Asymptotic Behavior of a Critical Fluid Model for a Processor Sharing Queue via Relative Entropy

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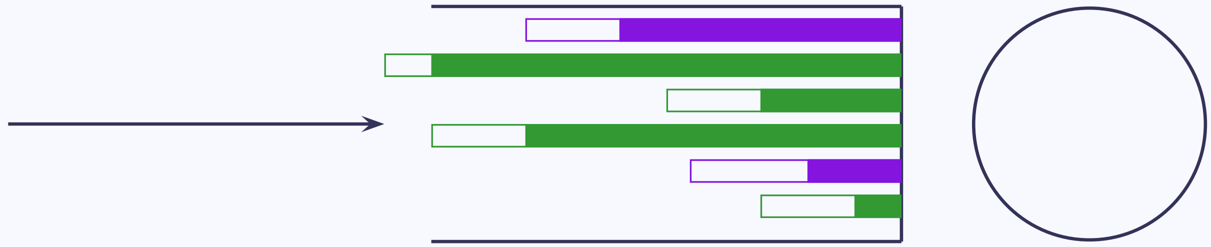
Joint work with Ruth J. Williams

Processor Sharing (PS) Queue



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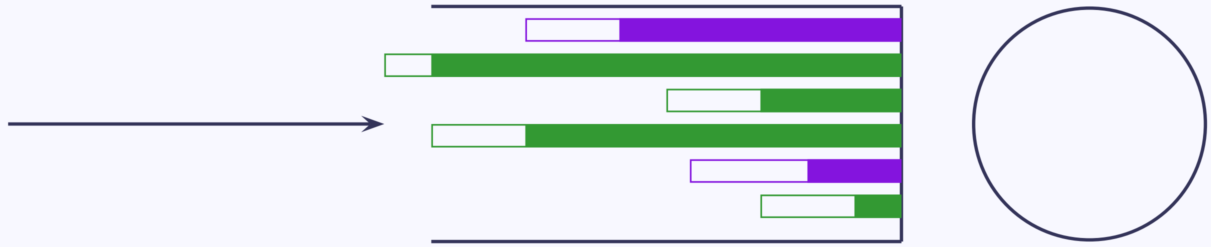

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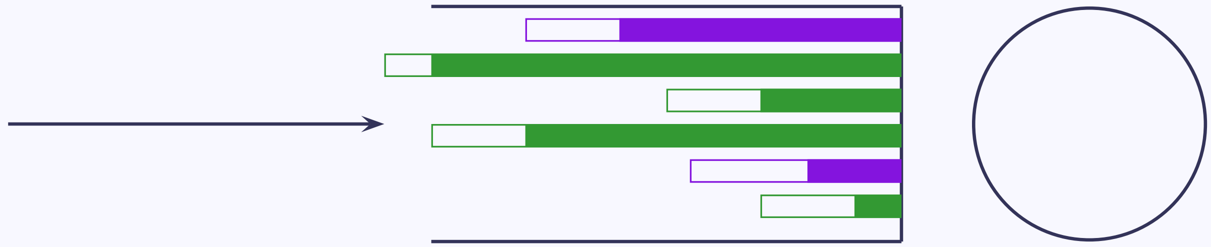

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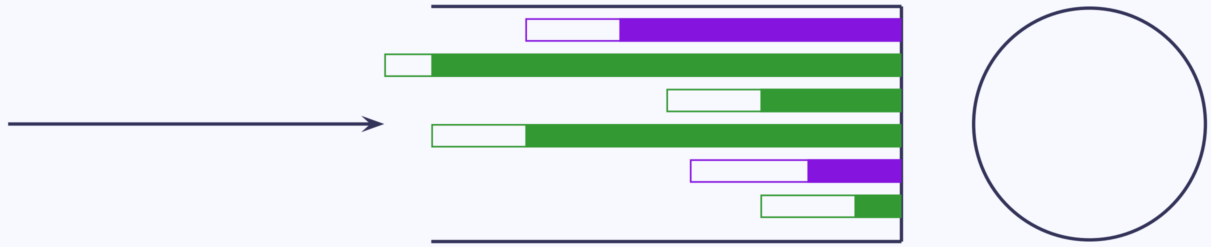

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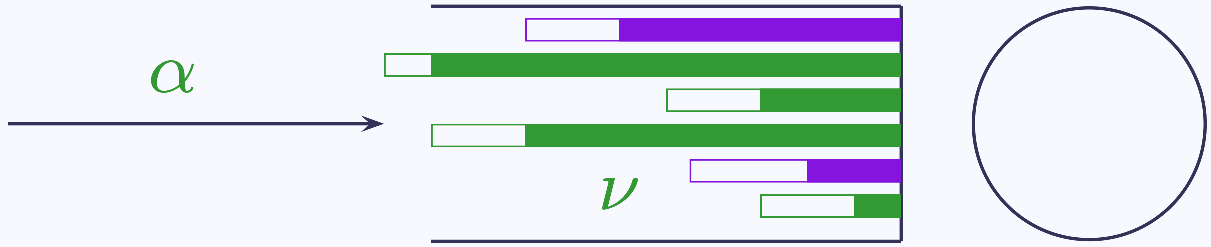
$$\frac{1}{\text{\# jobs in the system}}.$$

- Idealized model for computer time-sharing algorithms introduced by Kleinrock in ‘60’s.

- Until early 2000’s, only analyzed under restrictive distributional assumptions.

GI/GI/1 PS Queue



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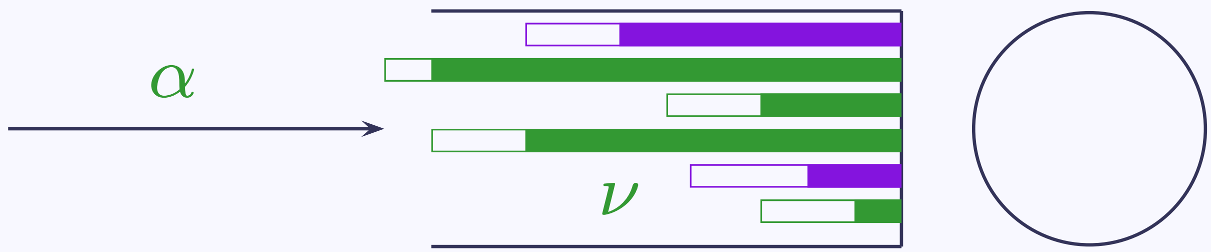

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  # jobs in the system a time 0, each with strictly positive residual service time

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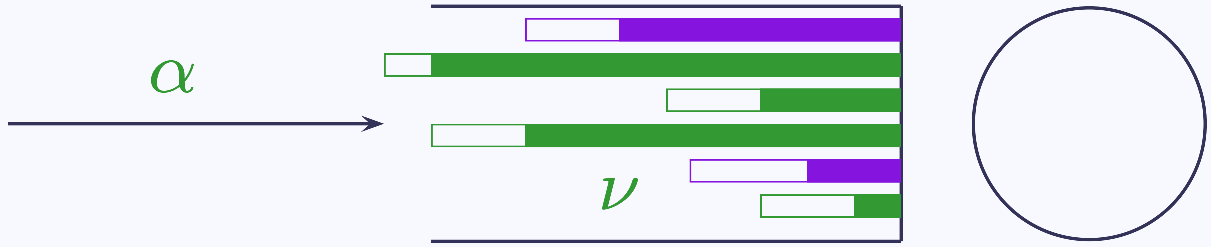

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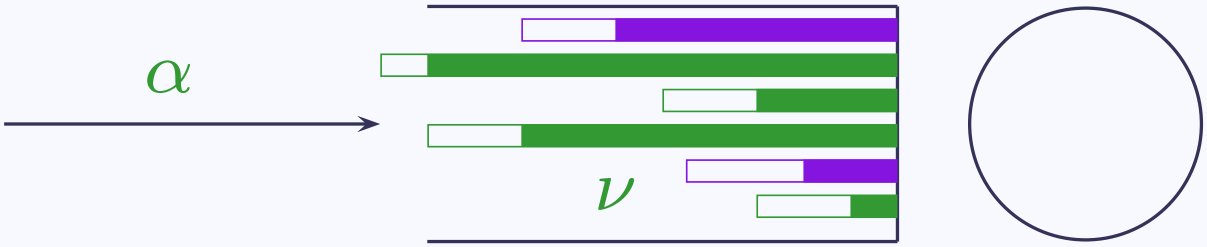

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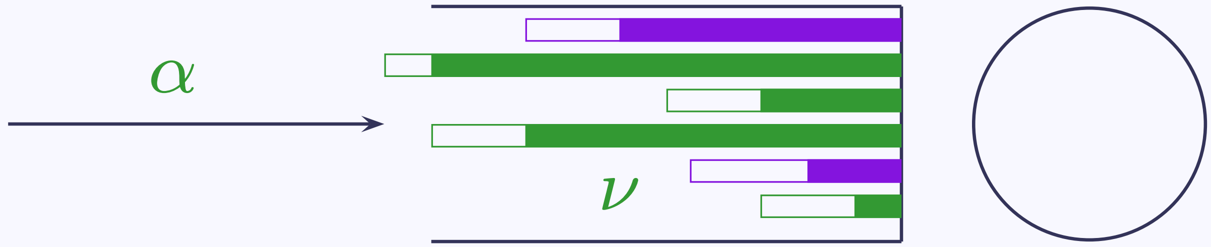
  # jobs in the system a time 0, each with strictly positive residual service time
- **Arrivals**:
  rate α delayed renewal process
- **Service times**:
  strictly positive, i.i.d. with distribution ν

GI/GI/1 PS Queue



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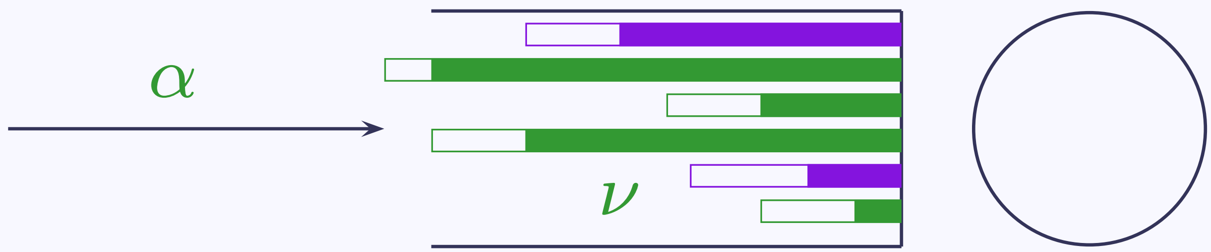

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  For each job in the system at time t, the residual service time at time t is the amount of processing time remaining at t

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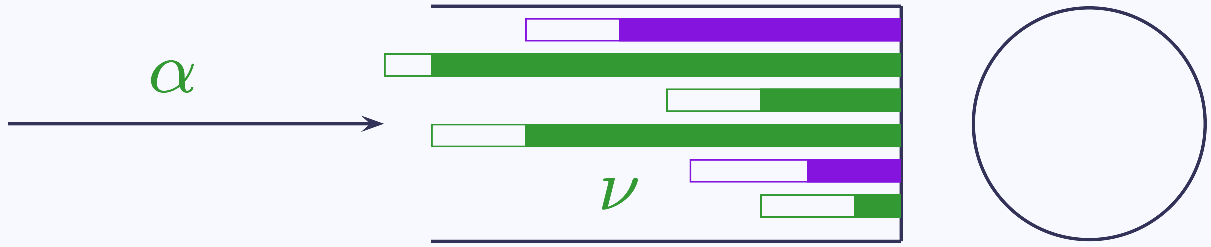

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- **Infinite dimensional system**:
  Must track all residual service times.

State Descriptor for GI/GI/1 PS Queue

$\mathcal{Z}(t)$

For each Borel set $A \subset [0, \infty)$,

$$\langle 1_A, \mathcal{Z}(t) \rangle \quad = \quad \text{\# jobs in the system at time } t$$
$$\text{with residual service time in } A.$$

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$\mathbf{M}$ endowed with the topology of weak convergence is a Polish space metrizable by the Prokhorov metric d.

State Descriptor for GI/GI/1 PS Queue

$\mathcal{Z}(t)$

Observe that

$$Q(t) \equiv \langle 1, \mathcal{Z}(t) \rangle = \#\text{ jobs in system at time } t,$$

$$W(t) \equiv \langle \chi, \mathcal{Z}(t) \rangle = \text{immediate workload at time } t,$$

where $\chi(x) = x, \; x \in [0, \infty)$.

PS Queues Literature

Survey

Yashkov '87 (mostly with restrictive distributional assumptions)

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More Recent (mostly with general distributions)

Baccelli & Towsley '90 (correlation of sojourn times)

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Limit Theorems via a Modular Approach

Gromoll, Puha & Williams '02 (fluid limit)

Puha & Williams '04 (analysis of critical fluid model solutions)

Gromoll '04 (diffusion limit)

Outline

1. Critical Fluid Model Solution (CMFS)
   a) Definition
   b) Existence & Uniqueness
   c) Invariant States
2. Statement of the Main Result in PW ‘16
3. Proof Strategy via Relative Entropy Arguments
4. Statement of Main Technical Result in PW ‘16
5. Proof of the Main Technical Result

Critical Fluid Model (GPW '02)

Model inputs: critical data (α, ν)

- $\alpha \in (0, \infty)$ is the arrival rate of fluid
- ν is a Borel probability measure on $[0, \infty)$ by which the fluid is distributed as it enters the system such that

$$\nu(\{0\}) = 0 \qquad \text{and} \qquad \rho = \alpha\langle \chi, \nu \rangle = 1.$$

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Initial Condition: $\xi \in \mathbf{M}$

ξ is a finite, nonnegative Borel measure on $[0, \infty)$ that gives the initial distribution of fluid

Critical Fluid Model (GPW '02)

A **Fluid Model Solution** for the critical data (α, ν) and initial condition $\xi \in \mathbf{M}$ is a function $\zeta : [0, \infty) \to \mathbf{M}$ with $\zeta(0) = \xi$ that is continuous, does not charge the origin, and

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for all $g \in \mathbf{C}_b^1$ with $g(0) = 0$ and $g'(0) = 0$, satisfies

$$\langle g, \zeta(t) \rangle = \langle g, \xi \rangle + \alpha\, t\, \langle g, \nu \rangle - \int_0^t \frac{\langle g', \zeta(u) \rangle}{\langle 1, \zeta(u) \rangle} du,$$

for $0 \le t < t^* = \inf\{u : \langle 1, \zeta(u) \rangle = 0\}$, and

$$\zeta(t) = \mathbf{0}, \qquad \text{for } t \ge t^*.$$

Existence and Uniqueness of CFMS

Let $\mathbf{K}$ be the set of continuous measures in $\mathbf{M}$:

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Theorem (*GPW '02).*
Given critical data (α, ν) *and* $\xi \in \mathbf{K}$, *there exists a unique fluid model solution* ζ^ξ *for the data* (α, ν) *such that* $\zeta^\xi(0) = \xi$.

Invariant States for CFMS

Definition. *Given critical data* (α, ν), $\xi \in \mathbf{K}$ *is an* *invariant state* *if* $\zeta^{\xi}(t) = \xi$ *for all* $t \geq 0$.

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$$f_e(x) = \frac{\langle 1_{(x,\infty)}, \eta \rangle}{\langle \chi, \eta \rangle}, \qquad \text{for } x \in [0, \infty).$$

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Theorem *(PW '04). The set of invariant states* $\mathbf{I}$ *for critical data* (α, ν) *is given by*

$$\mathbf{I} = \{ c\nu_e : c \in [0, \infty) \}.$$

Main Result

Given critical data (α, ν) *and* $u, l > 0$, *let*

$$\mathbf{M}_{u,l} = \{\eta \in \mathbf{M} \;:\; l \leq \langle \chi, \eta \rangle \text{ and } \langle 1_{(x,\infty)}, \eta \rangle \leq u \langle 1_{(x,\infty)}, \nu_e \rangle \text{ for all } x \in [0,\infty)\},$$

and set $\mathbf{K}_{u,l} = \mathbf{K} \cap \mathbf{M}_{u,l}$.

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Theorem 3.1 *(PW '16). Let* (α, ν) *be critical data such that* $\langle \chi^2, \nu \rangle < \infty$ *and* $u, l > 0$. *Then*

$$\lim_{t \to \infty} \sup_{\xi \in \mathbf{K}_{u,l}} d(\zeta^{\xi}(t), \mathbf{I}) = 0.$$

Relative Entropy

For absolutely continuous Borel probability measures η *and* γ *on* $\mathbb{R}_+$ *with densities* f *and* g*,*

$$\mathcal{E}(\eta,\gamma) = \int_0^\infty f(x) \ln\left(\frac{f(x)}{g(x)}\right) dx.$$

By convention, $0 \ln 0 = 0$ *and* $y \ln(y/0) = \infty$ *for* $y > 0$*.*

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Relative entropy is not a metric, but

1. $\mathcal{E}(\eta, \gamma) = 0$ *if and only if* $\eta = \gamma$, *and*
2. $d(\eta, \gamma) \leq \sqrt{\frac{\mathcal{E}(\eta,\gamma)}{2}}$.

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Since ν_e *is absolutely continuous, all invariant states are absolutely continuous.*

The value $\zeta(t)$ *of a fluid model solution* ζ *at time* t *is not necessarily absolutely continuous.*

Hence, it is possible that for all $t \geq 0$,

$$\mathcal{E}\left(\frac{\zeta(t)}{\langle 1, \zeta(t)\rangle}, \nu_e\right) = \infty.$$

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$$H(\eta) = \mathcal{E}(\eta_e, (\nu_e)_e).$$

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Then, given $\xi \in \mathbf{K}$ *such that* $0 < \langle \chi, \xi \rangle < \infty$, *let*

$$\mathcal{H}_\xi(t) = H(\zeta^\xi(t)) = \mathcal{E}(\zeta_e^\xi(t), (\nu_e)_e), \qquad \text{for } t \geq 0.$$

Strategy for Proving the Main Result

Show:

$\mathcal{H}_\xi(t) \to 0$ *uniformly as* $t \to \infty$ *on* $\mathbf{K}_{u,l}$ *for any* $u, l > 0$.

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Final Step:

Show that the Desired Conclusion follows.

Main Technical Result

Theorem 3.2 *(PW '16). Let* (α, ν) *be critical data such that* $\langle \chi^2, \nu \rangle < \infty$ *and let* $u, l > 0$*. For each* $\xi \in \mathbf{K}_{u,l}$, $\mathcal{H}_\xi$ *is nonincreasing. Furthermore,*

$$\lim_{t \to \infty} \sup_{\xi \in \mathbf{K}_{u,l}} \mathcal{H}_\xi(t) = 0.$$

Recall $\mathcal{H}_\xi(t) = \mathcal{E}(\zeta_e^\xi(t), (\nu_e)_e)$ *for* $t \geq 0$ *and* $\xi \in \mathbf{K}_{u,l}$.

Absolute Continuity of $\mathcal{H}_\xi$

Theorem 7.1 *(PW '16). Let* (α, ν) *be critical data such that* $\langle \chi^2, \nu \rangle < \infty$ *and let* $u, l > 0$. *For each* $\xi \in \mathbf{K}_{u,l}$, *there exists a continuous function* $\kappa_\xi : [0, \infty) \to (-\infty, 0]$ *such that for all* $0 \leq s < t < \infty$,

$$\mathcal{H}_\xi(t) - \mathcal{H}_\xi(s) = \int_s^t \kappa_\xi(u) du,$$

and $\kappa_\xi(u) = 0$ *if and only if* $\zeta^\xi(u) \in I$.

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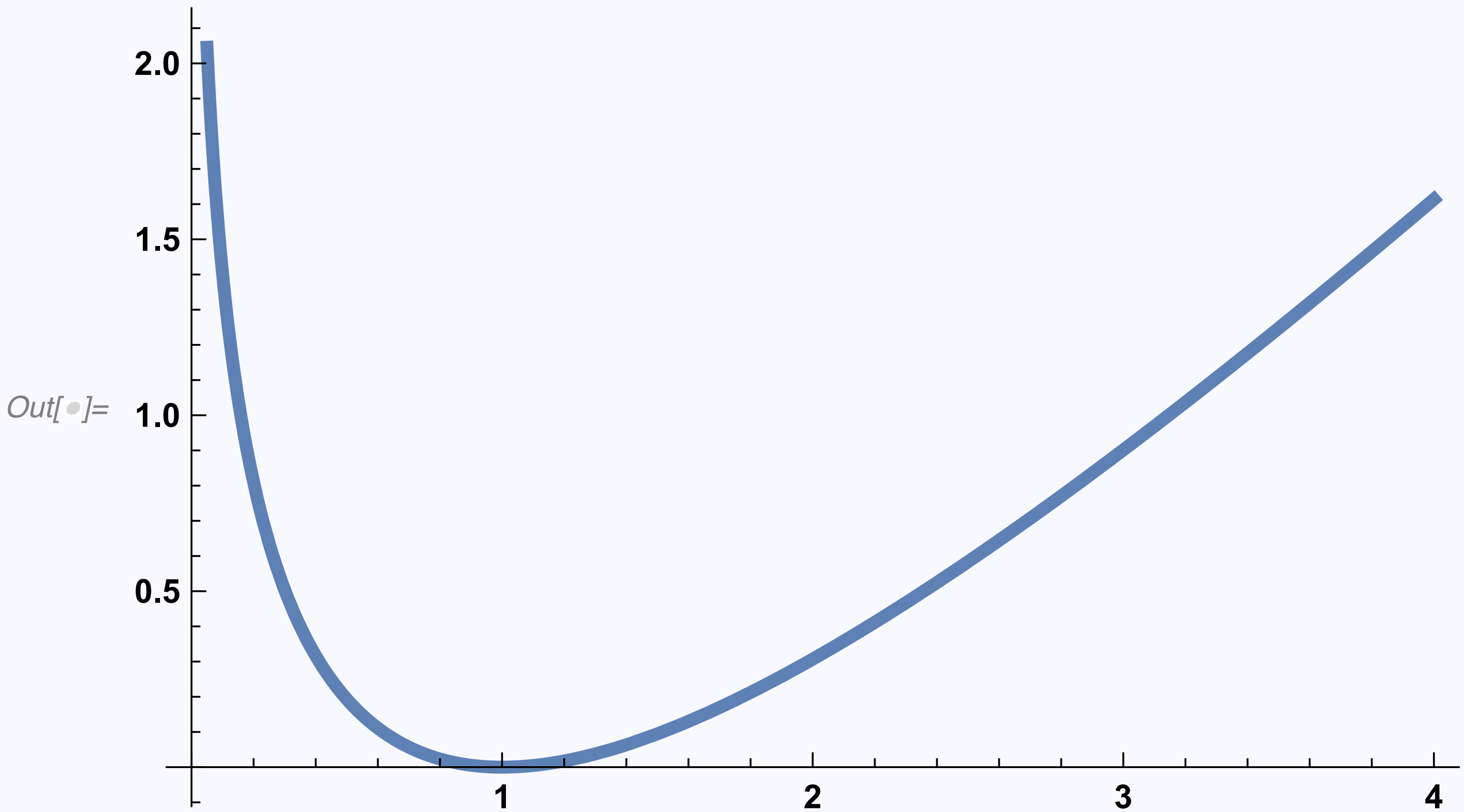
Henceforth, $u, l > 0$ *and critical data* (α, ν) *such that* $\langle \chi^2, \nu \rangle < \infty$ *are fixed .*

An Explicit Expression for κ_ξ



For $x \in (0, \infty)$, *set* $k(x) = x - 1 - \ln(x)$ *and set* $k(0) = \infty$.

An Explicit Expression for κ_ξ

Let $\xi \in \mathbf{K}_{u,l}$. *For* $t, x \in [0, \infty)$, *set*

$$\begin{aligned} q^\xi(t) &= \langle 1, \zeta^\xi(t) \rangle, \\ \overline{q}^\xi(t,x) &= \langle 1_{(x,\infty)}, \zeta^\xi(t) \rangle, \\ \overline{N}_e(x) &= \langle 1_{(x,\infty)}, \nu_e \rangle. \end{aligned}$$

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Then, for $t > 0$,

$$\kappa_\xi(t) = \frac{-1}{\langle \chi, \xi \rangle} \mathbb{E}_{\nu_e} \left[k \left(\frac{\overline{q}^\xi(t,X)}{q^\xi(t)\overline{N}_e(X)} \right) \right].$$

An Associated PDE

Corollary 7.1 *(PW '16)*

Let $\xi \in \mathbf{K}$*. Suppose that*

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Then for all $(t, x) \in [0, \infty)^2$,

$$\frac{\partial}{\partial t}\overline{q}^{\xi}(t,x) = \alpha\langle 1_{(x,\infty)}, \nu\rangle - \frac{\frac{\partial}{\partial x}\overline{q}^{\xi}(t,x)}{q^{\xi}(t)}.$$

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Remark. *Used by Paganini et. al. '12 to study stability properties of subcritical Bandwidth sharing models.*

Prf of Theorem 7.1: Absolute Continuity of $\mathcal{H}_\xi$

1. *Verify that* κ_ξ *is finite and continuous.*
2. *Restrict to absolutely continuous* $\xi \in \mathbf{K}_{u,l}$.
   a) *Prove that a weak formulation of the PDE holds.*
   b) *Use integration-by-parts together with the weak formulation of the PDE and other identities to verify that* κ_ξ *is the density of* $\mathcal{H}_\xi$.
3. *Use approximation arguments to extend to* $\xi \in \mathbf{K}_{u,l}$.

Prf of Theorem 3.2: $\mathcal{H}_\xi(t) \searrow 0$ uniformly on $\mathbf{K}_{u,l}$.

Fix $u, l, T, \varepsilon > 0$. *We show that there exist*

1. $B > 0$ *such that* $\mathcal{H}_\xi(t) \leq B$ *for all* $t \geq 0$ *and* $\xi \in \mathbf{K}_{u,l}$,

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1. $B > 0$ *such that* $\mathcal{H}_\xi(t) \leq B$ *for all* $t \geq 0$ *and* $\xi \in \mathbf{K}_{u,l}$,
2. *a compact set* $\mathbf{M}_{u,l,T}$ *that does not contain the zero measure and such that for all* $\xi \in \mathbf{K}_{u,l}$, $\zeta^\xi(t) \in \mathbf{M}_{u,l,T}$ *for all* $t \geq T$.

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3. *$\delta > 0$ such that if $t \geq T$ and $\mathcal{H}_\xi(t) \geq \varepsilon$, then $\kappa_\xi(t) \leq -\delta$.*

It follows by monotonicity of $\mathcal{H}_\xi$ that $\mathcal{H}_\xi(t) < \varepsilon$ for all $t \geq T + B/\delta$.

Prf of Main Result: Properties of H

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Proposition.

1. *For* $\eta \in \mathbf{M}$ *such that* $0 < \langle \chi, \eta \rangle < \infty$, $H(\eta) = 0$ *if and only if* $\eta \in \mathbf{J}$.
2. H *is continuous on* $\mathbf{M}_{u,l}$.

Prf of Main Result: An Intermediate Result

Corollary 6.1 *(PW '16).*

$$\lim_{t\to\infty} \sup_{\xi\in\mathbf{K}_{u,l}} d(\zeta^{\xi}(t), \mathbf{J}) = 0.$$

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By Theorem 3.2, $H(\zeta^\xi(t))$ *is uniformly close to zero.*

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Theorem 3.1 *(PW '16)*

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Using this and other properties of ζ^{ξ} *for* $\xi \in \mathbf{K}_{u,l}$, *it can be shown that* $\mathbf{J}$ *in Corollary 6.1 can be replaced by* $\mathbf{I}$.

Work in Progress

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Thank you for your attention.