TORSORS AND REPRESENTATION THEORY

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where $\mathfrak{g}_i$ are simple (no proper non-trivial ideals), $\mathfrak{z}$ is abelian.

By T we denote a maximal torus in G. All maximal tori in G are conjugate. The adjoint representations $Ad : G \to GL(\mathfrak{g})$ has a weight decomposition:

$$\mathfrak{g} = \mathfrak{t} \oplus \oplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

here $\mathfrak{t}$ is the Lie algebra of T and it coincides with $\mathfrak{g}_0$. The finite set $R \subset \hat{T} \setminus \{0\}$ is called the root system of $\mathfrak{g}$ and G.

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Important facts:

(1) $\dim \mathfrak{g}_\alpha = 1$;

(2) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$;

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Example 1.2. *Let $G = SL(2)$. Then*

$$T = \{ \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) | a \in k^* \}$$

is the subgroup of diagonal matrices. If $\omega \in \hat{T}$ is a generator, then

$$R = \{\alpha, -\alpha\}$$

with $\alpha = 2\omega$,

$$\mathfrak{g}_\alpha = \{ \left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right) | b \in k \}, \ \mathfrak{g}_{-\alpha} = \{ \left(\begin{smallmatrix} 0 & 0 \\ b & 0 \end{smallmatrix}\right) | b \in k \}.$$

Let $E = \hat{T} \otimes_{\mathbb{Z}} \mathbb{R}$. Consider a hyperplane in E which does not contain any root $\alpha \in R$. It divides E in two half-spaces and thus the set of roots R is divided accordingly: $R = R^+ \cup R^-$.

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Proposition 1.3. *There exists a unique linearly independent set $\alpha_1, \dots, \alpha_r$ such that every $\alpha \in R^+$ can be written uniquely as*

$$\alpha = \sum_{i=1}^{n} m_i \alpha_i$$

with $m_i \in \mathbb{Z}_{\geq 0}$.

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The Weyl group W can be defined as the normalizer $N_G(T)$ of T quotient by T. The action of W on T induces the action of W on $\hat{T}$. If $w \in W$ and $\alpha \in R$, then

$$\mathrm{Ad}_w(g_\alpha) = \mathfrak{g}_{w(\alpha)}.$$

Hence W permutes the roots.

Example 1.4. *Let $G = GL(n)$, T be the subgroup of diagonal matrices. Let $\varepsilon_1, \dots, \varepsilon_n$ be the natural basis in $\hat{T}$. Then*

$$R = \{\varepsilon_i - \varepsilon_j | 1 \leq i, j \leq n\}.$$

One can choose $\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n$ for a base. The corresponding Borel subgroup is the subgroup of upper triangular matrices. The Weyl group W is isomorphic to the symmetric group S_n.

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The Killing form. Let us assume now that $\mathfrak{g}$ is semisimple ($\mathfrak{z} = 0$). Then the bilinear symmetric form

$$(x, y) = \mathrm{tr}(\mathrm{ad}_x \, \mathrm{ad}_y)$$

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Theorem 1.5. *(1) The Weyl group W is generated by the simple reflections $s_1, \ldots, s_r$*

$$s_i(\xi) = \xi - 2\frac{(\xi, \alpha_i)}{(\alpha_i, \alpha_i)}\alpha_i,$$

where $\xi \in \hat{T}$; (2) W acts simply transitively on the set of bases.

Coroots To each simple root α_i we can choose $x_i \in \mathfrak{g}_{\alpha_i}, h_i \in \mathfrak{t}$ and $y_i \in \mathfrak{g}_{-\alpha_i}$ satisfying the relation $[x_i, y_i] = h_i$ and $\alpha_i(h_i) = 2$. They form an $sl(2)$-subalgebra. The following identity is very important:

$$\xi(h_i) = 2\frac{(\xi, \alpha_i)}{(\alpha_i, \alpha_i)}$$

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Dynkin diagram. The nodes correspond to simple roots. The number of edges joining i and j equals max $\{\alpha_i(h_j), \alpha_j(h_i)\}$. The arrow goes from the longest of the two roots to the shortest one.

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Connected Dynkin diagram are in bijection with simple Lie algebras and with simply connected simple algebraic groups. They are

classical: A_r $(sl(r+1))$, B_r $(so(2r+1))$, C_r $(sp(2r))$, D_r $(so(2r))$;

exceptional: G_2, F_4, E_6, E_7, E_8.

Representations.

Theorem 1.7. *(Lie) If V is a representation of a connected solvable group B, then there exists a full flag $0 \subset V_1 \subset \cdots \subset V_m = V$ invariant under the action of B.*

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Now let V be an irreducible representation of a reductive connected algebraic group G. By Lie's theorem V contains a B-invariant one-dimensional subspace kv. Let λ be the weight of this subspace with respect to the torus T. By irreducibility of V and the fact $\mathfrak{g}_\alpha v = 0$ for all $\alpha \in R^+$ every weight vector $w \in V_\mu$ can be obtained from v by application of $\mathfrak{g}_\alpha$ with $\alpha \in R^-$. That implies uniqueness of v (up to propotionality). So v is called a *highest vector* of V and λ is called the *highest weight* of V.

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Define

$$\hat{T}^+ = \{\lambda \in \hat{T} | \lambda(h_i) = 2\frac{(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \in \mathbb{Z}_{\geq 0}, i = 1, \ldots, r\}.$$

Weights from $\hat{T}^+$ are called *dominant*.

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Theorem 1.8. *For any dominant λ there exists a unique up to isomorphism irreducible representation V_λ with highest weight λ. Every finite-dimensional irreducible representation of G-module is isomorphic to V_λ for some $\lambda \in \hat{T}^+$.*

Example 1.9. *Let* $G = SL(2)$. *Then* $\hat{T}^+ = \{m\omega | m = 0, 1, 2 \dots\}$. *One can check that* $V_{m\omega} \simeq S^m(V)$, *where* V *is the natural 2-dimensional representation of* $SL(2)$.

It is in fact a key example. Indeed, the condition of dominance for a highest weight follows immediately from that for each of simple roots $sl(2)$*-triples.*

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Example 1.10. *If* $G = SL(r + 1)$, *the highest weight of the natural representation* V *is* ω_1, *the highest weight of the exterior power* $\Lambda^p(V)$ *is* ω_p *and the highest weight of the adjoint representation is* $\omega_1 + \omega_r$.

Exercise 2. Let $G = G_1 \times G_2$ be a direct product of two reductive groups. Then any irreducible representation V is isomorphic to a tensor product $V_1 \otimes V_2$ of the irreducible representations of G_1 and G_2.

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Exercise 3. An irreducible representation of a semisimple group G is called minuscule if all weights of V form one orbit under the action of W. Prove that the highest weight of a minuscule representation is fundamental and that the number of minuscule representations equals the order of $\hat{T}/Q$. Find all minuscule representations for all simple G.

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One can check that K coincides with the stabilizer of I. Let $f_1, \dots, f_n$ be generators of I and $W \subset k[G]$ be the minimal G-invariant subspace containing $f_1, \dots, f_n$. Then K coincides with the stabilizer of $W \cap I$. Set $V = \Lambda^d(W)$, $l = \Lambda^d(W \cap I)$ where $d = \dim(W \cap I)$. □

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A parabolic subgroup containing a fixed B as in (1) of Theorem **??** is called *standard*. All standard parabolic subgroups are in bijection with subsets $I \subset \{1, \dots, r\}$ of nodes of the Dynkin diagram. If $\omega_I = \sum_{i \in I} \omega_i$, then the corresponding standard parabolic P is the stabilizer of a highest vector in $\mathbb{P}(V_{\omega_I})$.

Now we are going to describe the structure of $\mathfrak{p}$. Let $h \in \mathfrak{t}^*$ be such that $\alpha_i(h) = 0$ if $i \notin I$ and $\alpha_i(h) = 1$ if $i \in I$. Define a $\mathbb{Z}$-grading

$$\mathfrak{g} = \bigoplus \mathfrak{g}_i$$

by setting

$$\mathfrak{g}_p = \bigoplus_{\alpha(h)=p} \mathfrak{g}_\alpha.$$

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Theorem 2.4. *Let V_λ be an irreducible representation of G and $G/P \subset \mathbb{P}(V_\lambda)$ be the orbit of a highest vector. Let $S^2(V_\lambda) = V_{2\lambda} \oplus N$. Let $\sigma : V_\lambda \to S^2(V_\lambda)$ be defined by $\sigma(v) = v \otimes v$ and $\psi : S^2(V_\lambda) \to N$ be the G-invariant projection. Then the affine cone $(G/P))_a$ is defined by the equations*

$$\psi\sigma(v) = 0.$$

Example 2.5. *Let* $G = SL(n)$, $I = \{p\}$. *Then* $G/P = Gr(n,p)$, $V_\lambda = \Lambda^p(V)$ *and the embedding* $G/P \to \mathbb{P}(\Lambda^p(V))$ *is the Plucker embedding.*

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We are especially interested in the case when P is a maximal parabolic, which is equivalent to the condition $I = \{i\}$. In this case we have the following properties

(1) $\mathbf{Pic} G/P$ has rank 1;

(2) $\mathfrak{g}_{-1}$ is an irreducible representation of G_0 with highest weight $-\alpha_i$;

(3) If $\mathfrak{g}$ is simple and θ is the highest root of $\mathfrak{g}$, then the depth of the $\mathbb{Z}$-grading equals the coefficient m_i in the decomposition $\theta = \sum_{j=1}^r m_j \alpha_j$;

(4) The subgroup G_+ is abelian iff $m_i = 1$.

3. GIT quotients and universal torsors

Let X be a projective variety and a reductive group T act on X. By linearization we understand a T-equivariant embedding $X \to \mathbb{P}(V)$ for some representation V of T. Recall that a point $x = kv \in X$ is called *stable* if the stabilizer of v in T is finite and the orbit Tv is closed in V. A point is *semistable* if the closure if Tv does not contain 0. Let X^s and X^{ss} denote the set of stable and semistable points in X respectively.

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The GIT quotient $X//T := X^{ss}//T = \operatorname{Proj} k[X_a]^T$ where $k[X_a]^T$ is the ring of T-invariant functions on the affine cone X_a over X. If the set of stable points in non-empty, X^s/T is an open subset in $X//T$.

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Assume that T is a torus. Then V has a weight decomposition

$$V = \oplus V_\mu,$$

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By Hilbert-Mumford criterion a point $x = kv \in \mathbb{P}(V)$ is semistable if the convex hull of $\mathrm{wt}(v)$ contains 0 and stable if 0 is an interior point of the convex hull of $\mathrm{wt}(v)$ in $E = \hat{T} \otimes_{\mathbb{Z}} \mathbb{R}$.

Example 3.1. *Let $G = SL(2) \times SL(n)$, $M = V_{\omega_1} \otimes V_{\omega_2}$ be the irreducible representation of G, and $T = T' \times T''$ is a maximal torus in G. Let $X = \mathbb{P}(M)$. One can identify M with the space of $2 \times n$ matrices (m_{ij}) with action of G given by multiplication on the right by elements of $SL(n)$ and on the left by elements of $SL(2)$. Let M° be the subset of matrices which have at most one zero entry. It is clear that $M^\circ \subset M^{ss}$ and the complement has codimension 2.*

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Define an open subset $U \subset X$ such that

$$U_a = \{m = (m_{ij}) \in M^s | m_{1j} \neq 0\}.$$

By application of T'' we can move any $m \in U$ to the subset

$$U'_a = \{m_{11} = \cdots = m_{1n}\}.$$

Thus,

$$U/T = U'/T' \simeq \mathbb{P}^{n-1} \setminus Z,$$

where codimension of Z is at least 2.

Let $D_i \subset X$ be defined by

$$(D_i)_a = \{m \in M^s | m_{1i} = 0\},$$

and $D_i' \subset D_i$ be given by the condition $m_{21} = \dots = m_{2n}$. Then $D_i/T = D_i'/T'$ is isomorphic to $\mathbb{P}^{n-2}$ with some set of codimension 2 removed.

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Then it is not difficult to see that X°/T *is* $\mathbb{P}^{n-1}$ *blown up at* n *points,* $D_1, \ldots, D_n$ *are exceptional divisors corresponding to these points and with some set of codimension* 2 *removed. Since* $X//T$ *is projective, we can conclude that it is isomorphic to* P^{n-1} *blown up at* n *points. The total number of exceptional divisors is* $2n$, *they are given by* T*-invariant hyperplane sections* $m_{ij} = 0$.

Universal torsor. Given a torus T. Recall that an X-*torsor* is a T variety $\mathcal{T}$ with an action of T and a morphism $f : \mathcal{T} \to X$ such that locally in étale topology $\mathcal{T}$ is equivariantly isomorphic to $T \times X$.

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Colliot-Thélène and Sansuc associated to a torsor $f : \mathcal{T} \to X$ the exact sequence

$$1 \to k[X]^*/k^* \to k[\mathcal{T}]^*/k^* \to \hat{T} \to \mathbf{Pic} X \to \mathbf{Pic} \mathcal{T} \to 0. \tag{1}$$

Here the second and the fifth arrows are induced by f. The forth arrow is called the *type* of $\mathcal{T} \to X$. To define it consider the natural pairing:

$$H^1(X, T) \times \hat{T} \to H^1(X, \mathbb{G}_m) = \mathbf{Pic} X,$$

where the cohomology groups are in étale topology. The type sends $\chi \in \hat{T}$ to $[\mathcal{T}] \cup \chi$, where $[\mathcal{T}] \in H^1(X, T)$ is the class of the torsor $\mathcal{T} \to X$. A torsor $\mathcal{T} \to X$ is called *universal* if its type is an isomorphism.

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If the variety X is projective, there is the following characterization of universal torsors: an X-torsor under is universal if and only if $\mathbf{Pic}\mathcal{T}=0$ and $k[\mathcal{T}]^*=k^*$, that is, $\mathcal{T}$ has no non-constant invertible regular functions.

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The Cox ring $\operatorname{Cox}X$ is isomorphic to $k[\mathcal{T}]$ if $\mathcal{T}\to X$ is a universal torsor.

Now let G be a simple connected simply algebraic group, P be a maximal parabolic. So $I = \{i\}$, and we will use the following notations: $\alpha = \alpha_i$, $\omega = \omega_i$, $V = V_{\omega_i}$ and v denotes a highest vector of V. We can assume without loss of generality that V is a faithful representation of G.

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Proposition 3.2. *Assume that the Dynkin diagram of G is simply laced and (G,ω) is not one of the following list:*

(G,ω_1), *where G is classical*, $(A_r,\omega_r), (A_3,\omega_2), (B_2,\omega_2), (C_2,\omega_2), (D_4,\omega_3), (D_4,\omega_4)$,

then the codimension of $(G/P)_a \setminus (G/P)_a^{sf}$ in $(G/P)_a$ is at least 2.

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The proof is based on the following arguments. Let $v \in V$. By use of Hilbert-Mumford criterion if $W\omega \backslash \mathrm{wt}(v)$ has at most two points, then v is stable. Since the intersection of two hyperplane section in $(G/P)_a$ has codimension 2, the set of unstable points has codimension 2.

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It is possible to show that if the stabilizer of v is non-trivial, then v belongs to the orbit of a highest vector for some simple subgroup $T \subset K \subset G$. All such K are known and one can check directly that $\dim K/(P\cap K) < \dim G/P - 1$.

Theorem 3.3. *Assume that G/P satisfies the conditions of Proposition* **??**. *Let $Y = (G/P)_a^{sf}/\mathbb{T}$ then $f : (G/P)_a^{sf} \to Y$ is a universal torsor.*

Proof. By construction $\mathbb{T}$ acts freely on $(G/P)_a^{sf}$. By GIT $f : (G/P)_a^{sf} \to Y$ is an affine morphism whose fibers are $\mathbb{T}$-orbits. Hence $f : (G/P)_a^{sf} \to Y$ is a torsor.

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The complement to $f : (G/P)_a^{sf}$ in $(G/P)_a$ has codimension at least 2. Since all invertible functions on $(G/P)_a$ are constant, and $\mathbf{Pic}(G/P)_a$ is trivial, the statement follows. □

4. The blow-up theorem

In this section we describe $G/P//T$ for certain G/P as a blow up of a projective space at some subvariety.

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Let us assume that the parabolic subgroup $P \subset G$ has abelian unipotent radical. Then the corresponding grading is of the form

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

and $\mathfrak{g}_0 = \mathfrak{g}' \oplus k$, where $\mathfrak{g}'$ is a semisimple Lie algebra whose Dynkin diagram is obtained from that of $\mathfrak{g}$ by removing the node α.

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Let us assume that the parabolic subgroup $P \subset G$ has abelian unipotent radical. Then the corresponding grading is of the form

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

and $\mathfrak{g}_0 = \mathfrak{g}' \oplus k$, where $\mathfrak{g}'$ is a semisimple Lie algebra whose Dynkin diagram is obtained from that of $\mathfrak{g}$ by removing the node α.

Here is the list of all such pairs (G, G'):

$$(A_r, A_i \times A_{r-i-1}), (B_r, B_{r-1}), (D_r, D_{r-1}), (D_r, A_{r-1}), (C_r, A_{r-1}), (E_6, D_5), (E_7, E_6).$$

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Define a grading on V

$$V = V_0 \oplus V_1 \oplus V_2 \oplus \cdots \oplus V_m,$$

by setting $V_0 = kv$, $V_{i+1} = \mathfrak{g}_{-1} V_i$. We have a G' equivariant isomorphism $\phi : \mathfrak{g}_{-1} \to V_1$ defined by $\phi(x) = xv$.

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Example 4.1. *For the pair* $(A_r, A_1 \times A_{r-2})$ *we have* $\omega = \omega_2$, $V = V_0 \oplus V_1 \oplus V_2$, $V_1 = k^2 \otimes k^{r-1}$, $V_2 = \Lambda^2(k^{r-1})$.

Define a one-parameter subgroup $D \subset \mathbb{T}$ of elements g_t which act on V_i as multiplication by t^{1-i}, $t \in k^*$.

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In our situation the open Schubert cell coincides with $G_{-1}v$. It is isomorphic to $\mathfrak{g}_{-1} \simeq V_1$ via the exponential map

$$\exp : \mathfrak{g}_{-1} \to G/P.$$

$$\exp(x) = v + xv + \frac{1}{2}x^2v + \cdots + \frac{1}{m!}x^m$$

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Example 4.2. *For the pair (E_7, E_6) we have $\omega = \omega_1$, $V = V_0 \oplus V_1 \oplus V_2 \oplus V_3$, $V_1 = V_{\omega_1}$ and $V_2 = V_{\omega_5}$ are fundamental representation of dimension 27. They are dual to each other. Finally, V_3 is the trivial one-dimensional representation of $G' = E_6$. The map $p_3 : V_1 \to V_3$ defines a G'-invariant cubic form on V_1. one can check that*

$$p_3(u) = \langle u, p_2(u) \rangle.$$

Lemma 4.3. *Let* $V' = V_1$, v' *be a highest vector with respect to* $G' \cap B$ *and* P' *be the stabilizer of* kv'. *Then*

$$(G'/P')_a = (G/P)_a \cap V_1 = p_2^{-1}(0).$$

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$$(G'/P')_a = (G/P)_a \cap V_1 = p_2^{-1}(0).$$

Proof. Let us prove the first equality. The tangent space to $u \in (G/P)_a$ is $ku + \mathfrak{g}u$. If $u \in (G'/P')_a \subset V_1$, then
$$T_{u,(G/P)_a} \cap V_1 = (ku + \mathfrak{g}u) \cap V_1 = ku + \mathfrak{g}'u = T_{u,(G'/P')_a}.$$
Hence $(G'/P')_a$ is an irreducible component of $(G/P)_a \cap V_1$. On the other hand, the closed set $(G/P)_a \cap V_1$ is a union of G'-orbits, but the closure of any non-zero orbit contains the unique closed orbit $(G'/P')_a$. Hence $(G'/P')_a = (G/P)_a \cap V_1$.

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If $p_2(u) = 0$, then obviously $p_n(u) = 0$ for all $n \geq 2$. Let $u = \phi(x)$ for some $x \in \mathfrak{g}_{-1}$. Then $\exp(x)v = v + u$ is in $(G/P)_a$. Hence $g_t \exp(x)v = tv + u$ is also in $(G/P)_a$ for any $t \in k^*$. But $(G/P)_a$ is a closed set, so that the limit point $u \in V_1$ is contained in it. By the first equality we see that u is actually in $(G'/P')_a$. On the other hand, $X_{-\alpha}v = v'$, $X_{-\alpha}^2 v = 0$, hence $p_2(v') = 0$. Since p_2 is G'-equivariant, p_2 vanishes on the orbit $G'(X_{-\alpha}v)$, and hence on $(G'/P')_a$. □

Theorem 4.4. *Let* $S = (G/P)_a \cap V_{>1}$ *and* $(G/P)_a^\circ = (G/P)_a \setminus S$. *Then*

$$\bar{\pi}_1 : (G/P)_a^\circ//D \to V_1 \setminus \{0\}$$

is well-defined. It is the inverse morphism of the blowing-up of $V_1\setminus\{0\}$ *at* $(G'/P')_a\setminus\{0\}$.

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Proof. If $u \notin (G'/P')_a$, the fiber $\pi_1^{-1}(u)$ lies in the open Schubert cell and the exponential map defines an isomorphism between V_1 and the big Schubert cell. Hence $\bar{\pi}_1$ is an isomorphism when restricted on $V_1 \setminus (G'/P')_a$.

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Now let us describe the fiber $\pi_1^{-1}(u)$ when $u \in (G'/P')_a$. It is not hard to see that $\pi_1^{-1}(u) = G_{-1}u$. Thus, since $\mathfrak{g}_{-1}$ is abelian,

$$\pi_1^{-1}(u) \simeq \pi_1^{-1}(u) \cap V_2 = \mathfrak{g}_{-1}u,$$

and

$$\bar{\pi}_1^{-1}(u) = \pi_1^{-1}(u)//D = \mathbb{P}(\mathfrak{g}_{-1}u).$$

We claim that for any $x \in \mathfrak{g}_{-1}$

$$xu = p_2(u, \phi(x)).$$

Indeed,

$$p_2(u, \phi(x)) = \frac{1}{2}(x\phi^{-1}(u)v + \phi^{-1}(u)xv) = x(\phi^{-1}(u)v) = xu,$$

since $[\phi^{-1}(u), x] = 0$ and $\phi^{-1}(u)v = u$. Note that $\mathbb{P}(p_2(u, V_1)) \simeq \mathbb{P}(N_u(G'/P')_a)$. That implies the statement. ∎

Corollary 4.5. *Assume that* $(G/P)_a^{sf} \subset (G/P)_a^\circ$. *Then* $(G/P)^{sf}/T$ *is isomorphic to* $\mathbb{P}(V')//T'$ *blown up at* $(G'/P')//T$.

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Apply this corollary to the case when $G = SL(n)$, $G' = SL(2) \times SL(n-2)$, $G/P = Gr(2,n)$. Then V' is as in Example **??**, and $(G'/P')_a$ is the locus of matrices of rank 1 which contains exactly one stable orbit. Thus,

Corollary 4.6. $Gr(2,n)//T$ *is isomorphic to* $\mathbb{P}^{n-3}$ *blown up at* $n-1$ *points. In particular,* $Gr(2,5)//T$ *is the Del Pezzo surface of degree 5.*

5. Batyrev's conjecture

Del Pezzo surfaces, classically defined as smooth surfaces of degree d in the projective space $\mathbb{P}^d$, $d \geq 3$, are among the most studied and best understood of algebraic varieties. Over an algebraically closed ground field such a surface is the quadric $\mathbb{P}^1 \times \mathbb{P}^1$ or the projective plane $\mathbb{P}^2$ with $r = 9 - d$ points in general position blown up; in this definition d can be any integer between 1 and 9.

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Despite the apparent simplicity the enumerative geometry of these surfaces displays amazing symmetries and puzzling coincidences. The 27 lines on a smooth cubic surface were discovered by Cayley and Salmon, and the symmetries of their configuration were explored by Schoutte, Coxeter and Du Val.

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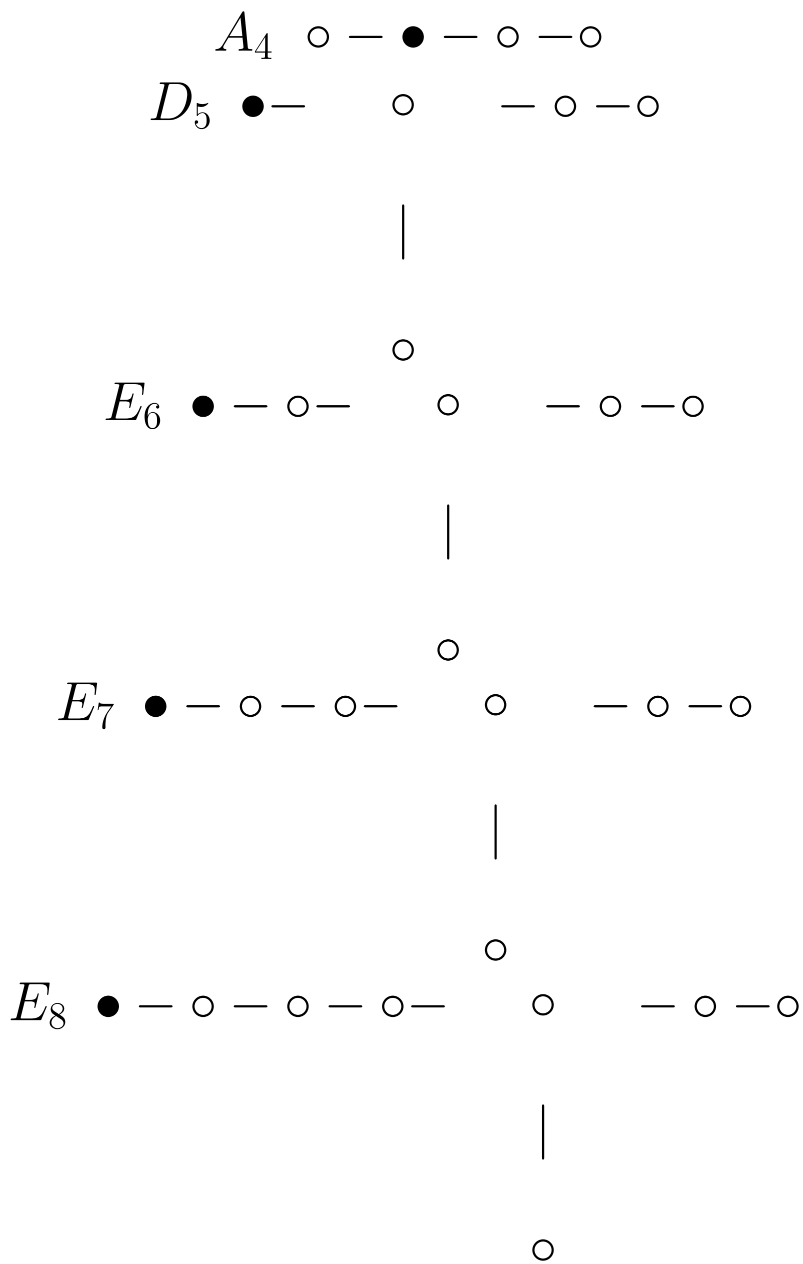
In the late 1960-s when Manin discovered that to a del Pezzo surface X of degree $d = 9 - r$, $d \leq 6$, one can attach a root system R_r in such a way that the automorphism group of the incidence graph of the exceptional curves on X is the Weyl group $W(R_r)$. The sequence R_r is

$$A_1 \times A_2 \subset A_4 \subset D_5 \subset E_6 \subset E_7 \subset E_8 \tag{2}$$


A_4
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The number of exceptional divisors on a Del Pezzo surface of degree $d > 1$ equals the dimension of the fundamental representation $V = V_\omega$ associated to a pair (R_r, R_{r-1}). These numbers are $6, 10, 16, 27, 56$. If $d = 1$, the number of exceptional divisors equals 240. That coincides with the number of roots of E_8. In this case $V = \mathfrak{g}$ is the adjoint representation and this is the only case when V is not minuscule.

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The following result was conjectured by Batyrev and proved by Popov and Derenthal for $d \geq 2$. It was reproved by Skorobogatov and myself by use of the blow up Theorem, and recently generalized to the case $d = 1$.

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Theorem 5.1. *Let X be a del Pezzo surface of degree $d \leq 5$, R_r be the corresponding root system, G be the corresponding simple algebraic group, P be the parabolic subgroup with semisimple part G' whose root system is R_{r-1}. Finally let $f : \mathcal{T} \to X$ be a universal X-torsor. There exists an equivariant embedding of $\mathcal{T}$ to the universal torsor $h : (G/P)_a^{sf} \to: Y = (G/P)_a^{sf}/\mathbb{T}$. The images under f of the T-invariant hyperplane sections of $\mathcal{T}$ corresponding to the weights $W\omega$ are the exceptional divisors of X.*

We will give a proof for $d > 1$ by induction on r with the base case $r = 4, d = 5$ already proven.

If $G' = SL(5)$, $G = \mathrm{Spin}(10)$ (a double cover of $SO(10)$), V is a 16-dimensional spinor representation of G. When restricted to G', V splits into direct sum $V = V_0 \oplus V_1 \oplus V_2$, where V_1 and V_2 are the second and the forth exterior power of the natural representation of $SL(5)$.

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The case $E_6 \subset E_7$ was already discussed.

If G is of type E_8, G' is of type E_7, then $V = \mathfrak{g}$ is the adjoint representation,

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

with $\mathfrak{g}_0 = \mathfrak{g}' \oplus k$, $\mathfrak{g}_{\pm 1}$ being the 56-dimensional representation of G', $\mathfrak{g}_{\pm 2}$ being the trivial representation of G'. This case requires a generalization of the blow up theorem and some additional tricks. We will discuss it later if time permits.

Let X' be a del Pezzo surface of degree $d+1$ such that X is obtained from X' by blowing up one point M. We assume that there is an embedding of the universal torsor $\mathcal{T}' \subset (G'/P')_a^{sf}$ satisfying all requirements of the theorem. Observe that $V_1 = V'$ and $\mathbb{T}' = T$. Let $x \in \mathcal{T}$ be such that $f'(x) = M$.

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We need to introduce one more torus $\mathbb{V}$ which is the centralizer of T in $GL(V)$. If we fix a T-eigen basis $\{e_\mu\}$ in V, then $\mathbb{V}$ is obtained from V by removing all coordinate hyperplanes and multiplication given by pointwise multiplication of coordinates. So we will think about $\mathbb{V}$ as a subset of V. $\mathbb{V}'$ denotes of course the corresponding torus for G'.

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Recall also the maps $p_i : V' \to V_i$. For a weight μ of V we denote by $p_i^\mu : V' \to V_i^\mu$ the polynomial function on V' such that

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Lemma 5.2. *There exists $s \in \mathbb{V}'$ such that $s(\mathcal{T}') \cap (G'/P')_a = Ts(x)$ is a single orbit and the restriction of p_2^μ on $s(\mathcal{T}')$ is not identically zero for all weights μ of V_2.*

Proof. We are looking for s in the form yx^{-1} for some $y \in (G'/P')_a$. First we claim that for each weight μ of V_2 there is y such that p_2^μ is not identically zero on $s(\mathcal{T}')$. Assume the contrary. Then $p_2^\mu(yx^{-1}u) = 0$ for all $u \in \mathcal{T}'$ and $y \in (G'/P')_a$. Look at $p_2^\mu(yx^{-1}u = 0)$ as a function of y. All quadratic forms in $I((G'/P')_a) \subset S^2(V'^*)$ of weight μ are proportional to some form $p_2^\mu(y)$. Write

$$p_2^\mu(y) = \sum_{\mu_1+\mu_2=\mu} c_{\mu_1,\mu_2} y^{\mu_1} y^{\mu_2}.$$

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To each weight μ of V we can associate now a divisor

$$l'_\mu = f'(s(\{u \in s(\mathcal{T}')|p_i^\mu(u) = 0\}$$

on X'. If μ is a weight of V_1, then by induction assumption l'_μ is an exceptional divisor. If μ is a weight of V_2, then l'_μ is a conic (self-intersection index is zero). In fact, since $\mathbf{Pic}X'$ is generated by exceptional divisors we can compute the intersection indices for all $[l'_\mu]$ and $[l'_\nu]$.

To check that $s(\mathcal{T}') \cap (G'/P')_a = Ts(x) = Ty$ choose μ and ν so $[l'_\mu]$ and $[l'_\nu]$ have intersection index 1. Then $u \in s(\mathcal{T}')$ and $p_2^\mu(u) = p_2^\nu(u) = 0$ implies $f'(s(u)) = f'(s(x))$. $\square$

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By use of the blow up theorem and functoriality of blow ups we obtain the following commutative diagram

$$\begin{array}{ccccc} \mathcal{T} & \longrightarrow & \mathrm{Bl}_{Ty}(s(\mathcal{T}')) & \longrightarrow & X \\ & \searrow & \downarrow & & \downarrow \\ & & s(\mathcal{T}') & \longrightarrow & X' \end{array}$$

where the horizontal arrows are torsors under tori, and the vertical arrows are smooth contractions. The map $\mathcal{T} \to \mathrm{Bl}_{Ty}(s(\mathcal{T}'))$ is a torsor with torus D. Therefore the composition $f : \mathcal{T} \to X$ of two torsors is a torsor with torus $\mathbb{T}$.

Next we show that $f : \mathcal{T} \to X$ is a universal torsor. For each weight μ of V let l_μ be the image under f of the corresponding hyperplane section. Then l_ω is the exceptional divisor corresponding to the blown up point M. Furthermore, l_μ for all weights μ of V_1 and l_ω generate $\mathbf{Pic}X$.

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Recall that $(G/P)_a^{sf} \to Y$ is a universal torsor, i.e. its type $\hat{\mathbb{T}} \to \mathbf{Pic}Y$ is an isomorphism. The restriction map $\mathbf{Pic}Y \to \mathbf{Pic}X$ is an isomorphism, since l_μ for all weights μ of V_1 and l_ω lie in the image. Since $\mathbf{Pic}Y$ and $\mathbf{Pic}X$ have the same rank the restriction map is an isomorphism and hence the type $\hat{\mathbb{T}} \to \mathbf{Pic}X$ of f is an isomorphism as well.

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Finally, we need to show that the set of exceptional divisors of X coincides with l_μ for all weights μ of V. That follows from the fact that the surjective homomorphism

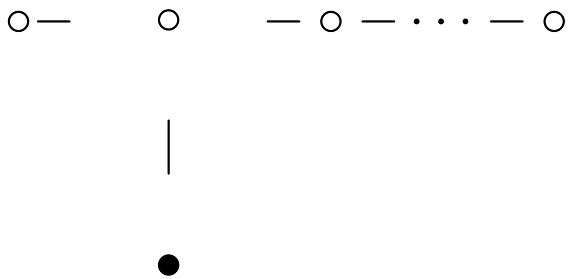
$$\theta : k[(G/P)_a^{sf}] = k[(G/P)_a] \to k[\mathcal{T}]$$

is $\mathbb{T}$-equivariant. Consider the weight decomposition

$$k[(G/P)_a] = \oplus_{\chi\in\hat{\mathbb{T}}} k[(G/P)_a]_\chi, \;\; k[\mathcal{T}] = \oplus_{\chi\in\hat{\mathbb{T}}} k[\mathcal{T}]_\chi$$

If $-\mu$ is a weight of V^*, then $k[(G/P)_a]_{-\mu}$ is one-dimensional and spanned by the coordinate function u^μ. By surjectivity $\theta(u^\mu) \neq 0$, and hence the image of the hyperplane section $u^\mu = 0$ under f is an exceptional curve.

$\mathbb{P}^{n-3}$ **blown up at** n **points.**(Sturmfels, Velasco) Consider the pair $A_{r-1} \subset D_r$. Then $\omega = \omega_r$ and $V = V_\omega$ is one of the two spinor representation of $Spin(2r)$ which is a double cover of $SO(2r)$ and G/P is a connected components of the grassmanian of maximal isotropic subspaces in the natural representation of $SO(2r)$.



Let us recall construction of the spinor representation. The natural representation Γ of G is isomorphic to k^{2r} equipped with G-invariant quadratic form b. Consider the Clifford algebra $\mathrm{Cliff}(\Gamma) = T(\Gamma)/(xy + yx - b(x, y)$. Obviously G is a subgroup in the group of automorphisms of $\mathrm{Cliff}(\Gamma)$.

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It is known that $\mathrm{Cliff}(\Gamma)$ is isomorphic to a matrix algebra. Thus it has a unique simple module. Write $\Gamma = L \oplus L'$ a sum of two isotropic subspaces. Then $G' = GL(L)$ is the subgroup that preserves this decomposition and $\mathrm{Cliff}(\Gamma)$ is isomorphic to the algebra of $\mathbb{Z}_2$-graded differential operators in $\Lambda(L)$. Hence the group of automorphisms of $\mathrm{Cliff}(\Gamma)$ has a projective representations in $\Lambda(L)$. The restriction of this projective representation to $SO(2n)$ gives a sum of two irreducible representations of $Spin(2r)$:

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We set $V = \Lambda_{ev}(L)$. Then the $\mathbb{Z}$-grading on V is given by $V_i = \Lambda^{2i}(L)$. In particular, $V_1 = V' = \Lambda^2(L)$ and the maps $p_i : V_1 \to V$ are given by the formula

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$(G'/P')_a$ coincides with bivectors in $\Lambda^2(L)$ of rank 2. Indeed, this is the image of $Gr(2, n)$ under the Plucker embedding.

Theorem 5.3. *Let X be $\mathbb{P}^{n-3}$ blown up at n points and $f : \mathcal{T} \to X$ be a universal torsor. There exists a closed equivariant embedding of $\mathcal{T}$ to the universal torsor $f : (G/P)_a^{sf} \to : Y = (G/P)_a^{sf}/\mathbb{T}$. The images under f of the T-invariant hyperplane sections of $\mathcal{T}$ are the exceptional divisors of X.*