Techniques for calculating Cox rings

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Table of Contents

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Mori dream spaces

Definition. A *Mori dream space* X is:

- a normal variety with finitely generated class group $\mathrm{Cl}(X)$;
- with finitely generated *Cox ring*

$$\mathcal{R}(X) := \bigoplus_{[D]\in\mathrm{Cl}(X)} H^0(X, \mathcal{O}_X(D)).$$

Table of Contents

$f_1, \ldots, f_r \in \mathcal{R}(X)$ generate $\mathcal{R}(X) \iff$ for any $D \geq 0$ the following map is surjective

$$\begin{aligned} \bigoplus_{i=1}^r H^0(D - E_i) &\xrightarrow{\partial_1} H^0(D) \\ \sum_{i=1}^r \sigma_i e_i &\longmapsto \sum_{i=1}^r \sigma_i f_i, \end{aligned}$$

where $E_i := \mathrm{div}_{w_i}(f_i)$ is the divisor defined by f_i.

A criterion for surjectivity of ∂_1:

- if $E_1 \cap E_2 = \emptyset$ there is an exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_X(-E_1 - E_2) \longrightarrow \bigoplus_{i=1}^{2} \mathcal{O}_X(-E_i) \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

  which gives the following

$$\bigoplus_{i=1}^{2} H^0(D - E_i) \longrightarrow H^0(D) \longrightarrow H^1(D - E_1 - E_2),$$

- $H^1(D - E_1 - E_2) = 0$ if $D - E_1 - E_2 - K_X = N + \Delta$ with N nef and big, Δ s.n.c. and $\lfloor \Delta \rfloor = 0$.

Techniques for calculating Cox rings

Degrees of generators and relations

Observation. The previous criterion can be applied to check if the primitive generators of the extremal rays of the effective cone are the degrees of all the generators of the Cox ring of a Mori dream surface.

Examples.

- Del Pezzo surfaces,
- some K3 surfaces,
- rational surfaces with elliptic fibration like

$$x^3 + y^3 + z^3 + \lambda\, xyz = 0,$$

- very general degree $d \geq 4$ surfaces of $\mathbb{P}^3$ containing a line.

Techniques for calculating Cox rings

Degrees of generators and relations

Suppose that $\{f_1, \ldots, f_r\}$ is a minimal generating set for $\mathcal{R}(X)$.

- Let $\mathcal{I}(X)$ be the *ideal of relations* of $\mathcal{R}(X)$:

$$0 \longrightarrow \mathcal{I}(X) \longrightarrow \mathbb{K}[X] \longrightarrow \mathcal{R}(X) \longrightarrow 0.$$

- $\mathbb{K}[X]$ is a $\mathrm{Cl}(X)$-graded polynomial ring in $x_1, \ldots, x_r$ variables.
- We say that $\mathcal{I}(X)$ is *generated in degree* $w \in \mathrm{Cl}(X)$ if any set of generators of $\mathcal{I}(X)$ contains an element of degree w.

Consider the $\mathrm{Cl}(X)$-graded minimal resolution of $\mathcal{R}(X)$:

$$\bigoplus_{w\in\mathrm{Cl}(X)} \mathbb{K}[X](-w)^{b_{2,w}} \longrightarrow \bigoplus_{w\in\mathrm{Cl}(X)} \mathbb{K}[X](-w)^{b_{1,w}} \longrightarrow \mathbb{K}[X] \longrightarrow 0,$$

- $b_{1,w} \neq 0 \iff \mathcal{I}(X)$ is generated in degree w.
- $b_{1,w} = \dim_{\mathbb{K}} \operatorname{Tor}_1^{\mathbb{K}[X]}(\mathcal{R}(X), \mathbb{K})_w$.

  Thus $b_{1,w}$ can be calculated by tensoring the Koszul complex of $x_1, \dots, x_r$ with $\mathcal{R}(X)$ and taking homology.

Theorem (LaVe09) [1]. $\mathcal{I}(X)$ has a generator in degree $w = [D]$ if and only if the following complex is not exact

$$\bigoplus_{1 \leq i < j \leq r} H^0(D - E_i - E_j) \xrightarrow{\partial_2} \bigoplus_{i=1}^{r} H^0(D - E_i) \xrightarrow{\partial_1} H^0(D),$$

where the maps are explicitely given by:

$$\partial_1(e_i) = f_i \qquad \partial_2(e_{ik}) = f_i e_k - f_k e_i.$$

[1] A. Laface, M. Velasco, *Picard graded Betti numbers and the defining ideal of Cox rings* J. Algebra, 322(2):353–372, 2009.

Table of Contents

Let $\pi : X \to Y$ be a, not étale, degree n cyclic covering with both X and Y normal.

- π is a finite map of degree n.
- $\pi_*\mathcal{O}_X \cong \bigoplus_{i=0}^{n-1} \mathcal{O}_Y(-iB)$.

The divisor $B = \pi(R)$ is the *branch divisor*, where R is the *ramification divisor*. Denote by $f_R \in H^0(R)$ a section which defines R.

Techniques for calculating Cox rings

Cyclic coverings

Let $\pi : X \to Y$ be a degree n cyclic covering of a Mori dream space Y. Same notation as before.

- $\mathcal{O}_X(\pi^* D) \cong \mathcal{O}_Y(D) \otimes_{\mathcal{O}_Y} \pi_* \mathcal{O}_X \cong \bigoplus_{i=0}^{n-1} \mathcal{O}_Y(D - iB)$.
- One can prove that every element $g \in H^0(\pi^* D)$ can be uniquely written as

$$g = g_0 + f_R \cdot g_1 + \cdots + f_R^{n-1} \cdot g_{n-1}$$

where $g_i \in \pi^* H^0(D - iB)$.

Techniques for calculating Cox rings

Cyclic coverings

Theorem (ArHaLa10)[2] Same notation as before with $\mathrm{Cl}(Y)$ free. Let $\mathcal{R}(X)_\pi := \bigoplus_{w \in \pi^* \mathrm{Cl}(Y)} \mathcal{R}(X)_w$. Then

$$\mathcal{R}(X)_\pi \cong \mathcal{R}(Y)[t]/\langle t^n - f_B \rangle.$$

[2] M. Artebani, J. Hausen, A. Laface, *On Cox rings of K3-surfaces* Compositio Math., 146:964–998, 2010.

Observation. The previous theorem can be used to calculate the Cox ring of a double cover $\pi : X \to Y$ of a rational surface Y branched along a very general $B \in |-2K_Y|$, where Y is:

- a Del Pezzo surface, $\operatorname{Cl}(X) = \pi^* \operatorname{Cl}(Y)$,
- a Hirzebruch surface $\mathbb{F}_4$, $|\operatorname{Cl}(X) : \pi^* \operatorname{Cl}(Y)| = 2$.

Table of Contents

Let $X \subseteq Z$ be a hypersurface in a projective Mori dream space Z. The inclusion map $i : X \to Z$ is *good* if:

- $w := [X] \in \mathrm{Cl}(Z)$ is ample and spanned;
- X is general in $|w|$.

With the above hypothesis, by generalized Lefschetz theorem (RaSr06[3]) the map

$$i^* : \mathrm{Cl}(Z) \to \mathrm{Cl}(X)$$

is well defined and is an isomorphism if $\dim(Z) \geq 4$.

[3] G. V. Ravindra, V. Srinivas, *The Grothendieck-Lefschetz theorem for normal projective varieties*, J. Algebraic Geometry, 15:563–590, 2006

Denote by $\bar{Z} = \mathrm{Spec}(\mathcal{R}(Z))$ and by $\bar{X} = V(s) \subseteq \bar{Z}$ where s is the section defining X in Z.

Theorem (ArLa)[4] Let $i : X \to Z$ be good with Z projective Mori dream space of dimension ≥ 4 and let $\{f_1, \ldots, f_r\}$ be a minimal generating set for $\mathcal{R}(Z)$. If all the

$$V(s, f_1), \ldots, V(s, f_r) \subseteq \bar{Z}$$

are distinct and irreducible, then $\mathcal{R}(X) \cong \mathcal{R}(Z)/\langle s \rangle$.

[4] In progress, gereralizes S.Y. Jow *a generalized Lefschetz hyperplane theorem for Mori dream spaces*, Math. Z., published online on 22.1.2010.

Example. Let $\pi : Z \to \mathbb{P}^n$ be the blow-up of $\mathbb{P}^n$, with $n \geq 4$, at a codimension 2 linear subspace L and let X_d be the strict transform of a general degree d hypersurface containing L.

- $\mathcal{R}(Z) \cong \mathbb{K}[x_1, \ldots, x_{n+2}]$ with grading matrix

$$\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 & 0 \\ -1 & -1 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

- The irrelevant ideal of Z is

$$\mathfrak{Irr}(Z) = (x_1, x_2) \cap (x_3, \ldots, x_{n+2}).$$

- The equation of $\bar{X}_d$ in $\bar{Z}$ is: $x_1 f + x_2 g = 0 \Rightarrow \mathcal{O}(\bar{X}_d)$ is not a factorial ring.

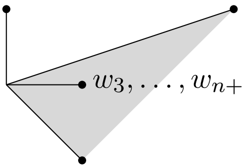
Techniques for calculating Cox rings

Hypersurfaces in Mori dream spaces

Example (continued). Consider the toric variety Z_1 defined by:

- $\mathcal{R}(Z_1) \cong \mathbb{K}[x_0, \ldots, x_{n+2}]$,
- $\mathfrak{Irr}(Z_1) = (x_1, \ldots, x_{n+1}) \cap (x_0, x_3, \ldots, x_{n+2})$,

where $w_0 = \deg(x_0) = \deg(f/x_2)$ and w_3 is ample. The movable cone of Z_1 is the shaded region.



Example (continued). $\mathcal{R}(X_d) \cong \dfrac{\mathbb{K}[x_0, \ldots, x_{n+2}]}{(x_0x_2 - f, x_0x_1 + g)}$.

Proof.