Mini-course
Cherednik algebras
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Finite dimensional representations of rational Cherednik algebras of type A_{n-1} when $t \neq 0$ $(t=1)$

References: math.RT/0208138 Y. Berest, P. Etingof, V. Ginzburg
math.RT/0208126 I. Gordon

Type A_{n-1}

$W = S_n$ $\quad \mathfrak{h} \cong \mathbb{C}^{n-1}$

$y \in \mathfrak{h}$, $x \in \mathfrak{h}^*$ $\quad s_{ij} \in S_n$: transposition

$\mathfrak{h} \cong \operatorname{span}\{y_i - y_{i+1} \mid 1 \le i \le n-1\}$

$\mathfrak{h}^* \cong \operatorname{span}\{x_i - x_{i+1} \mid 1 \le i \le n-1\} \xrightarrow{\sim} \operatorname{span}\{x_1, \dots, x_n\} / (x_1 + \dots + x_n = 0)$ $\quad x_i(\ )=$

→ It may be more convenient to use the fundamental weights as a basis of $\mathfrak{h}^*$.

$\langle \cdot , \cdot \rangle : \mathfrak{h}^* \oplus \mathfrak{h} \to \mathbb{C}$

In $H_{t=1,c}(S_n)$ ($= {}_nH_c$),
$$yx - xy = \langle x, y \rangle - c \sum_{1 \le i < j \le n} \langle y, x_i - x_j \rangle \langle y_i - y_j, x \rangle s_{ij}.$$

Theorem: i) If $H_{t=1,c}(S_n)$ admits finite dimensional representations, then $c = \pm \frac{r}{n}$ with $r \in \mathbb{Z}_{\ge 1}$, $(r,n)=1$.

ii) If $c = \frac{r}{n} > 0$, $(r,n)=1$, the only irreducible finite dimensional representation of $H_{t=1,c}(S_n)$ is $L(\mathrm{triv})$.

$H_{t,c}(W) \cong H_{t,-c}(W)$ $\quad$ If $c = -\frac{r}{n} < 0$, $(r,n)=1$, the only irreducible finite dimensional representation of $H_{t=1,c}(S_n)$ is $L(\mathrm{sign})$.

Recall that $\mathrm{Ext}^1_{\mathcal{O}}(L(\tau), L(\tau)) = 0$. This implies that, if $c = \pm\frac{r}{n}$, $(r,n)=1$, any finite dimensional representation of $H_c(S_n)$ is isomorphic to $L(\mathrm{triv})^{\oplus k}$ or $L(\mathrm{sign})^{\oplus k}$ for some $k \ge 1$, depending on the sign of c.

When does H_c have a one-dimensional representation $V=\mathbb{C}v$?

$yv = 0 = xv$, $\sigma v = v$.

$$0=(yx-xy)v=\Big(\langle y,x\rangle - c\sum_{1\le i<j\le n}\langle y, x_i-x_j\rangle\langle y_i-y_j, x\rangle\Big)v \quad \forall x\in\mathfrak{h}^*, \forall y\in\mathfrak{h}.$$

$$\Leftrightarrow\ 0=\big(\langle y,x\rangle - c\cdot n\cdot\langle y,x\rangle\big)v \ \Leftrightarrow\ c=\frac{1}{n}$$

n is the Coxeter number of S_n.

Reminder on the Koszul complex: R: commutative ring

$r_1,\dots,r_k$ a sequence of elements in R.

The Koszul complex associated to this data is: $R^k = R^{\oplus k}$: free R-module.

$$0\to \Lambda^k R^k \to \Lambda^{k-1}R^k \to \dots \to \Lambda^p R^k \xrightarrow{\partial_p} \dots \to \Lambda^1 R^k \to \overset{\overset{R}{\parallel}}{\Lambda^0 R^k} \to 0$$

and ∂_p is given by:

$$\partial_p(e_{i_1}\wedge\dots\wedge e_{i_p}) = \sum_{j=1}^{p}(-1)^{j+1} r_{i_j}\, e_{i_1}\wedge\dots\wedge \overset{\text{not there}}{\widehat{e_{i_j}}}\wedge\dots\wedge e_{i_p} \qquad 1\le i_1<\dots<i_p\le k.$$

$e_1,\dots,e_k$ is a basis of the free R-module $R^k = \underbrace{R\oplus\dots\oplus R}_{k \text{ times}}$

Theorem: If $r_1,\dots,r_k$ is a regular sequence in R ($\Leftrightarrow r_j$ is not a zero divisor in $R/(r_1,\dots,r_{j-1})R$), then the Koszul complex is a free resolution of $R/(r_1,\dots,r_k)R$.

Take $R=\mathbb{C}[\mathfrak{h}]$, $r_i = x_i - x_{i+1}$, $1\le i\le n-1$, so $R/(r_1,\dots,r_{n-1})R \cong \mathbb{C}$. The Koszul resolution of $\mathbb{C}$ is: