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1. Hilbert Schemes & Quiver varieties

A. Defn: $X = \mathbb{C}^2$, $n \in \mathbb{N}$

$X^{[n]} = \{ I \triangleleft \mathbb{C}[x,y] : \dim_{\mathbb{C}} \frac{\mathbb{C}[x,y]}{I} = n \}$ is the Hilbert scheme of n pts in the plane.

Eg. $n=2$. $\frac{\mathbb{C}[x,y]}{I}$ has basis $\{1+I, x+I\}$ or $\{1+I, y+I\}$

case: $\{1+I, x+I\}$. $x^2 = \alpha + \beta x \mod I$
$y = \gamma + \delta x \mod I$.

$U_{\text{II}} \underset{\text{open}}{\subset} X^{[2]}$, $U_{\text{II}} \cong \mathbb{C}^4$

also for $\{1+I, y+I\} \to U_{\boxminus} \cong \mathbb{C}^4$.

Thm (Fogarty '68) $X^{[n]}$ is smooth, irreducible variety of dimn $2n$.

$I \in X^{[n]} \rightsquigarrow V(I) \subseteq \mathbb{C}^2$ a finite set of points.

$\frac{\mathbb{C}[x,y]}{I} \cong \bigoplus_{p \in V(I)} \frac{\mathbb{C}[x,y]_p}{I_p}$ multiplicity of I at $p := \dim_{\mathbb{C}} \frac{\mathbb{C}[x,y]_p}{I_p}$

$$\pi : X^{[n]} \longrightarrow X \times \dots \times X / S_n = X^{(n)}$$
$$I \longmapsto \sum_{p \in V(I)} \text{mult}_p I \, [p]$$

($X^{(n)}$ ← n unordered points on plane)

Ex. calculate when $n=2$.

I generically ~~consists of~~ has support consisting of n distinct points.

and these n distinct points determine I, so π is generically

an isomorphism.

$X^{(n)}$ is a singular variety. ($\mathcal{O}(X^{(n)}) = \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]^{S_n}$)

$\mathrm{Sm}\, X^{(n)}$ = n-distinct points. π is a resolution of sing.

Worst point is $n \cdot [0] \in X^{(n)}$

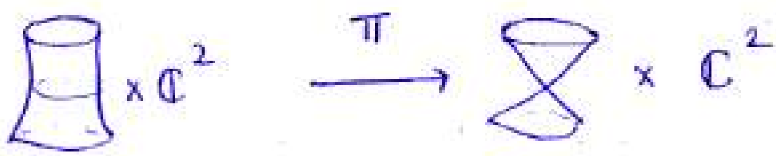
$$\pi^{-1}(n \cdot [0]) = \left\{ I \triangleleft \mathbb{C}[x,y] : \dim_{\mathbb{C}} \frac{\mathbb{C}[x,y]}{I} = n \; ; \; I \supset (x,y)^n \right\}$$

Eg. $n=2$. $\quad U_{\text{III}} \cap \pi^{-1}(2 \cdot [0]) = \left\{ I(\delta) \triangleleft \mathbb{C}[x,y] : I = \langle x^2, y - \delta x \rangle \right\}$ $\quad \alpha = 0 = \beta \quad \gamma = 0$

$$\cong \mathbb{C}$$

$$U_{\text{II}} \cap \pi^{-1}(2 \cdot [0]) = \left\{ J(\epsilon) \triangleleft \mathbb{C}[x,y] : J(\epsilon) = \langle y^2, x - \epsilon y \rangle \right\}.$$

If $\delta \neq 0$ $\quad I(\delta) = J(\delta^{-1})$ so $\pi^{-1}(2 \cdot [0]) = \mathbb{P}^1$.



Thm (Iarribino) $\pi^{-1}(n \cdot [0])$ is reduced, irreducible of dim $n-1$,

but singular when $n > 2$.

Analogue: $\quad \mathcal{N}$ nilpotent cone in $M_n(\mathbb{C})$

$$T^* \mathcal{B} \xrightarrow[\text{springer res.}]{} \mathcal{N} \quad , \quad \mathcal{B} \text{ flag manifold}.$$

For $n=2$, $X^{[n]} \to X^{(n)}$ is a springer resolution.

But in Lie theory fibre of 0 is flag manifold + important

In Hilbert scheme case, above Thm says it is different.

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$T = (\mathbb{C}^\times)^2 \curvearrowright \mathbb{C}^2 = X \Rightarrow T \curvearrowright \mathbb{C}[x,y], X^{[n]}, X^{(n)}$ + π is T-equiv.

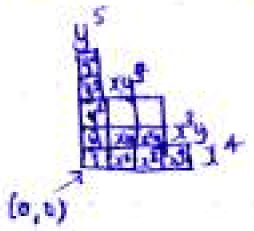
e.g. $(\sigma,\tau)\cdot x = \sigma x$, $(\sigma,\tau)\cdot y = \tau y$.

$I \in (X^{(n)})^T \Leftrightarrow (\sigma,\tau)\cdot I = I \quad \forall (\sigma,\tau) \in I$.

$\Leftrightarrow I$ is generated by monomials.

Monomial ideals of colength n $\overset{1-1}{\longleftrightarrow}$ partitions of n.

$\lambda = (4,3,3,1,1) \vdash 12$



$I_\lambda = \langle x^4, x^3y, xy^3, y^5 \rangle$

In fact $I_\lambda \in U_\lambda \underset{\text{open}}{\subset} X^{[n]}$

$U_\lambda = \left\{ I \triangleleft \mathbb{C}[x,y] : \{x^iy^j + I : (i,j) \in \lambda\} \text{ basis for } \frac{\mathbb{C}[x,y]}{I} \right\}$

$\bigcup_{\lambda \vdash n} U_\lambda = X^{[n]}$

Ex. $T_{I_\lambda} X^{[n]}$ – tangent space on I_λ, T acts.

What's the character?

$\exists$ tautological bundle $\mathcal{B} \to X^{[n]}$ where

$\mathcal{B}(I) = \dfrac{\mathbb{C}[x,y]}{I}$ rank n vector bundle.

$\mathcal{B}$ has $\mathbb{C}[x,y]$-action, T-action, bundle is

T-equivariant.

e.g. $n=2$, $\mathcal{B}(I_{\square\square}) = \dfrac{\mathbb{C}[x,y]}{\langle x^2, y\rangle}$ $T: (0,0) \oplus (1,0)$

$\mathcal{B}(I_{\boxminus}) = \dfrac{\mathbb{C}[x,y]}{\langle x, y^2\rangle}$ $T: (0,0) \oplus (0,1)$.

B. Combinatorics around the Hilbert scheme.

$\Lambda =$ symmetric functions over $\mathbb{Q}$.

Basis - monomial symmetric functions m_λ

- Schur polynomials s_λ.

Transition matrix K; $\lambda \vdash n$, $s_\lambda = \sum_{\mu \vdash n} K_{\lambda,\mu} m_\mu$.

$K_{\lambda\mu}$ = Kostka number

= no. of semistandard tableaux of shape λ, content μ.

$\Lambda_q = \Lambda \otimes \mathbb{Q}(q) \ni P_\lambda(q)$ Hall-Littlewood polynomials.

$P_\lambda(0) = s_\lambda$, $P_\lambda(1) = m_\lambda$. Transition matrix $K(q)$

$\lambda \vdash n$ $\quad s_\lambda = \sum_{\mu \vdash n} K_{\lambda\mu}(q) P_\lambda(q)$, $K_{\lambda\mu}(q)$ Kostka-Foulkes (polynomial)

$K_{\lambda\mu}(0) = \delta_{\lambda,\mu}$, $K_{\lambda\mu}(1) = K_{\lambda\mu}$.

Lascoux - Schutzenberger: add charge.

Geometric Repⁿ Thy: $\tilde{K}_{\lambda\mu}(q) = q^{n(\mu)} K_{\lambda\mu}(q^{-1})$

$n(\mu) = \sum (i-1)\mu_i$.

[$\mu \vdash n$ $\mathcal{B}$ flag manifold. $x_\mu \in M_n(\mathbb{C})$ Jordan shape μ.
$\{0 \subset F_1 \subset F_2 \subset \dots \subset F_{n-1} \subset \mathbb{C}^n : \dim F_i = i\}$

e.g. $\mu = (1,\dots,1) \rightsquigarrow x_\mu = 0$

$\mathcal{B} \supset \mathcal{B}_\mu = \{F \in \mathcal{B} : x_\mu F_i \subseteq F_{i-1} \ \forall i\}$

$H^*(\mathcal{B}_\mu, \mathbb{C}) \circlearrowleft S_n$ (springer) and $\sum_i [H^{2i}(\mathcal{B}_\mu, \mathbb{C}), S(\lambda)] q^i = \tilde{K}_{\lambda\mu}(q)$ ($S(\lambda)$: simple of S_n)]

$R_\mu(\underline{x}) := H^*(\mathcal{B}_\mu, \mathbb{C}) \twoheadleftarrow H^*(\mathcal{B}, \mathbb{C}) \cong \dfrac{\mathbb{C}[x_1,\dots,x_n]}{\langle \mathbb{C}[x_1,\dots,x_n]_+^{S_n} \rangle}$

↑ top degree $n(\mu)$ with repⁿ $S(\mu)$ appearing (Garnier polynomials) $V_\mu \subseteq \mathbb{C}[\underline{x}]$.

$$R_\mu(\underline{x}) = \frac{\mathbb{C}[\underline{x}]}{L_\mu}$$

Garson-Procesi: L_μ is largest homogeneous S_n-invariant ideal s.t. $V_\mu \cap \underline{L_\mu} = 0$

$\Lambda_{q,t} = \mathbb{Q}(q,t) \otimes \Lambda$ $\quad \tilde{M}_\lambda(q,t)$.

$\tilde{M}_\lambda(0,t) = \tilde{P}_\lambda(t)$

$$s_\lambda = \sum_{\lambda,\mu} \tilde{K}_{\lambda,\mu}(q,t)\, \tilde{M}_\mu(q,t)$$

(↖ Kostka - Macdonald polynomials.)

$$\tilde{K}_{\lambda\mu}(0,t) = \tilde{K}_{\lambda\mu}(t)$$

FACT : $\tilde{K}_{\lambda\mu}(1,1) = \dim_{\mathbb{C}} S(\lambda)$; $\tilde{K}_{\lambda\mu}(q,t) = \tilde{K}_{\lambda\mu'}(t,q)$

μ' is transpose of μ.

Qn: Is $K_{\lambda\mu}(q,t) \in \mathbb{N}[q,t]$?

IDEA : $R_\mu(\underline{x}) \otimes R_{\mu'}(y) \twoheadleftarrow \mathbb{C}[x_1,\ldots,x_n; y_1,\ldots,y_n]$.

Factor out $J_\mu \subseteq \mathbb{C}[\underline{x},\underline{y}]$, J_μ is largest homogeneous S_n-inv. ideal such that $J_\mu \cap \text{sign}_\mu = 0$.

$\text{sign}_\mu \overset{!}{\subset} V_\mu \otimes V_{\mu'}$

n! conjecture (Thm Haiman) : $\dfrac{\mathbb{C}[x,y]}{J_\mu}$ is a bigraded repn of S_n producing as its character the $\tilde{K}_{\lambda\mu}(q,t)$

i.e. $\displaystyle\sum_{i,j} \left[\left(\frac{\mathbb{C}[x,y]}{J_\mu}\right)_{i,j} : S(\lambda) \right] q^i t^j = \tilde{K}_{\lambda\mu}(q,t)$

Observations : Each $\dfrac{\mathbb{C}[x,y]}{J_\mu}$ has only 1 copy of trivial repn

∴ each should be a ~~copy~~ quotient of

$$\frac{\mathbb{C}[x,y]}{\langle \mathbb{C}[x,y]_+^{S_n} \rangle} = \mathbb{C}[x,y]^{\text{co}\, S_n}.$$

$(n+1)^{n-1}$ - conjecture (Thm Haiman)

$$\dim_{\mathbb{C}} \mathbb{C}[\underline{x}, \underline{y}]^{co\, S_n} = (n+1)^{n-1}.$$

Procesi's strategy:

$$\begin{array}{ccc} & & (\mathbb{C}^n)^2 = (\mathbb{C}^2)^n \\ & \nearrow & \downarrow \\ (\mathbb{C}^2)^{[n]} & \xrightarrow[\pi]{} & (\mathbb{C}^2)^n / S_n = (\mathbb{C}^2)^{(n)} \end{array}$$

$$\mathrm{Iso}(n) = \left((\mathbb{C}^2)^{[n]} \times_{(\mathbb{C}^2)^{(n)}} \mathbb{C}^{2n} \right)_{red.} \xrightarrow{p} (\mathbb{C}^2)^{[n]}$$

$$\mathbb{C}[\underline{x}, \underline{y}]^{co\, S_n} = \Gamma\left(p^*(\pi^{-1}(n\cdot[0])), \mathcal{O}_{\mathrm{Iso}(n)}\right)$$
$$= \Gamma\left(\pi^{-1}(n\cdot[0]), p_* \mathcal{O}\right)$$

Haiman's Thm: i) $\mathrm{Iso}(n)$ is Cohen-Macaulay.

$\Rightarrow \mathcal{P} = p_* \mathcal{O}$ is a vector bundle of rank $n!$ on $(\mathbb{C}^2)^{[n]}$

(since generically has rank $n!$)

ii) $\mathcal{P}(I_\lambda) = \dfrac{\mathbb{C}[x, y]}{J_\lambda}$ T-equivariant (looks like reg. rep^n of S_n)

Procesi bundle.

iii) $\dim \Gamma(\pi^{-1}(0), \mathcal{P}) = (n+1)^{n-1}$

$\rightsquigarrow$ generalise to $\mathfrak{h} \oplus \mathfrak{h}^* / G$.

Thm (Ginzburg - Kaledin)

A crepant resolution exists iff $G = S_n \ltimes \mu_m^n \curvearrowright \mathbb{C}^{2}$.

RCA of S_n

$c \in \mathbb{C}$. Def^n: A_c subalgebra of $\mathrm{End}(\mathbb{C}[x_1, \ldots, x_n])$

gen. by $x_1, \ldots, x_n$, $\sigma \in S_n$, $T_1, \ldots, T_n$

$$T_i = \frac{\partial}{\partial x_i} + c \sum_{j \neq i} \frac{1}{x_i - x_j} (1 - (ij))$$

Def^n: A_c is quotient of $\mathbb{C}\langle x_1, \ldots, x_n; y_1, \ldots, y_n \rangle \rtimes S_n$

$[x_i, x_j] = 0 = [y_i, y_j]$

$[y_i, x_i] = 1 + c \sum_{j \neq i} (ij)$, $[y_i, x_j] = -c(ij)$, $i \neq j$.

There is a filtration Λ on A_c, $\deg_\Lambda x_i = \deg_\Lambda \sigma = 0$

$\deg_\Lambda y_i = 1$. PBW theorem: $\mathrm{gr}\, A_c \cong \mathbb{C}[x, y] \rtimes S_n$.

$U_c := e A_c e$.

Def^n: R ring, ${}_R M$. p.dim = proj dim^n. $\rightsquigarrow$

gl. dim R := sup $\{$p.dim M : $M\}$

e.g. gl.dim $\mathbb{C}S_n = 0$, gl. dim $\mathbb{C}[x_1, \ldots, x_n] = n$.

$\Rightarrow$ gl. dim $\mathbb{C}[x, y] \rtimes S_n = 2n$

Thm (Serre). X affine variety, $R = \mathcal{O}(X)$

gl. dim $R = n \Leftrightarrow X$ smooth dim^n n.

Say R "smooth" if $\mathrm{gl.dim}\, R = n$.

Lemma: $\mathrm{gl\,dim}\, A_c \leq 2n$.

Proof: Check on f.gen. ${}_{A_c}M$ generators $m_1, \dots, m_t$.

Filter M by $\Lambda^i M = \Lambda^i A_c m_1 + \dots + \Lambda^i A_c m_t$.

$\mathrm{gr}_\Lambda M \rightsquigarrow$

$$0 \to K_{2n} \to F_{2n-1} \to \dots \to F_1 \to F_0 \to \mathrm{gr}_\Lambda M \to 0$$

F_i graded free, K_{2n} projective.

F_0 has generators $e_1, \dots, e_t$, $\hat{F}_0 = A_c \hat{e}_1 + \dots + \hat{A}_c \hat{e}_t$.

$\mathrm{gr}_\Lambda \hat{F}_0 = F_0$ etc. but can lift K_{2n}!

So $0 \to \hat{K}_{2n} \to \hat{F}_{2n-1} \to \dots \to \hat{F}_0 \to 0$. □

Qn: $\mathrm{gl\,dim}\, A_c = 2n - \min\{\dim L_c(\lambda) : \lambda \in \mathrm{Irr}\, S_n\}$?
(" = GK dim^n)

Proof shows $\mathrm{gl.\,dim}\, U_c \leq \infty$.

Thm (G-Stafford) $c \geq 0$, $c \notin \frac{1}{2} + \mathbb{Z}$. Then

$$H_c\text{-mod} \cong U_c\text{-mod}.$$

Proof: $\mathcal{I} : A_c\text{-mod} \to U_c\text{-mod}$, $\mathcal{I}' : U_c\text{-mod} \to A_c\text{-mod}$

$M \mapsto eM$. $\quad N \mapsto A_c e \otimes_{U_c} N$

Obviously $\mathcal{I}\mathcal{I}'(N) = eA_c e \otimes_{U_c} N \cong N$

$I'I(M) = A_c e \otimes_{eA_ce} eA_c \otimes_{A_c} M \rightsquigarrow$ need $A_c e A_c = A_c$

Thm (Ginzburg's Duflo Thm) Any two-sided proper ideal of A_c annihilates some $L_c(\lambda)$.

$A_c e A_c \subsetneq A_c$ then $eL_c(\lambda)=0$ some $\lambda \in Irr(S_n)$

$[L_c(\lambda) : triv]_{S_n} = 0$ for some λ.

$R \hookrightarrow \Delta_c(\lambda) \twoheadrightarrow L_c(\lambda)$

R is filtered by simples $L_c(\mu)$, $\mu < \lambda$ in the ordering $z(c,-)$. The trivial rep^n appears in $\mathbb{C}[x] \otimes \lambda$ in the ~~big~~ lowest degree copy of $\lambda \otimes \lambda \subseteq \mathbb{C}[x] \otimes \lambda$. Weight $n(\lambda) + c(n(\lambda) - n(\lambda^t))$

↑ $n(\lambda)$ degree $\quad (n(\lambda) = \sum (i-1)\lambda_i)$.

So trivial rep^n appears for the first time in $\Delta_c(\lambda)$ in weight space of weight $n(\lambda) + c(n(\lambda) - n(\lambda^t))$.

Other copies of triv in $L_c(\mu)$ appear first in $n(\mu) + c(n(\mu) - n(\mu^t))$. Refine partial order on $\mathcal{O}_c$:

$\mu \lneq \lambda$ dominance order on partitions:

(FACT: $\text{Hom}(\Delta_c(\mu), \Delta_c(\lambda)) \overset{KZ}{\hookrightarrow} \text{Hom}_{\mathcal{H}_q}(Sp_c(\mu), Sp_c(\lambda))$

$\underset{\substack{\text{Dipper-James} \\ q\neq -1}}{\cong} \quad \mathrm{Hom}_{S_q(n,n)}(W_q(\lambda), W_q(\mu)).$

It's quick to check that

$$n(\lambda) + c(n(\lambda) - n(\lambda^t)) < n(\mu) + c(n(\mu) - n(\mu^t))$$

$\forall\, \mu < \lambda$ so $\exists$ a copy of triv in $L_c(\lambda)$ for all λ. $\square$

Cor. $gl.\dim U_c = gl.\dim A_c \leq 2n \quad (c \geq 0,\ c \notin \frac{1}{2} + \mathbb{Z})$

$$U_c\text{-mod} \underset{gr}{\rightsquigarrow} \mathbb{C}[\underline{x}, \underline{y}]^{S_n}\text{-mod} = \mathcal{O}_{(\mathbb{C}^2)^{(n)}}\text{-mod}$$

$$\uparrow$$

$$\rightsquigarrow \mathcal{O}_{(\mathbb{C}^2)^{[n]}}\text{-mod}$$

Want analogy of BGG:

$$\begin{array}{ccc} \mathcal{D}_{\mathcal{B}} & \xrightarrow{gr} & T^*\mathcal{B} \\ \wr & & \downarrow \text{Springer} \\ \mathcal{U}(\mathfrak{g})_\lambda & \xrightarrow[gr]{} & \mathcal{N} \end{array}$$

Thm (Haiman) $(\mathbb{C}^2)^{[n]} = \mathrm{Proj} \underbrace{\bigoplus_{m\geq 0} R_m}_{= R}$

where $R_m = H^0((\mathbb{C}^2)^{[n]}, \mathcal{L}^{\otimes m}) \qquad \mathcal{L} = \Lambda^n \mathcal{B}$

$= (\mathbb{C}[\underline{x}, \underline{y}]^{\mathrm{sign}})^m$, $m = 0$ take fixed ring.

Thm (Serre) $\mathrm{Coh}(\mathrm{Proj}\, R) \cong R\ gr\ mod / tors$.

(where torsion means $M = \bigoplus M_m$, $M_m = 0 \quad m \gg 0$)

This generalises to noncommutative setting.

We need analogues of $H^0((\mathbb{C}^2)^{[n]}, \mathcal{L}^{\otimes m})$.

$m=0$ $\quad \mathbb{C}[\underline{x},\underline{y}]^{S_n} \longleftrightarrow U_c$

$m=1$ $\quad \mathbb{C}[\underline{x},\underline{y}]^{\text{sign}}$
$\quad\quad = e(\mathbb{C}[\underline{x},\underline{y}]\rtimes S_n)e_-$ where e_- = sign idempotent

• Identify $U_c, A_c \hookrightarrow \mathcal{D}(\mathbb{C}^n_{reg})\rtimes S_n$ with image.

Thm (Heckman-Opdam, BEG)

$$U_c := eA_ce = e\delta^{-1}A_{c+1}\delta e \quad , \quad \delta = \prod_{i<j}(x_i - x_j)$$

$$P_c^{c+1} := {}_{U_{c+1}}\big|eA_{c+1}\delta e\big|_{U_c}.$$

$$S = P_c^{c+1} \otimes_{U_c} - : U_c\text{-mod} \to U_{c+1}\text{-mod}.$$

Heckman-Opdam shift functor.

NB $\quad \mathrm{gr}_{\mathcal{A}}\, P_c^{c+1} \cong \mathbb{C}[\underline{x},\underline{y}]^{\text{sign}}$.

order filtration $P_c^{c+1} \subset \mathcal{D}(\mathbb{C}^n_{reg})\rtimes S_n$.

$m>1$ $\quad P_c^{c+m} = P_{c+m-1}^{c+m} \otimes_{U_{c+m-1}} \cdots \otimes P_c^{c+1}$

If $c>0$, $c \notin \frac{1}{2}+\mathbb{Z}$ then $P_c^{c+m} : U_c\text{-mod} \to U_{c+m}\text{-mod}$

is an equivalence of categories. (Same proof as $A_c \overset{\text{mor}}{\sim} U_c$).

NC version of Haiman's ring R

$$P_c := \bigoplus_{m\geq 0} P_c^{c+m} \quad \text{not a ring!}$$

NCAG: $\mathbb{Z}$-algebras.

$$\hat{P}_c = \bigoplus_{i\geq j\geq 0} P_{c+j}^{c+i} = \begin{bmatrix} P_c^c & 0 & 0 & \\ P_c^{c+1} & P_{c+1}^{c+1} & 0 & \cdots \\ P_c^{c+2} & P_{c+1}^{c+2} & & \\ & \vdots & & \end{bmatrix}$$

$$\mathrm{Coh}\,\hat{P}_c = \hat{P}_c\text{-grmod}/\text{tors}.$$

$\hat{P}_c$-grmod $\ni M = \bigoplus_{j\geq 0} M_j \quad P_{c+j}^{c+i} \cdot M_j \subseteq M_i$.

Lemma: U_c-mod $\longrightarrow \mathrm{Coh}\,\hat{P}_c$ is an equivalence.

$$M \longmapsto \bigoplus_{j\geq 0} P_c^{c+j} \otimes_{U_c} M.$$

Proof: $\Phi : \bigoplus_{j\geq 0} M_j \longmapsto (P_c^{c+r})^* \otimes_{U_{c+r}} M_r \quad r \gg 0.$

(inverse shift functor.) □

<u>Thm</u>: $\exists$ a filtration on each P_{c+j}^{c+i} inherited from $D(\mathbb{C}^n_{reg}) \rtimes S_n$ we can form $\mathrm{gr}\,P_{c+j}^{c+i}$, $\mathrm{gr}\,\hat{P}_c$.

$\mathrm{gr}\,\hat{P}_c \cong \hat{R}$, where $\hat{R} = \bigoplus_{i\geq j\geq 0} R_{ij}$ where $R_{ij} = R_{i-j}$.

Proof: Projectivity of P_{c+j}^{c+i} & $\Rightarrow P_{c+j}^{c+i} := P_{c+i-1}^{c+i} \otimes \cdots \otimes P_{c+j}^{c+j+1}$

$= P_{c+i-1}^{c+i} \cdots P_{c+j}^{c+j+1} \subseteq D(\mathbb{C}^n_{reg}) \rtimes S_n.$

$$\mathrm{gr}(P_{c+i-1}^{c+i})\,\mathrm{gr}(P_{c+i-2}^{c+i-1}) \cdots \mathrm{gr}(P_{c+j}^{c+j+1}) \subseteq \mathrm{gr}(P_{c+i-1}^{c+i} \cdots P_{c+j}^{c+j+1})$$

$$\overset{\shortparallel}{(\mathbb{C}[x,y]^{\mathrm{sign}})(\mathbb{C}[x,y]^{\mathrm{sign}}) \cdots (\mathbb{C}[x,y]^{\mathrm{sign}})}$$

Then show equality! □

$$\mathrm{Coh}\,\hat{R} = \hat{R}\text{-grmod}/\text{tors} \overset{\text{tault}}{=} R\text{-grmod}/\text{tors}$$

$$\underset{\text{serre}}{\simeq} \mathrm{Coh}(\mathbb{C}^2)^{[n]}$$

So $\Psi(U_c\text{-mod}, \text{filt}) \longmapsto (\mathrm{Coh}\,\hat{P}_c, \text{filt}) \overset{gr}{\longmapsto} \mathrm{Coh}(\mathbb{C}^2)^{[n]}$

This depends on choice of filtration.

$\Psi(U_c, \Lambda) = \mathcal{O}_{(\mathbb{C}^2)^{[n]}}$, $\Psi(eA_c, \Lambda) = \mathcal{P}$ Procesi bundle.

$\Psi(e\Delta_c(\lambda), \mathcal{F}) = (\mathcal{P}/\mathcal{P}K)_\lambda$ $\quad K = \langle y_1, \dots, y_n\rangle \mathbb{C}[x,y] \triangleleft \mathbb{C}[\underline{x}, \underline{y}]$

($\mathcal{F}$: degree filtration. λ: isotypic component of λ.)

$L_c(\text{triv}) = \mathbb{C}$ for $c = 1/n$.

$\rightsquigarrow$ filt^n $\mathcal{F}$ on $L_{k+1/n}(\text{triv})$

$\Psi_{k+1/n}(L_{k+1/n}(\text{triv})) = \mathcal{O}_{\pi^{-1}(0)} \otimes \mathcal{L}^k$.

Qn: What's $\Psi_{r/n}(L_{r/n}(\text{triv}))$ for other r with $(r,n) = 1$?.

To remove dependence on filtration

$\Psi_c(M, \mathcal{F}) = \tilde{M}_{\mathcal{F}} \circlearrowleft R$, $\quad I = \sqrt{\mathrm{ann}_R(\tilde{M}_{\mathcal{F}})}$

$V(I) \subseteq (\mathbb{C}^2)^{[n]}$. Count with multiplicity

$\underline{\mathrm{Ch}}\, M = \sum_{\mathcal{Z}} \mathrm{mult}_{\mathcal{Z}} \tilde{M}\,[\mathcal{Z}]$ characteristic cycle.

This is independent of the choice of filtration.

$M \in \mathcal{O}_c$, $\Phi_c(M)$ supported on $Z = \pi^{-1}(\mathbb{C}^n \times \{0\}/S_n)$.
$\subseteq (\mathbb{C}^2)^{[n]}$.

(because of local nilpotence).

Grojnowski - Nakajima: There are strata in $\mathbb{C}^n \times \{0\}/S_n$

$\lambda = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_t$. $\quad G_\lambda = \{ \sum \lambda_i [p_i] : p_i \neq p_j \}$.

$\dim G_\lambda = t$. $\quad \pi_{\lambda_i} : (\mathbb{C}^2)^{[\lambda_i]} \longrightarrow (\mathbb{C}^2)^{(\lambda_i)}$

Iarribino $\longrightarrow$ $\dim \pi_{\lambda_i}^{-1}(\lambda_i[p_i]) = \lambda_i - 1$ and irreducible.

- $\pi^{-1}(G_\lambda)$ is a fibration with fibres

$$\pi_{\lambda_1}^{-1}(\lambda_1[0]) \times \dots \times \pi_{\lambda_t}^{-1}(\lambda_t[0])$$

So $\dim G_\lambda = t + \sum(\lambda_i - 1) = \sum \lambda_i = n$.

$Z^n_\lambda = \overline{\pi^{-1}(G_\lambda)}$ irred of dim^n n, $Z^n = \bigcup_{\lambda \vdash n} Z^n_\lambda$.

Thm (Grojnowski; Nakajima)

$$\bigoplus_{n \geq 0} H^{BM}_{2n}(Z^n, \mathbb{C}) \xrightarrow{\sim} \Lambda \otimes_{\mathbb{Q}} \mathbb{C}, \quad \Lambda \text{ symmetric f}^{n}\text{s}.$$

$\circlearrowleft$ Heisenberg

(Fock space)

$$[Z_\lambda] \longmapsto m_\lambda.$$

$$M \in \mathcal{O}_c \longmapsto \underline{Ch}(M) \longmapsto H^{BM}_{2n}(Z^n, \mathbb{C}) \hookrightarrow \Lambda \otimes_{\mathbb{Q}} \mathbb{C}$$

$$K(\mathcal{O}_c) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow \Lambda \otimes_{\mathbb{Q}} \mathbb{C}$$

$\nearrow$ Grothendieck gp.

$$\bigoplus_{n\geq 0} K(\mathcal{O}_c(S_n)) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} \Lambda \otimes_{\mathbb{Q}} \mathbb{C}.$$

Propn: $\underline{Ch}\ e\Delta_c(\lambda) = \sum_{\mu} K_{\lambda\mu} [Z_\mu]$

So $[\Delta_c(\lambda)] \longmapsto \sum_{\mu} K_{\lambda\mu} m_\mu = s_\lambda$.

Thm (Rouquier) $\mathcal{O}(S_n) \simeq S_q(n,n)$-mod $q = e^{2\pi i c}$.

$S_q(n,n)$ decomposition matrix known by Varagnolo-Vasserot.

described by ~~an act~~ canonical basis for action of

$U_v(\hat{gl}_e)$ on $\Lambda \otimes_{\mathbb{Q}} \mathbb{C}_v$ where e = order of q.

$[L_c(\lambda)] \longmapsto$ lower canonical basis (at $v=1$) of Leclerc-Thibon.

Qn: Find raising & lowering operators on $\mathcal{O}_c(S_n)$ which will produce the canonical basis.

References for $(\mathbb{C}^2)^{[n]}$:

- math.berkeley.edu/~mhaiman
  (·Combinatorics, symmetric functions & Hilbert schemes
  · t,q Catalan numbers & Hilbert schemes)

- www.maths.ed.ac.uk/~igordon.

$t=0$: jt. w/M. Martino. G Complex ref'n gp.

$$[y_i, x_j] = \sum_{s\in S} c(s) \frac{\langle \alpha_s, x_j\rangle \langle y_i, \alpha_s^\vee\rangle}{\langle \alpha_s, \alpha_s^\vee\rangle} s$$

$A_{0,c}$

$c=0$: $A_{1,0} = D(\mathfrak{h}) \rtimes G$, $Z(A_{1,0}) = \mathbb{C}$

$A_{0,0} = \mathbb{C}[\mathfrak{h}\times\mathfrak{h}^*] \rtimes G$, $Z(A_{0,0}) = \mathbb{C}[\mathfrak{h}\times\mathfrak{h}^*]^G$.

Theorem (Etingof - Ginzburg)

$Z_c = Z(A_{0,c})$: i) $\operatorname{gr} Z_c \cong \mathbb{C}[\mathfrak{h}\times\mathfrak{h}^*]^G$.

ii) $A_{0,c}$ is a finite module over Z_c.

Lemma: Every simple $A_{0,c}$-module is finite dimensional over $\mathbb{C}$.

Proof: S simple needs one generator as $A_{0,c}$-module.

$\Rightarrow S$ is finitely gen. Z_c-module (ii).

Z_c is a Noetherian ring by (i) so S has a maximal Z_c-submodule, S'. S/S' irred. as Z_c-module so $\exists$ a maximal $m \triangleleft Z_c$ s.t $m(S/S') = 0$ i.e. $mS \subseteq S' \subsetneq S$.

But mS is an $A_{0,c}$-module so $mS = 0$ i.e.

S is a fin. gen. $Z_c / m Z_c$ -module. □
$\quad\quad\quad\quad\quad\quad \cong \mathbb{C}$.

Strategy: $\chi_c : \text{Irr}\, A_{0,c} \longrightarrow \text{Max}\, Z_c =: X_c$.

X_c variety attached to Z_c.

Thm (Etingof - Ginzburg)

i) All simple $A_{0,c}$-modules have dim^n $\leq |G|$.

ii) If $\dim S = |G|$ then $S \cong_G \mathbb{C}G$.

iii) $\chi_c^{-1}(\text{Sm}\, X_c) \xrightarrow{\sim} \text{Sm}\, X_c$

$\quad\quad \{S : \dim S = |G|\}$

Proof: $A_{0,c}[\delta^{-1}] \cong \mathbb{C}[\mathfrak{h}^{reg} \times \mathfrak{h}^*] \rtimes G$.

G now acts on $\mathfrak{h}^{reg} \times \mathfrak{h}^*$ freely and it's not hard to show that $\mathbb{C}[\mathfrak{h}^{reg} \times \mathfrak{h}^*] \rtimes G \cong \text{Mat}_{|G|}(\mathbb{C}[\mathfrak{h}^{reg} \times \mathfrak{h}^*]^G)$. □

Ex. $\mathbb{C}[\mathfrak{h}]^G, \mathbb{C}[\mathfrak{h}^*]^G \hookrightarrow Z_c$.

So $\mathbb{C}[\mathfrak{h}]^G \otimes \mathbb{C}[\mathfrak{h}^*]^G \subset Z_c \subset A_{0,c}$.

$$\Upsilon_c : \text{Max}\, Z_c = X_c \longrightarrow \mathfrak{h}/G \times \mathfrak{h}^*/G.$$

Defn: Reduced RCA: $p \in \mathfrak{h}/G \times \mathfrak{h}^*/G$

$A_{0,c}(p) := \dfrac{A_{0,c}}{m_p A_{0,c}}$. $p = (0,0)$, $A_{0,c}(0,0) = \bar{A}_{0,c}$ restricted R.C.A.

$A_{0,c}^{(p)} \cong \frac{\mathbb{C}[\mathfrak{h}]}{m_{p'}} \otimes \mathbb{C}G \otimes \frac{\mathbb{C}[\mathfrak{h}^*]}{m_{p''}}$ $\quad p = (p', p'')$

So $\dim A_{0,c}(p) = |G|^3$.

$A_{0,c}(p) = B_1 \oplus \dots \oplus B_t$, $Z_c/m_p Z_c \hookrightarrow A_{0,c}(p)$

Thm (Müller): # blocks in $A_{0,c}(p)$ = # blocks in $Z_c/m_p Z_c$

$$= |\Upsilon^{-1}(p)|.$$

Thm (Brown-Gordon) If all points in $\Upsilon^{-1}(p)$ are smooth then $A_{0,c}(p) \cong \mathrm{Mat}_{|G|}\left(\frac{Z_c}{m_p Z_c}\right)$.

Qn: How do we describe blocks of $A_{0,c}(p)$? $\Upsilon_c^{-1}(p)$?

$p = (0,0)$: concentrate on $\overline{A}_{0,c}$.

$A_{0,c}$ graded with $\deg x = 1 = -\deg y$, $\deg \sigma = 0$.

$\Upsilon_c^{-1}((0,0))$, equiv. $\overline{A}_{0,c}$ inherits grading / $\mathbb{C}^*$-action.

$\lambda \in \mathrm{Irr}\, G$ $\quad \Delta_c(\lambda) = A_{0,c} \otimes_{\mathbb{C}[\mathfrak{h}^*] \rtimes G} \lambda$

$$\overline{\Delta}_c(\lambda) = A_{0,c} \otimes_{\mathbb{C}[\mathfrak{h}]^G \otimes \mathbb{C}[\mathfrak{h}^*] \rtimes G} \lambda$$

$(f_1 \otimes f_2 \times g) \cdot v = f_1(0) f_2(0) g \cdot v$, $\dim \overline{\Delta}_c(\lambda) = |G| \dim \lambda$

$\overline{\Delta}_c(\lambda) \overset{!}{\twoheadrightarrow} \overline{L}_c(\lambda)$ simple $\therefore$ indecomposable, so belongs to a block.

$$\operatorname{Irr} G \xrightarrow{\Theta_c} \Upsilon_c^{-1}((0,0))$$

Produces a partition of irreducibles of G. $\lambda \underset{c}{\sim} \mu \Leftrightarrow \Theta_c(\lambda) = \Theta_c(\mu)$

Conjecture (Gordon – Martino): Associated to c, $\exists$ another partition of irreducibles of G into families (or 2-sided cells).

i) These should be the same.

ii) $\dim \left(\Upsilon_c^{*}((0,0)) \right)_{\mathcal{C}} = |\mathcal{C}|$.

"Proof"

$$\begin{array}{ccc}
A_{1,c} & \overset{?}{\rightsquigarrow} & A_{0,c} \\
\cup & & \cup \\
\mathcal{O}_c & & \text{standard } \bar{A}_{0,c} \\
\downarrow \text{KZ} & & ?\downarrow ? \\
\mathcal{H}_q(G) & \longrightarrow & J_c
\end{array} \quad \square.$$

Evidence: 1) $G \neq G(m,1,n) = S_n \ltimes \mu_m \curvearrowright \mathbb{C}^n$.

i) X_c is never ever smooth.

ii) $G = G(m,1,n)$ $\qquad \operatorname{Irr} G = m$-multipartitions of n.

$X_c \cong \mathcal{X}_c$ – a quiver variety.

HK rotation $\wr$ classify $\mathbb{C}^\times$-fixed pts here.

$\mathcal{M}_c \underset{\text{Nak refl'n}}{\sim} ((\mathbb{C}^2)^N_c)^{\mu_m} \longrightarrow$ partitions of N with m-core depending on c.

$\exists$ conj. of Bonnafé - Geck - Iancu - Lam in type B to describe 2-sided cells. Agree here!