n-transposition groups and vertex operator algebras

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n-transposition group

$\mathbf{G} = \langle \mathbf{D} \rangle$: group

$\mathbf{D}$: normal subset of involutions in $\mathbf{G}$ (union of conjugacy classes)

Definition $(\mathbf{G}, \mathbf{D})$: $\mathbf{n}$-transposition group

$$\overset{\text{def}}{\iff} \ {}^{\forall}\mathbf{a}, \mathbf{b} \in \mathbf{D} \quad {}^{\exists}\mathbf{k} \leq \mathbf{n} \quad \text{s.t.} \quad (\mathbf{ab})^{\mathbf{k}} = \mathbf{1}$$

$$({}^{\exists}\mathbf{x}, \mathbf{y} \in \mathbf{D} \quad \text{s.t.} \quad \text{order of } \ \mathbf{xy} = \mathbf{n})$$

n-transposition group

$G = \langle D \rangle$: group

D : normal subset of involutions in G (union of conjugacy classes)

Definition (G, D) : n-transposition group

$\overset{\text{def}}{\Longleftrightarrow}$ ${}^{\forall}a, b \in D \quad {}^{\exists}k \leq n \quad \text{s.t.} \quad (ab)^k = 1$

$({}^{\exists}x, y \in D \quad \text{s.t.} \quad \text{order of } xy = n)$

Targets

$\mathbb{M}$ $(n = 6)$

$\mathbb{B}$ $(n = 4)$

Fi_{24} $(n = 3)$

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Targets

$\mathbb{M}$ $(n = 6)$ $\rightsquigarrow$ affine E_8-diagram

$\mathbb{B}$ $(n = 4)$ $\rightsquigarrow$ affine E_7-diagram

Fi_{24} $(n = 3)$ $\rightsquigarrow$ affine E_6-diagram

Outline

- Notation and Terminology
- Automorphisms of VOA
- Miyamoto involutions
- Sakuma's theorem
- n-transposition property $(n = 6, 4, 3)$
- Application to the sporadic simple finite groups
- Automorphisms of commutant subalgebras

Notation and Terminology

$(V, Y(\cdot, z), \mathbb{1}, \omega)$: Vertex Operator Algebra (VOA)

$V = \bigoplus_{n \geq 0} V_n$: $\mathbb{N}$-graded vector space over $\mathbb{R}$

$Y(\cdot, z) : V \otimes V \to V((z))$: vertex operator map

$$Y(a, z)b = \sum_{n \in \mathbb{Z}} a_{(n)} b\, z^{-n-1}$$

$\mathbb{1}$: the vacuum vector $\quad V_0 = \mathbb{R}\mathbb{1}$

ω : the conformal vector $\quad \omega \in V_2, \quad Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$

$$\Longrightarrow [L(m), L(n)] = (m - n)L(m + n) + \delta_{m+n,0} \frac{m^3 - m}{12} c$$

Notation and Terminology

$\mathbf{e} \in \mathbf{V}_2$: Virasoro vector $\qquad \left(\mathsf{Y}(\mathsf{e},\mathsf{z}) = \sum_{\mathsf{m}\in\mathbb{Z}} \mathsf{e}_{(\mathsf{m})}\mathsf{z}^{-\mathsf{m}-1} = \sum_{\mathsf{n}\in\mathbb{Z}} \mathsf{L}^{\mathsf{e}}(\mathsf{n})\mathsf{z}^{-\mathsf{n}-2} \right)$

$$\overset{\text{def}}{\Longleftrightarrow} \quad \mathsf{e}_{(1)}\mathsf{e} = 2\mathsf{e}$$

$$\Longleftrightarrow \quad \mathsf{Y}(\mathsf{e},\mathsf{z})\mathsf{Y}(\mathsf{e},\mathsf{w}) \sim \frac{\mathsf{c}_{\mathsf{e}}}{2(\mathsf{z}-\mathsf{w})^4} + \frac{2\mathsf{Y}(\mathsf{e},\mathsf{w})}{(\mathsf{z}-\mathsf{w})^2} + \frac{\partial_{\mathsf{w}}\mathsf{Y}(\mathsf{e},\mathsf{w})}{(\mathsf{z}-\mathsf{w})}$$

$$\Longleftrightarrow \quad [\mathsf{L}^{\mathsf{e}}(\mathsf{m}),\mathsf{L}^{\mathsf{e}}(\mathsf{n})] = (\mathsf{m}-\mathsf{n})\mathsf{L}^{\mathsf{e}}(\mathsf{m}+\mathsf{n}) + \delta_{\mathsf{m}+\mathsf{n},0}\frac{\mathsf{m}^3-\mathsf{m}}{12}\mathsf{c}_{\mathsf{e}}$$

$$(\mathbf{U},\mathsf{e}) : \text{sub VOA} \quad \overset{\text{def}}{\Longleftrightarrow} \quad \begin{cases} \mathbf{U} : \text{subalgebra} \ni \mathbb{1} \\ \mathsf{e} \in \mathbf{U}_2 : \text{Virasoro vector} \\ (\mathbf{U}, \mathbf{Y}(\cdot,\mathsf{z})|_{\mathsf{U}}, \mathbb{1}, \mathsf{e}) : \text{VOA} \end{cases}$$

$\mathsf{e} \in \mathbf{V}$: Virasoro vector

$\rightsquigarrow$ $\mathbf{Vir}(\mathsf{e})$: Virasoro sub VOA generated by e

Commutant subalgebra (a.k.a. coset construction)

$\mathbf{V}$: VOA $\supset$ $\mathbf{U}$: sub VOA

$$\mathbf{Com_V(U)} := \{\mathbf{a} \in \mathbf{V} \mid [\mathbf{Y(a, z_1), Y(u, z_2)}] = \mathbf{0} \ \text{ for } {}^\forall \mathbf{u} \in \mathbf{U}\}$$

$\Longrightarrow$ $\mathbf{Com_V(U)}$: sub VOA s.t. $\mathbf{U} \otimes \mathbf{Com_V(U)} \hookrightarrow \mathbf{V}$

Commutant subalgebra (a.k.a. coset construction)

$\mathbf{V}$: VOA $\quad \supset \quad$ $\mathbf{U}$: sub VOA

$$\mathrm{Com}_V(U) := \{a \in V \mid [Y(a, z_1), Y(u, z_2)] = 0 \text{ for } {}^\forall u \in U\}$$

$\Longrightarrow$ $\mathrm{Com}_V(U)$: sub VOA $\quad$ s.t. $\quad U \otimes \mathrm{Com}_V(U) \hookrightarrow V$

Example [GKO] (unitary Virasoro VOAs)

$$V_{k+1}(\mathfrak{sl}_2) \subset V_k(\mathfrak{sl}_2) \otimes V_1(\mathfrak{sl}_2)$$

$\mathrm{Com}_{V_k(\mathfrak{sl}_2) \otimes V_1(\mathfrak{sl}_2)}(V_{k+1}(\mathfrak{sl}_2)) \simeq L(c_k, 0)$ (simple Virasoro VOA)

where $c_k = 1 - \dfrac{6}{(k+2)(k+3)}, \ k = 1, 2, 3, \ldots .$

$$c_1 = \frac{1}{2}, \quad c_2 = \frac{7}{10}, \quad c_3 = \frac{4}{5}, \quad c_4 = \frac{6}{7}, \quad \ldots$$

Automorphisms of VOA

$\mathbf{V}$: simple VOA $\qquad \mathbf{G} < \mathbf{Aut}(\mathbf{V})$: finite subgroup

$\mathbf{V}^{\mathbf{G}}$: $\mathbf{G}$-invariant subalgebra

$\chi \in \mathrm{Irr}(\mathbf{G}) \qquad \mathbf{M}_\chi$: irr $\mathbf{G}$-mod affording χ

$$\Longrightarrow \quad \mathbf{V} = \bigoplus_{\chi \in \mathrm{Irr}(\mathbf{G})} \mathbf{M}_\chi \underset{\mathbb{C}}{\otimes} \mathbf{Hom}_{\mathbf{G}}(\mathbf{M}_\chi, \mathbf{V}) \quad \curvearrowleft \quad \mathbf{G} \otimes \mathbf{V}^{\mathbf{G}}$$

$\mathbf{V}_\chi := \mathbf{Hom}_{\mathbf{G}}(\mathbf{M}_\chi, \mathbf{V})$: $\mathbf{V}^{\mathbf{G}}$-module

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Theorem [Dong-Li-Mason]

(1) $\mathbf{V}_\chi \neq \mathbf{0}$ $({}^\forall \chi \in \mathrm{Irr}(\mathbf{G}))$ $\qquad$ (2) $\mathbf{V}_\chi \simeq \mathbf{V}_\mu \iff \chi = \mu$

(3) $\mathbf{V}_\chi$: irr $\mathbf{V}^{\mathbf{G}}$-module $\quad (\Longrightarrow \mathbf{V}^{\mathbf{G}}$: simple subalgebra)

Galois theory

Theorem [Dong-Mason, Hanaki-Miyamoto-Tambara]

$$\{\text{subgroups of } \mathbf{G}\} \stackrel{1:1}{\longleftrightarrow} \{\text{subalgebras of } \mathbf{V} \supset \mathbf{V}^{\mathbf{G}}\}$$

by $\quad \mathbf{G} > \mathbf{H} \quad \longmapsto \quad \mathbf{V}^{\mathbf{H}} > \mathbf{V}^{\mathbf{G}}$

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The simplest case : $\mathbf{G} = \langle \mathbf{g} \rangle$ let $\zeta = \mathbf{e}^{2\pi\sqrt{-1}/|\mathbf{g}|}$

$$\begin{array}{ccccccccc} \mathbf{V} & = & \mathbf{V}^0 & \oplus & \mathbf{V}^1 & \oplus & \cdots & \oplus & \mathbf{V}^{|\mathbf{g}|-1} \\ \mathbf{g} & : & \mathbf{1} & & \zeta & & \cdots & & \zeta^{|\mathbf{g}|-1} \end{array}$$

$\mathbf{V}^0 = \mathbf{V}^{\langle \mathbf{g} \rangle}$: simple sub VOA $\mathbf{V}^{\mathbf{i}}$: ineq irr $\mathbf{V}^0$-submod

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$\Longrightarrow$ any finite automorphism of $\mathbf{V}$ is induced by an irreducible decomposition w.r.t. its simple sub VOA

Miyamoto involution

Definition e : Ising vector $\iff \mathbf{Vir}(\mathbf{e}) \simeq \mathbf{L}(1/2, \mathbf{0})$

Fact irr $\mathbf{L}(1/2, \mathbf{0})$-mod : $\mathbf{L}(1/2, \mathbf{0})$, $\mathbf{L}(1/2, 1/2)$, $\mathbf{L}(1/2, 1/16)$

Every $\mathbf{L}(1/2, \mathbf{0})$-mod is semisimple

$\Longrightarrow \mathbf{V} = \mathbf{V_e}(\mathbf{0}) \oplus \mathbf{V_e}(1/2) \oplus \mathbf{V_e}(1/16)$ (isotypical decomposition)

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Define a linear automorphism $\tau_{\mathbf{e}}$ of V by

$$\tau_{\mathbf{e}} := \begin{cases} 1 & \text{on } \mathsf{V}_{\mathbf{e}}(0) \oplus \mathsf{V}_{\mathbf{e}}(1/2) \\ -1 & \text{on } \mathsf{V}_{\mathbf{e}}(1/16) \end{cases}$$

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Theorem [Miyamoto] $\tau_e \in \mathbf{Aut}(\mathbf{V})$

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Definition $\mathbf{e}$: σ-type on $\mathbf{V}$ $\iff$ $\tau_{\mathbf{e}} = \mathrm{id}_{\mathbf{V}}$

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Remark $\mathbf{e} \in \mathbf{V} \subset \mathbf{W} \implies \tau_{\mathbf{e}} \in \mathrm{Aut}(\mathbf{W})$: well-defined

(cf. $\mathbf{r}_\alpha \in \mathrm{Aut}(\mathfrak{h}) \rightsquigarrow \tilde{\mathbf{r}}_\alpha = \mathbf{e}^{\mathrm{ad}(\mathbf{f}_\alpha)}\mathbf{e}^{-\mathrm{ad}(\mathbf{e}_\alpha)}\mathbf{e}^{\mathrm{ad}(\mathbf{f}_\alpha)} \in \mathrm{Aut}(\mathfrak{g}(\mathbf{A}))$)

However, "e is of σ-type on V" $\not\Longrightarrow$ "e is of σ-type on W"

Generalization

$$c_m = 1 - \frac{6}{(m+2)(m+3)}, \quad c_1 = \frac{1}{2}, \quad c_2 = \frac{7}{10}, \quad c_3 = \frac{4}{5}, \quad c_4 = \frac{6}{7}, \ldots$$

$e \in V$: simple $c = c_m$ Virasoro vector $\quad (Vir(e) \simeq L(c_m, 0))$

$L(c_m, 0)$-modules: $L(c_m, h^{(m)}_{r,s})$ where

$$h^{(m)}_{r,s} := \frac{(r(m+3) - s(m+2))^2 - 1}{4(m+2)(m+3)}, \quad 1 \le s \le r \le m+1$$

$$\Longrightarrow \quad V = \bigoplus_{1 \le s \le r \le m+1} V_e(h^{(m)}_{r,s}) \qquad \text{(isotypical decomposition)}$$

$$\text{Define } \tau_e := \begin{cases} (-1)^{r+1} & \text{on} \quad V_e(h^{(m)}_{r,s}) \quad (m : \text{even}) \\ (-1)^{s+1} & \text{on} \quad V_e(h^{(m)}_{r,s}) \quad (m : \text{odd}) \end{cases}$$

Theorem [Miyamoto] $\quad \tau_e \in Aut(V)$

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Examples

$(\mathbf{Sym_r}, \mathbf{D} = \{(\mathbf{i}\,\mathbf{j}) \mid \mathbf{1} \leq \mathbf{i} \neq \mathbf{j} \leq \mathbf{r}\})$: 3-transposition group

$(\mathbf{Alt_r}, \mathbf{D} = \{(\mathbf{i}\,\mathbf{j})(\mathbf{k}\,\ell)\})$: 6-transposition group

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$\mathbf{Fi_{24}}$: Fischer's largest 3-transposition group

$\mathbb{M}$: Monster $(\mathbf{n} = \mathbf{6})$ $\quad$ $\mathbb{B}$: Babymonster $(\mathbf{n} = \mathbf{4})$

3-transposition property

$\mathbf{V} = \bigoplus_{\mathbf{n} \geq \mathbf{0}} \mathbf{V_n}$: VOA (over $\mathbb{R}$)

If $\mathbf{V_1} \neq \mathbf{0} \Longrightarrow \langle \mathbf{V_1} \rangle$: affine VOA, $\mathbf{V}$: $\hat{\mathfrak{g}}$-mod, $\exp(\mathfrak{g}) \subset \mathbf{Aut}(\mathbf{V})$

In the following we assume $\mathbf{V_1} = \mathbf{0}$ and $\mathbf{V}$ has a pos def inv form

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Theorem [Miyamoto]

$\langle \sigma_\mathbf{e} \in \mathbf{Aut}(\mathbf{V}) \mid \mathbf{e} : \sigma$-type Ising vector$\rangle$: 3-transposition group

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i.e., $\mathbf{g}\sigma_\mathbf{e}\mathbf{g}^{-1} = \sigma_\mathbf{ge}$ and $\mathbf{g}\tau_\mathbf{e}\mathbf{g}^{-1} = \tau_\mathbf{ge}$ $({}^\forall \mathbf{g} \in \mathbf{Aut}(\mathbf{V}))$

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Theorem [Matsuo]

$\langle \sigma_{\mathbf{e}} \in \mathbf{Aut}(\mathbf{V}) \mid \mathbf{e} : \sigma\text{-type Ising vector} \rangle$: symplectic type $(\not\ni \mathbf{Fi_{24}})$

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Theorem [Sakuma, Lam-Yamada-Y]

Sub VOA generated by two Ising vectors are classified

(9 cases, "$\#$ of nodes in $\mathbf{E}_8^{(1)}$"!!)

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Dihedral VOAs : $U_{1A}, U_{2A}, U_{3A}, U_{4A}, U_{5A}, U_{6A}, U_{4B}, U_{2B}, U_{3C}$

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Dihedral VOAs : $U_{1A}, U_{2A}, U_{3A}, U_{4A}, U_{5A}, U_{6A}, U_{4B}, U_{2B}, U_{3C}$

$$U_{1A} = L(1/2, 0) \longleftrightarrow C_{\mathbb{M}}(1A) = \mathbb{M}$$

$$U_{2A} \supset L(1/2, 0) \otimes L(7/10, 0) \longleftrightarrow C_{\mathbb{M}}(2A) = 2A.\mathbb{B}$$

$$U_{3A} \supset L(4/5, 0) \otimes L(6/7, 0) \longleftrightarrow C_{\mathbb{M}}(3A) = 3A.Fi_{24}$$

4-transposition property

Fix an Ising vector $\mathbf{e} \in \mathbf{V}$ and set

$$\mathbf{I} := \{\mathbf{f} \in \mathbf{V} \mid \mathbf{f} : \text{Ising vector}, \quad (\mathbf{e}|\mathbf{f}) = 2^{-5}\}$$

$$\mathbf{G} := \langle \tau_{\mathbf{f}} \in \mathbf{Aut}(\mathbf{V}) \mid \mathbf{f} \in \mathbf{I} \rangle$$

$$\mathbf{W} := \mathbf{Com}_{\mathbf{V}}(\mathbf{Vir}(\mathbf{e})) \subset \mathbf{V}$$

Define

$$\begin{aligned} \varphi_{\mathbf{e}} : \quad G &\longrightarrow \mathbf{Aut}(\mathbf{W}) \\ \mathbf{g} &\longmapsto \mathbf{g}|_{\mathbf{W}} \end{aligned}$$

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Lemma $\varphi_{\mathbf{e}}$: well-defined group homomorphism

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Fix an Ising vector $\mathbf{e} \in \mathbf{V}$ and set

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$$\mathbf{G} := \langle \tau_{\mathbf{f}} \in \mathbf{Aut}(\mathbf{V}) \mid \mathbf{f} \in \mathbf{I} \rangle$$

$$\mathbf{W} := \mathbf{Com}_{\mathbf{V}}(\mathbf{Vir}(\mathbf{e})) \subset \mathbf{V}$$

Define $$\begin{array}{rccc} \varphi_{\mathbf{e}} : & G & \longrightarrow & \mathbf{Aut}(\mathbf{W}) \\ & \mathbf{g} & \longmapsto & \mathbf{g}|_{\mathbf{W}} \end{array}$$

Lemma $\varphi_{\mathbf{e}}$: well-defined group homomorphism

Theorem [Höhn-Lam-Y]

$\langle \varphi_{\mathbf{e}}(\tau_{\mathbf{f}}) \in \mathbf{Aut}(\mathbf{W}) \mid \mathbf{f} \in \mathbf{I} \rangle$: 4-transposition group

3-transposition property

Fix an Ising vector $\mathbf{e} \in \mathbf{V}$ and set

$$\mathbf{J} := \{\mathbf{f} \in \mathbf{V} \mid \mathbf{f} : \text{Ising vector}, \quad (\mathbf{e}|\mathbf{f}) = 13 \cdot 2^{-10}, \underline{(\cdots)}\}$$

$$\mathbf{H} := \langle \tau_{\mathbf{f}} \in \mathbf{Aut}(\mathbf{V}) \mid \mathbf{f} \in \mathbf{J} \rangle$$

$$\mathbf{X} := \mathbf{Com}_{\mathbf{V}}(\mathbf{Vir}(\mathbf{a}_{\mathbf{e},\mathbf{J}})) \subset \mathbf{V} \quad (\mathbf{Vir}(\mathbf{a}_{\mathbf{e},\mathbf{J}}) \simeq \mathbf{L}(4/5, 0))$$

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Application to the sporadic simple groups

$\mathbf{V}^\natural$: the moonshine VOA over $\mathbb{R}$ [FLM]

$$\mathbf{V}^\natural = \bigoplus_{n \geq 0} \mathbf{V}^\natural_n, \quad \mathbf{V}^\natural_0 = \mathbb{R}\mathbb{1}, \quad \mathbf{V}^\natural_1 = 0,$$ has a pos def inv form

$\mathrm{Aut}(\mathbf{V}^\natural) = \mathbb{M}$: Monster, the largest sporadic finite simple group

There are two conjugacy classes of involutions in $\mathbb{M}$:

$$C_{\mathbb{M}}(2A) \simeq 2.\mathbb{B} \qquad C_{\mathbb{M}}(2B) \simeq 2^{1+24}_{+}.Co_1$$

where $A.B$ stands for an extension $1 \to A \to A.B \to B \to 1$

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Theorem [Conway, Miyamoto, Matsuo, Höhn]

Ising vectors in $\mathbf{V}^\natural \ni e \overset{1:1}{\longleftrightarrow} \tau_e \in \mathbb{M}$: 2A-elements

$(\mathbb{M}, \{\tau_e\}_{e\in \mathbf{V}^\natural})$: 6-transposition group [Sakuma]

Application to the sporadic simple groups

Fix an Ising vector $\mathbf{e} \in V^\natural$

$V\!B^{\natural,0} := \mathrm{Com}_{V^\natural}(\mathrm{Vir}(\mathbf{e}))$: the Babymonster VOA

Theorem [Höhn, Y] $\mathrm{Aut}(V\!B^{\natural,0}) \simeq \mathbb{B}$: Babymonster

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Recall $\mathbf{I} = \{\mathbf{f} \in V^\natural \mid \mathbf{f}$: Ising vector, $(\mathbf{e}|\mathbf{f}) = 2^{-5}\}$

$\Longrightarrow (\mathbb{B}, \{\varphi_{\mathbf{e}}(\tau_{\mathbf{f}})\}_{\mathbf{f}\in\mathbf{I}})$: 4-transposition group

Lemma $\varphi_{\mathbf{e}}(\tau_{\mathbf{f}})$: 2A-element of $\mathbb{B}$ $(C_{\mathbb{B}}(2A) \simeq 2.^2E_6(2).2)$

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Theorem [Höhn-Lam-Y]

2A-elements of $\mathbb{B}$ $\overset{1:1}{\longleftrightarrow}$ $\{\varphi_{\mathbf{e}}(\tau_{\mathbf{f}}) \mid \mathbf{f} \in I \subset V^\natural\}$

$\overset{1:1}{\longleftrightarrow}$ $L(7/10, 0) \hookrightarrow VB^{\natural,0}$ (σ-type)

Application to the sporadic simple groups

Recall $J = \{f \in V^\natural \mid f : \text{Ising vector}, \ (e|f) = 13 \cdot 2^{-10}, \underline{(\cdots)}\}$

$\rightsquigarrow a_{e,J} \in V^\natural : c = {}^4/_5$ Virasoro vector

$$VF^\natural := \mathrm{Com}_{V^\natural}(\mathrm{Vir}(a_{e,J})) \subset V^\natural$$

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Proposition [Höhn-Lam-Y]

$\mathrm{Aut}(VF^\natural) \supset \mathrm{Fi}_{24}$: the largest Fischer group

$\Longrightarrow (\mathrm{Fi}_{24}, \{\psi_{\mathbf{a}_{\mathbf{e},\mathbf{J}}}(\tau_{\mathbf{f}})\}_{\mathbf{f}\in\mathbf{J}})$: 3-transposition group

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Proposition [Höhn-Lam-Y]

$\mathrm{Aut}(VF^\natural) \supset Fi_{24}$: the largest Fischer group

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Theorem [Höhn-Lam-Y]

2C-elements of Fi_{24} $\overset{1:1}{\longleftrightarrow}$ $\{\psi_{a_{e,J}}(\tau_f) \mid f \in J \subset V^\natural\}$

$\overset{1:1}{\longrightarrow}$ $L({}^6\!/_7, 0) \hookrightarrow VF^\natural$ (σ-type)

McKay's observation

$$\begin{array}{ccc} \mathbb{M} \supset \text{2A-elements} & \rightsquigarrow & \text{affine } \mathbf{E_8} \text{ diagram} \\ \Updownarrow & & \Updownarrow \\ \mathbf{L}(^1/_2, 0) \hookrightarrow \mathbf{V}^\natural & \longleftrightarrow & \mathbf{L}(^1/_2, 0) \hookrightarrow \mathbf{V}_{\sqrt{2}\mathbf{E_8}} \end{array}$$

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$$\mathbf{Fi_{24}} \supset \text{2C-elements} \quad \rightsquigarrow \quad \text{affine } \mathbf{E_6} \text{ diagram}$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$\mathbf{L}(6/7, 0) \hookrightarrow \mathbf{VF}^\natural \quad \longleftrightarrow \quad \mathbf{L}(6/7, 0) \hookrightarrow \mathbf{V}_{\sqrt{2}\mathbf{E_6}}$$

Automorphisms of commutant subalgebras

$\mathsf{N}_{\mathbb{M}}(\mathsf{5A}) = \mathsf{D}_{10} \times \mathsf{HN}$ $\Longrightarrow$ ${}^{\exists}\mathsf{VH} \subset \mathsf{V}^{\natural}$

U_{5A} : known (W_5-algebra) $\qquad$ $\mathsf{Aut}(\mathsf{VH}) \supset \mathsf{HN}$

$\mathsf{N}_{\mathbb{M}}(\mathsf{3C}) = \mathsf{Sym}_3 \times \mathsf{Th}$ $\Longrightarrow$ ${}^{\exists}\mathsf{VT} \subset \mathsf{V}^{\natural}$

U_{3C} : known (W_9-algebra) $\qquad$ $\mathsf{Aut}(\mathsf{VT}) \supset \mathsf{Th}$

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$N_{\mathbb{M}}(5A) = D_{10} \times HN$

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$\Longrightarrow$

${}^{\exists}VH \subset V^{\natural}$

$\mathrm{Aut}(VH) \supset HN$

$N_{\mathbb{M}}(3C) = \mathrm{Sym}_3 \times Th$

U_{3C} : known (W_9-algebra)

$\Longrightarrow$

${}^{\exists}VT \subset V^{\natural}$

$\mathrm{Aut}(VT) \supset Th$

To Be Continued!