$

CoR Assume that k is such that $-C_\tau(k) \notin \mathbb{N}$ (for all $\tau \in \hat{G}$). Then

$$\text{Ker } d(k) = (\text{Ker } d(k) \cap \text{Im } d(k)) \oplus K_0^{\circ}$$

In particular $\text{Ker } d(k) \cap P = K_0^{\circ} = \mathbb{C}$

proof Let $\omega \in \text{Ker } d(k) \cap K_{m,\tau}^l$ with $l+m > 0$

Then $E(k)\omega = (l+m+C_\tau(k))\omega := \lambda\omega$ ($\lambda \neq 0$)

Thus $\omega = \lambda^{-1} E(k)\omega = \lambda^{-1} (\partial d(k) + d(k)\partial)\omega$
 $= \lambda^{-1} d(k)\partial\omega \in \text{Im}(d(k))$ \square

COR Assume k as above: $-C_c(k) \in \mathbb{N}$

⑨

There exists a unique linear map

$$I(k): K^\bullet \rightarrow K^\bullet \text{ such that:}$$

- (i) completely homogeneous $I(k)(K_m^l) \subset K_m^l$
and $I(k)(1 \otimes 1) = 1 \otimes 1$.
- (ii) $I(k)(p \otimes \omega) = (I(k)(p)) \otimes \omega \quad \forall \omega \in \wedge^k V^*$
- (iii) $d(k)I(k) = I(k)d(0)$

Moreover, $I(k)$ is a G -equivariant isomorphism.

proof Easy to see (using $\text{Ker } d(k) \cap P_+ = \text{Ker } d(0) \cap P_+ = 0$)

that if $I(k)$ exists it is ~~unique~~ isomorphism.

The uniqueness and equivariance follow.

Existence: Induction on m : assume $I(k)$

already defined on K_i^\bullet ($i < m$). Let

$p \in P_m = K_m^0$. Then $d(k)I(k)(d(0)(p)) = 0$

$\Rightarrow \exists! q \in P_m$ with $d(k)q = I(k)(d(0)(p))$.

Define $I(k)(p) := q$ and extend to K_m^\bullet by (ii). \square

THM (D0) $(K^\bullet, d(k))$ is a complex, In other words, the "curvature" $d(k)^2 = 0$.

COR (D0) $\forall \xi, \eta \in V: T_\xi(k)T_\eta(k) = T_\eta(k)T_\xi(k)$.

proof: $d(k)^2(0) = \sum (T_i(k)T_j(k) - T_j(k)T_i(k)) p \otimes dx_i \wedge dx_j$

COR (of above proof) Let $H^i(k)$ be the 10
 cohomology of $(K^\circ, d(k))$. Equivalent is:

- triv. $\left\{ \begin{array}{l} \text{(a)} \quad H^i(k) = 0 \quad \forall i > 0 \\ \text{(b)} \quad H^0(k) = \mathbb{C} \\ \text{(c)} \quad \text{There exists a quasi-isomorphism} \end{array} \right. \begin{array}{l} \Downarrow \text{Euler char.} \\ \Downarrow \text{above construct.} \end{array}$
- $I(k): (K^\circ, d(0)) \rightarrow (K^\circ, d(k))$

III Singular parameters

DEF We call $k = (k_{c,j})$ singular for Eriv if there exists $i > 0$ with $H^i(k) \neq 0$. (above equiv. cond.)

DEF (contravariant form) Define antilinear isom $\sharp: V \rightarrow V^*$ by $\sharp^*(\eta) = (\sharp, \eta) \quad \forall \sharp, \eta \in V$.

Extend to $\sharp: S \rightarrow P$ ($S := S(V)$; $P := P(V)$)

By commutativity of $T_\sharp(k)$: we can extend

$\sharp \rightarrow T_\sharp(k)$ to an alg. hom. $S \rightarrow \text{End}(P)$
 $S \rightarrow S(T(k))$

Define sesquilinear pairing $(\cdot, \cdot)_k$ on P by

$$(p, q)_k := (p^\sharp(T(k))(q))(0)$$

PROP (DO) (I) $(P^g, q^g)_k = (P, q)_k \quad \forall g \in G.$ (11)

(II) $(P_{m, \tau}, P_{\ell, \sigma})_k = 0$ unless $m = \ell, \sigma = \tau$

(III) $(\xi^* P, q)_k = (P, T_\xi(k) q)_k \quad \forall P, q, \xi.$

(IV) Symmetry $(P, q)_k = \overline{(q, P)_k}$ (!)

COR (DO) Equivalent are:

(a) k singular.

(b) $(\cdot, \cdot)_k$ degenerate.

(c) There exists a subspace $I \subsetneq P$ stable for $G, T_\xi(k)$ and multiplication by $p \in P$

(d) \bar{k} singular.

COR (I) $\det (\cdot, \cdot)_k$ on $P_{m, \tau}$ is a product of linear forms of the form

$$l + C_\tau(k) \quad l > 0, P_{\ell, \tau} \neq 0$$

(II) The set $\mathcal{K}_{\text{triv}}^{\text{sing}}$ of singular parameters for triv is a locally finite union of hyperplanes of the form $\mathcal{H}_{\ell, \tau} = \{k : l + C_\tau(k) = 0 \ \& \ P_{\ell, \tau} \neq 0\}$

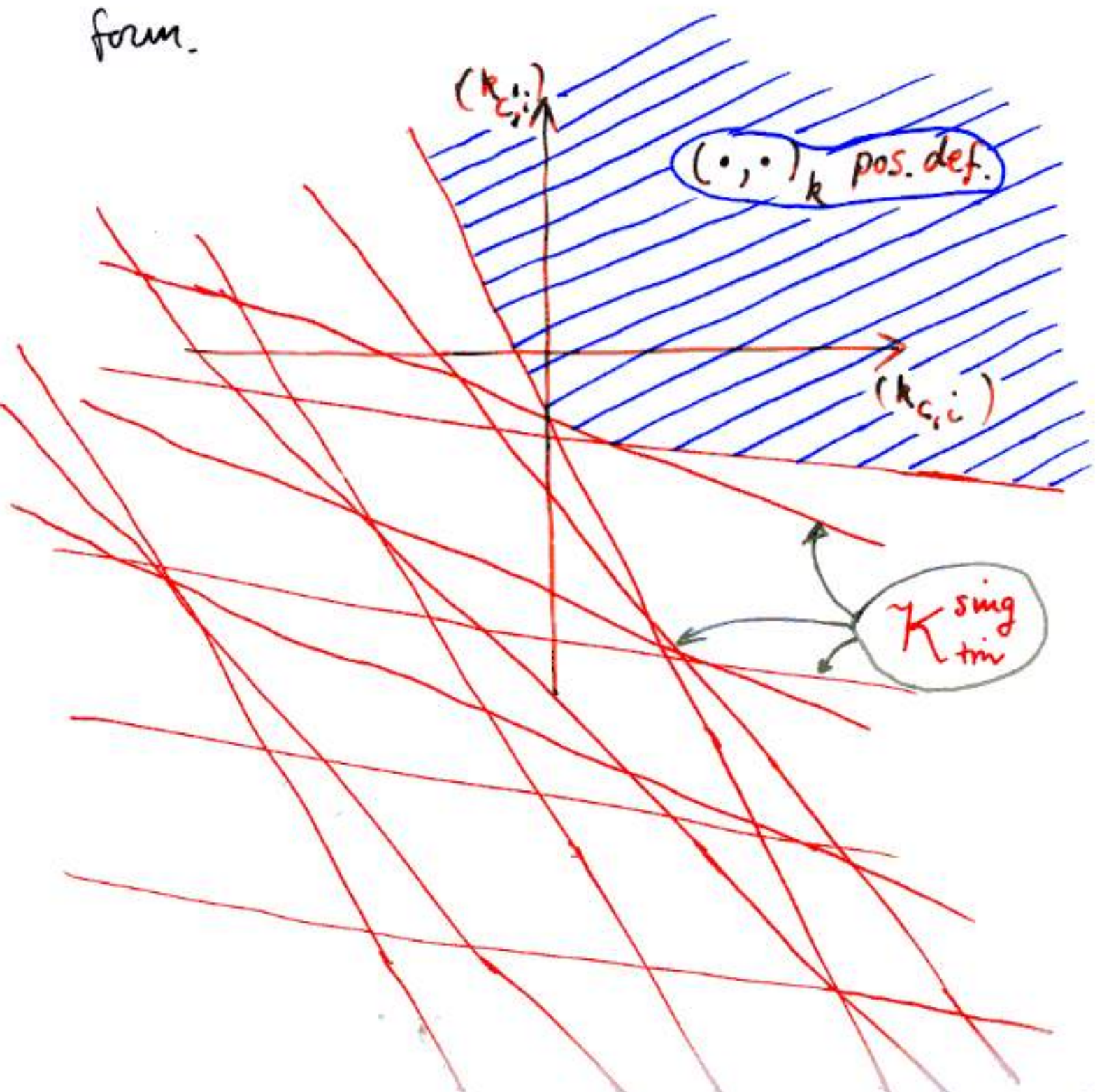
Remark $l + C_\tau(k) = 0 \ \& \ P_{\ell, \tau} \neq 0 \not\Rightarrow k \in \mathcal{K}_{\text{triv}}^{\text{sing}}$

COR If $k_{c,i} \in \mathbb{R}$ and

$$\forall \tau \in \hat{G} : c_\tau(k) + m(\tau) > 0$$

lowest embedding degree
of τ in P

Then $(\cdot, \cdot)_k$ is a positive definite Hermitian form.



IV Rational Cherednik algebra

(13)

The rational Cherednik algebra $A(k)$ is the complex ass. algebra given by a presentation of the form:

$$A(k) = T(V \oplus V^*) \rtimes G / \text{rels}$$

$$\text{with rels} = \begin{cases} \bullet [\xi, \eta] = 0 & \forall \xi, \eta \in V \\ \bullet [x, y] = 0 & \forall x, y \in V^* \\ \bullet [\xi, x] = x(\xi) + \sum_{H \in A} \sum_{\substack{g \in G_H \\ g \neq 1}} c_g(k) \frac{d_H(\xi) \chi(V_H)}{d_H(V_H)} \end{cases}$$

$V_H^* = d_H$
a constant depending lin. on the $k_{\xi, i}$

DEF We filter $A(k)$ by $\deg(x) = \deg(\xi) = 1, \deg(g) = 0$

THM (Ginzburg, Etingof) $A(k)$ has the PBW-property:

$$\text{gr} A(k) \cong (P \otimes S) \rtimes G$$

In particular: $A(k) \cong P \otimes S \otimes \mathbb{C}[G]$
as vector spaces

THM (Dunkel representation of $A(k)$) P is a faithful

$A(k)$ -module via $\xi \rightarrow T_\xi(k), x \rightarrow m_x, g \rightarrow g$

(multiplication by x)

REMARK If fact $P \simeq \text{Ind}_{S \rtimes G}^{A(k)}(\text{triv})$ (14)

where $S \rtimes G$ acts on triv via the augmentation

$$S \rtimes G \rightarrow \mathbb{C}[G] = (S/S_+) \rtimes G$$

$$S \otimes \mathfrak{g} \rightarrow S(0) \otimes \mathfrak{g}.$$

DEF A standard module of type $\tau \in \hat{G}$

for $A(k)$ is the module

$$\Delta(\tau) = \text{Ind}_{S \rtimes G}^{A(k)}(\tau)$$

In particular, the Dunkl representation is $P \simeq \Delta(\text{triv})$

PROP (i) $\Delta(\tau) \cong P \otimes_{\mathbb{C}} V_{\tau}$

(ii) $\Delta(\tau)$ is graded module $\Delta(\tau)_m := P_m \otimes V_{\tau}$

for the grading of $A(k)$ with $\deg(\xi) = -\deg(x) = -1$
 $\deg(\mathfrak{g}) = 0$

(iii) S act locally nilpotently on $\Delta(\tau)$

(iv) $E(k) = E(0) + Z(k) = \sum_{i=1}^n x_i T_i(k)$ acts on

$(\Delta(\tau))_{m, \sigma}$ by scalar $m + C_{\sigma}(k) - C_{\tau}(k)$

(v) All submodules of $\Delta(\tau)$ are graded.

COR $\Delta(\tau)$ is indecomposable and has unique irreducible quotient $L(\tau)$.

DEF We define ordering on \hat{G} by (15)

$$\sigma < \tau \iff C_\sigma(k) - C_\tau(k) \in \mathbb{N}$$

COR $\Delta(\tau)$ has finite Jordan-Hölder series whose irreducible ^{sub}quotients are isomorphic to $L(\sigma)$'s with $\sigma \leq \tau$ and $[\Delta(\tau):L(\tau)] = 1$.

PROBLEM Compute and interpret the lower uni-triangular matrix $[\Delta(\tau, k), L(\sigma, k)]$

REMARK Again we can show that

$$[\Delta(\tau, k), L(\sigma, k)] = \delta_{\tau, \sigma} \text{ unless } k \in \mathcal{K}_\tau^{\text{sing}}$$

which is some locally finite union of hyperplanes $l - C_\tau(k) + C_\sigma(k) = 0$ & $(\Delta(\tau))_{l, \sigma} \neq 0$

REMARK $\mathcal{K}_\tau^{\text{sing}}$ was computed for $\tau = \text{triv}$ and $G = W$ real reflection group (D-deJeu-0)

For $K_{C, i} = k \in \mathbb{C} \forall C, i$ it equals the zero set of the Bernstein-Sato polynomial b_W of the discriminant Δ_W of W . (up to shift by $\frac{1}{2}$)

$\mathcal{K}_{\text{triv}}^{\text{sing}}$ was computed for $G(m, p, n)$. (D-0)

Interpretation $[\Delta(\tau) : L(\sigma)]$ (via KZ-functor) (6)

(a) Assume G has Coxeter-like presentation (G, S)

DEF Let $R = \mathbb{C}[q_{c,i}^{\pm 1}]$ and put:

$$\mathcal{H}(G) := R[B^{(G,S)}] / ((\sigma_s - 1)(\sigma_s - q_{c,1}) \dots (\sigma_s - q_{c,e_c - 1}))$$

(b) Assume $\mathcal{H}(G)$ is R -free of rank $|G|$

THM (G.G.O.R.) Let $k = (k_{c,i})$ with $k_{c,i} \in \mathbb{C}$

and put $q_{c,i} = \exp(2\pi i(j - e_c k_{c,i})/e_c) \in \mathbb{C}^\times$

(1) The simple $\mathcal{H}(G, q) := \mathcal{H}(G) \otimes_R \mathbb{C}_q$ modules

$L(\sigma)$ are parametrized by $\sigma \in \hat{G}$ such that

$\text{Support}(L(\sigma)) \not\subseteq V \setminus V^{\text{reg}}$
 as a $P(V)$ -module \leftarrow simple quotient of $\Delta(\sigma)$

(2) Let K be the quotient field of R , extended such that $\mathcal{H}(G) \otimes_R K \cong K[G]$. Let $\tau \in \hat{G}$ and let $M(\tau) \in \mathcal{H}(G)\text{-mod}$ be R free s.t. $M(\tau) \otimes_R K$ is isomorphic to $\tau \otimes_{\mathbb{C}} K$. Then

$$[M(\tau) \otimes_R \mathbb{C}_q : L(\sigma)] = [\Delta(\tau) : L(\sigma)]$$

\swarrow $\mathcal{H}(G, q)\text{-mod}$ \searrow $\mathbb{A}(K)\text{-mod}$
 $\sigma \in \hat{G}$ s.t. $\text{supp}(L(\sigma)) \not\subseteq V^{\text{sing}}$

COR The decomposition matrix of $\mathcal{H}(G)$ is lower uni-triangular w.r.t. the order $<$ on \hat{G} . (provided (a), (b)).

REM Such results were known for:

\hat{G}_n
 Dipper-James

W Weyl gp
 \mathcal{H} unital
 Geck '98

$G(m, 1, n)$
 Ariki '96
 (proof LLT-conj.)

(17)

REM Equivalent is

(1) $k \not\subseteq K^{\text{sing}} = \bigcup_{\tau \in \hat{G}} K_{\tau}^{\text{sing}}$

(2) $\forall \tau \in \hat{G}: \Delta(\tau)$ is simple

(3) \mathcal{O}_k (this morning's lecture by Nicolas) is semisimple (by reciprocity formulas).

V The spherical algebra

DEF The spherical subalgebra $U_k \subset A(k)$

is the algebra $U_k = e A(k) e \subset A(k)$

where $e = \frac{1}{|G|} \sum_{g \in G} g$ is trivial idemp. G .

PROP The natural action of U_k on $e \Delta(\text{triv}) \simeq P^G$ is a faithful representation of U_k by polynomial differential operators on P^G . We write $U_k \ni f \rightarrow L_f \in D(G \setminus V)$

EX $p \in S(V)^G \Rightarrow L_p(k) \in D(G \setminus V)$

G -inv. diff. op. V^{reg} s.t. $P^G \rightarrow P^G$

Second lecture: Change of notational

conventions and context:

Yesterday	Today
\mathbb{C}	k , a \mathbb{C} -algebra
G a C.R.G	W , a Weyl gp.
$k = (k_{c,i,j})_{c,i,j}$	$C \in k = k_{c,1}$ (constants $\forall C$ equal)
$A(k)$	A_c (or A_c)

(Remo Numbering starts with 4!)

Rem All construction we'll discuss

today work in the general setup of

Yesterday's talk, but we will not

burden the notations by pursuing generality

Rational Cherednik algebra

(4)

k \mathbb{C} -algebra (Noetherian)

$c \in k$

DEF

$$A = A(c) = (k \otimes_{\mathbb{C}} T(V \oplus V^*) \rtimes W) / \text{Rel}$$

Relations :

- $[x, y] = 0 \quad \forall x, y \in V^*$

- $[\xi, \eta] = 0 \quad \forall \xi, \eta \in V$

- W acts diag. on $V \oplus V^*$

- $[\xi, x] = \langle \xi, x \rangle + c \sum_{\alpha \in R_+} \alpha(\xi) \kappa(\alpha) S_{\alpha}$

THM (Etingof, Ginzburg; Cherednik)

$$A \cong_{k\text{-mod}} P \otimes kW \otimes S \quad \begin{aligned} P &= k \otimes P(V) \\ &= k \otimes S(V^*) \\ S &= k \otimes S(V) \end{aligned}$$

- A graded by $\deg(V^*) = 1 - \deg(V) = 1$
 $\deg(W) = 0$

- A filtered by $\deg(V^*) = \deg(V) = 1$
 $\deg(W) = 0$

Then $gr(A) \cong k \otimes S(V \oplus V^*) \rtimes W$ (PBW) ⑤

PROP Grading of A is inner

$$\text{Put } \Sigma_c := \sum_{\text{basis } V} b^* b \in P_+ \otimes S_+$$

$$Z_c := c \sum_{\alpha \in R_+} (1 - s_\alpha) \in Z(kW)$$

$$E := \Sigma_c - Z_c$$

Then: $A_i = \{a \in A : [E, a] = i \cdot a\}$ \square

DEF For $E \in \hat{W}$, let $c_E \in k$ be the scalar action of Z_c on $k \otimes E$

REM $c_E = n_E \cdot c$ with $n_E \in \mathbb{Z}_{\geq 0}$

REM $n_E = a_E + A_E$ (lowest and highest powers of q in $\varphi_E(q)$, gen. degree of E) (due to Brion-Michel)

(but n_E is much easier to compute!)

(compared to a_E and A_E)

Category $\mathcal{O} = \mathcal{O}_c$ of H.W.M. of A (6)

(A) (ungraded)

\mathcal{O} full subcat. of A -mod M s.t.

1) $\mathfrak{S} = \mathfrak{k} \otimes \mathfrak{S}(V)$ locally nilpotent

2) gen. \mathfrak{E} -^{gen.}weight spaces are f.g.

$$\text{and } M = \sum_{\alpha \in \mathfrak{k}} W_{\alpha}(M)$$

Examples $E \in \hat{W}$;

① $\mathfrak{k} \otimes E$ is $\mathfrak{S} \rtimes W$ -mod via $\mathfrak{S} \xrightarrow{\text{aug.}} \mathfrak{k}$

DEF Put $\Delta(E) := \text{Ind}_{\mathfrak{S}W}^A(\mathfrak{k}E)$

Standard modules

Rem $\Delta(E)$ graded via $\Delta(E) \simeq P \otimes E$

② let $\nabla(E) = \text{grhom}_{PW}(A, \mathfrak{k}E)$
 $= \bigoplus_{i \geq 0} \text{grhom}_{PW}^i(A, \mathfrak{k}E)$

$$\cong_{S\text{-mod}} \text{grhom}_k(S, k) \otimes E \cong P \otimes E \quad \textcircled{7}$$

where A -action is: $(a \cdot f)(b) = f(ba)$

Hence $\nabla(E)$ also graded, in \mathcal{O} is called: costandard

Lemma $\text{Ext}_A^i(\Delta(E), \nabla(F)) =$

$$\cong \text{Ext}_{S_W}^i(E, \nabla(F))$$

$$\cong \text{Ext}_S^i(k, \text{grhom}(S, k)) \otimes \text{Hom}_W(E, F)$$

$$= \begin{cases} k & \text{if } E \cong F, i=0 \\ 0 & \text{else} \end{cases}$$

③ Generalised standard modules

$$\Delta_n(E) := \text{Ind}_{S_W}^A ((S/S_{>n}) \otimes_k E)$$

- Rem
- Also graded (put $S_0 \otimes_k E$ deg 0)
 - has Δ -filtration
 - All M in \mathcal{O} are quot. of a sum of Δ 's

(B) (graded)

(2)

$\tilde{\mathcal{O}} =$ graded modules in \mathcal{O}

DEF $\tilde{\mathcal{O}}^\alpha =$ graded mods in \mathcal{O}
($\alpha \in k$) with $M_i \subseteq W_{i-\alpha}(M)$

Hence: $M \in \tilde{\mathcal{O}}^\alpha$ means that

$M_i(F) \subset$ gen. Σ_c -weight sp.
of weight $i - \alpha + C_F$

At bottom of grading of M :

$$i_0 = \alpha - C_F$$

COR $\tilde{\mathcal{O}}^\alpha$ nonempty $\Rightarrow \alpha \equiv C_F \pmod{\mathbb{Z}}$
for some $F \in \hat{W}$

COR $\tilde{\mathcal{O}}(E)$ graded version of $\Delta(E)$
with $S_0 \otimes E$ in degree 0

Let $\tilde{\Delta}(E)(\tau)$ object with shifted
grading ($S_0 \otimes E$ deg $-\tau$)

Then $\tilde{\Delta}(E)(\tau) \in \tilde{\mathcal{O}}(C_E - \tau)$

Prop

$$\mathbb{P} := \bigcup_{E \in \hat{W}} (C_E + \mathbb{Z}) / \sim$$

9

PROP GR

$x \sim y \Leftrightarrow x - y$ not invertible in k .

$$\bullet \tilde{\mathcal{O}} = \bigoplus_{a \in \mathbb{P}} \tilde{\mathcal{O}}^a$$

$$\bullet \mathcal{O} = \bigoplus_{a \in \mathbb{P} / \mathbb{Z}} \mathcal{O}^{a + \mathbb{Z}} \quad \text{with}$$

$\mathcal{O}^{a + \mathbb{Z}}$ image $\tilde{\mathcal{O}}^a$ (forget grading)

$$\bullet \tilde{\mathcal{O}}^a \xrightarrow[\text{forget}]{\sim} \mathcal{O}^{a + \mathbb{Z}} \quad \text{equiv cat.}$$

(grading invar)

COR 1 (\sim Sörgel) $E \in \hat{W}$. For $n \in \mathbb{Z}_+$

suff. large, $\Delta_n(E)$ contains indecomp. projective summand $P(E)$ with Δ -filtration and $\Delta(E)$ as quotient.

proof By **PROP GR** we may work ¹⁰

in $\mathcal{O}^{cE+\mathbb{Z}}$ -block, hence in $\tilde{\mathcal{O}}^{cE}$

Let $p: \tilde{\mathcal{O}} \rightarrow \tilde{\mathcal{O}}^{cE}$ projection

For $M \in \tilde{\mathcal{O}}^{cE}$:

$$\begin{aligned} \textcircled{1} \quad \text{Hom}_{\tilde{\mathcal{O}}} (p(\tilde{\Delta}_n(E)), M) &= \\ \text{Hom}_{\tilde{\mathcal{O}}} (\tilde{\Delta}_n(E), M) &= \\ \text{Hom}_{S_W}^{\text{gr}} (S/S^{>n} \otimes E, M) &\textcircled{2} \end{aligned}$$

$\textcircled{1}$: $\exists n$ s.t. $p(\tilde{\Delta}_n(E)) = p(\tilde{\Delta}_m(E)) \forall m \geq n$

Since added $\tilde{\Delta}(F) \langle \tau \rangle$'s at bottom
are no longer in $\tilde{\mathcal{O}}^{cE}$

$\textcircled{2}$: So may take $n \gg 0$, then by $\textcircled{2}$

$$\text{Hom}_{\tilde{\mathcal{O}}} (p(\tilde{\Delta}_n(E)), M) = M_0(E)$$

|
S'loc. inlp.

Hence $\text{Hom}_{\tilde{\mathcal{O}}} (p(\tilde{\Delta}_n(E)), -)$ exact

Take $P(E)$ suitable summand of $\tilde{\Delta}(E)$

COR 2 $\exists P(E)$ cover $L(E)$, and $P(E)$ has Δ -filtration with subquot $\Delta(F)$ s.t. $C_F - C_E \in \mathbb{Z}_{>0}$ 11.

COR 3 (of Prop 6r) All submodules $\Delta(E)$ are graded $\Rightarrow \Delta(E)$ has unique irred quotient $L(E)$.

PROP $\Delta(E)$ has a finite J-H series with subquot. $\approx L(F)$'s with $C_E - C_F \in \mathbb{Z}_{>0}$ (highest weight theory)

THM k field Then \mathcal{O}_c is a highest weight category in sense of C.P.S., with \hat{W} param. the standard modules Δ , with ordering $E < F \Leftrightarrow C_F - C_E \in \mathbb{Z}_{>0}$

COR If $C_E - C_F \notin \mathbb{Z}_{>0} \forall E, F \Rightarrow \mathcal{O}_c$ is s.s.

EOR (CPS) in \mathcal{O}_c we have

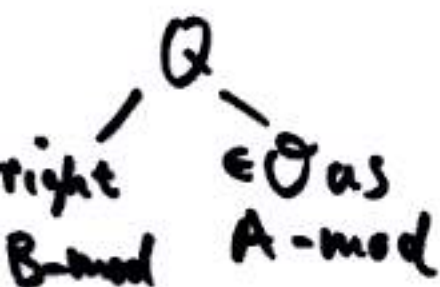
- enough proj. $P(E)$, $E \in \hat{W}$
 $P(E)$ Δ -filtration
- Inj. $I(E)$, ∇ -filtration
- BGG reciprocity

$$[L(E); L(F)] = [I(F); \nabla(E)]$$

$$[P(E); \Delta(F)] = [\nabla(F); L(E)]$$

- \mathcal{O}_c has finite global hom. dim.
- Unitriangular transitions between $[L(E)]$ -basis and $[\Delta(E)]$ -basis. in $K_0(\mathcal{O}_c)$ (**Geometric Meaning?**)

COR Put $B = \text{End}_{\mathcal{O}}(Q)^{\text{opp}}$ Q proj. gen. of \mathcal{O} .



Morita equiv:

$$\Gamma: \mathcal{O} \xrightarrow{\sim} B\text{-mod}$$

$$M \rightarrow \text{Hom}_{\mathcal{O}}(Q, M)$$

$$Q \otimes X \leftarrow X$$

Localization and monodromy

1.2 1/2

Localization of A

Recall the Dwork representation

$$\delta: A(\mathbb{C}) \longrightarrow \text{End}_k(k \otimes_{\mathbb{C}} P)$$

$$\left\{ \begin{array}{l} W \ni W \longrightarrow W \text{ (acting on } P) \\ k \otimes_{\mathbb{C}} P \ni P \longrightarrow m_P \text{ (mult. by } P) \\ V \ni \mathfrak{z} \longrightarrow T_{\mathfrak{z}}(\mathbb{C}) \text{ (Dwork operator)} \end{array} \right.$$

Observation After localization on V^{reg}

we get (with $A_{\text{reg}}(\mathbb{C}) := \underbrace{P[\alpha_H^{-1}: H \in A]}_{:= P_{\text{reg}}} \otimes S \otimes kW$)

$$\delta_{\text{reg}}: A_{\text{reg}}(\mathbb{C}) \xrightarrow{\sim} k \otimes D(V^{\text{reg}}) \rtimes W$$

Pf Put $\delta_{\text{reg}}^{-1}(\partial_{\mathfrak{z}}) := T_{\mathfrak{z}} - 2C \sum_{H \in A} \frac{\alpha_H(\mathfrak{z})}{\alpha_H} \underbrace{E_{H,1}}_{= \frac{1-S_H}{2}}$

Clearly this extends to inverse of δ_{reg} \square

