

Mapping class group representations and conformal boundary conditions

Joint work with J Fuchs & C Stigner

Christoph Schweigert

Department of Mathematics
and Center for Mathematical Physics
University of Hamburg

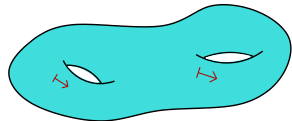
September, 2009

Introduction: Topological field theory

Fact [Reshetikhin-Turaev]:

Modular tensor category \mathcal{C} gives 3d Topological Field Theory
i.e. tensor functor

$$\text{tft}_{\mathcal{C}} : 3\text{-cobord}(\mathcal{C}) \rightarrow \text{Vect}_{\text{fin}}(\mathbb{C})$$

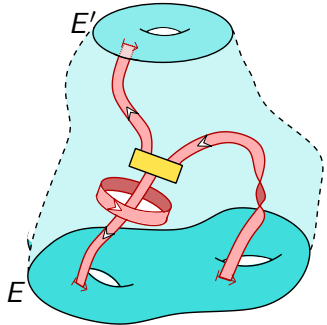


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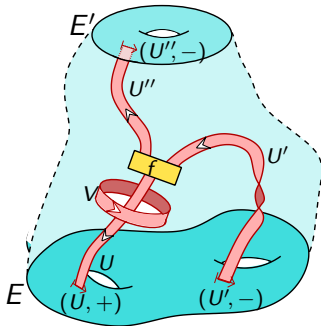
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$$\text{tft}_{\mathcal{C}} : 3\text{-cobord}(\mathcal{C}) \rightarrow \text{Vect}_{\text{fin}}(\mathbb{C})$$

In particular:

- $\text{tft}_{\mathcal{C}}(\emptyset) = \mathbb{C}$
- $\text{tft}_{\mathcal{C}}(\emptyset \xrightarrow{M} \partial M) 1 \in \text{tft}_{\mathcal{C}}(\partial M)$
- Representations of mapping class groups of X on $\text{tft}_{\mathcal{C}}(X)$



Introduction: TFT construction of RCFT correlators

TFT-construction of Rational Conformal Field Theory (\mathcal{C}, F) :

- Correlators of full RCFT on X :

$\text{Cor}(X) \in \text{tft}_{\mathcal{C}}(\hat{X}) = \text{space of conformal blocks on } \hat{X}$

- TFT-construction: $\text{Cor}(X) = \text{tft}_{\mathcal{C}}(\emptyset \xrightarrow{M_X} \hat{X}) \mathbf{1}$ Invariant under mapping class group of X , compatible with factorization

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- Decoration data: Modular tensor category \mathcal{C} with special symmetric *Frobenius algebra* F :
 - Algebra and coalgebra F in \mathcal{C}
 - Coproduct $\Delta \in \text{Hom}(F, F \otimes F)$ morphism of F -bimodules
 - F also special and symmetric

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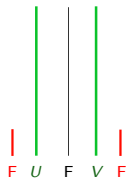
- Dictionary:

Physics	Mathematics
Boundary conditions	Left F -modules
\vdots	\vdots

- \rightsquigarrow Study modules over F internally in \mathcal{C}

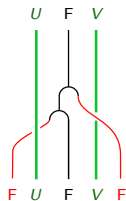
A classifying algebra

- $\{U_i \mid i \in \mathcal{I}\}$: representatives of the isomorphism classes of simple objects of \mathcal{C}
- F special symmetric Frobenius algebra in \mathcal{C}
- For any $U, V \in \text{Obj}(\mathcal{C})$: F -bimodule $U \otimes^+ F \otimes^- V$:



A classifying algebra

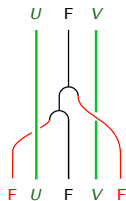
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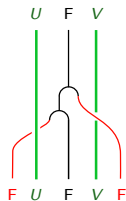
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- For any pair of F -bimodules D, \tilde{D} :

$$\text{Hom}_{F|F}(D, \tilde{D}) := \{\phi \in \text{Hom}(D, \tilde{D}) \mid \phi \text{ intertwining actions of } F\}$$



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Idea:

Endow the vector space $\mathcal{A} := \bigoplus_{i \in \mathcal{I}} \text{Hom}_{F|F}(U_i \otimes^+ F \otimes^- U_i, F)$

with the structure of a commutative semisimple associative algebra over \mathbb{C} such that the irreps of \mathcal{A} describe elementary boundary conditions, including reflection coefficients

A classifying algebra

- Homogeneous component $\mathcal{A}_i := \text{Hom}_{F|F}(U_i \otimes^+ F \otimes^- U_i, F)$ of dimension $Z_{i\bar{i}}$

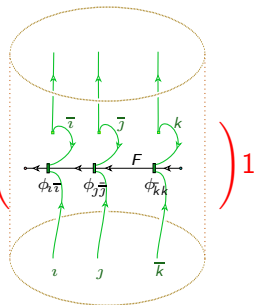
A classifying algebra

- Homogeneous component $\mathcal{A}_i := \text{Hom}_{F|F}(U_i \otimes^+ F \otimes^- U_{\bar{i}}, F)$ of dimension $Z_{i\bar{i}}$
- Non-degenerate bilinear pairing

$$c_{k\bar{k}}^{\text{bulk}} : \mathcal{A}_k \otimes \mathcal{A}_{\bar{k}} \rightarrow \mathbb{C}$$

- Define multiplication: $\mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ for $\Phi_i \in \mathcal{A}_i$ by:

$$c^{\text{bulk}}(\Phi_{i\bar{i}} \star \Phi_{j\bar{j}}, \Phi_{\bar{k}k}) := \frac{\theta_k \dim(U_k)}{S_{0,0}} \text{tftc} \left(\begin{array}{c} \text{Diagram} \end{array} \right) 1$$



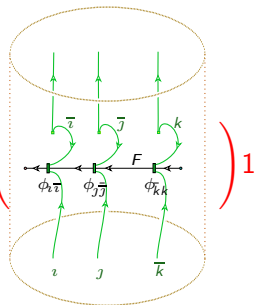
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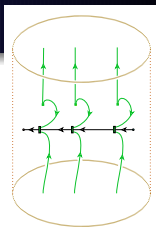
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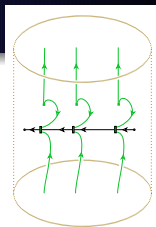
- F Morita equivalent to $\mathbf{1} \Rightarrow$ the Verlinde algebra
- The invariant of the ribbon graph in $S^2 \times S^1$ is given by the trace of an endomorphism of conformal 3-point blocks on the sphere

Mapping class group representations

- Consequence of the TFT-axioms: space of conformal blocks carries a representation of the mapping class group

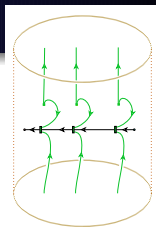


Mapping class group representations



- Consequence of the TFT-axioms: space of conformal blocks carries a representation of the mapping class group
- Properties of F and $\phi_\alpha, \phi_\beta, \phi_\gamma$ morphisms of bimodules \Rightarrow intertwiners of the action of the mapping class group
- Analogously: intertwiners of for conformal n -point blocks on the sphere for any $n \rightsquigarrow n$ -ary product on \mathcal{A}

Mapping class group representations



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Theorem, the classifying algebra

\mathcal{C} - modular tensor category

F - special symmetric Frobenius algebra in \mathcal{C}

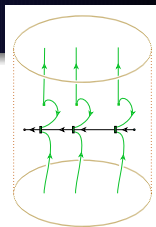
- The vector space $\mathcal{A} := \bigoplus_{i \in \mathcal{I}} \text{Hom}_{F|F}(U_i \otimes^+ F \otimes^- U_i, F)$

with the product from intertwiners of mapping class group action is a semisimple commutative associative algebra over \mathbb{C}

- The representation functions of irreps are reflection coefficients

Mapping class group representations

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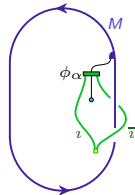
Theorem, the classifying algebra

\mathcal{C} - modular tensor category

F - special symmetric Frobenius algebra

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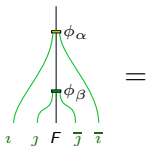
$$\rho_M(\phi_{i\alpha}) = \frac{1}{\dim(M)}$$



- The representation functions of irreps are reflection coefficients

Commutativity of \mathcal{A}

Using F Frobenius and ϕ_α, ϕ_β bimodule morphisms:



Commutativity of \mathcal{A}

Using F Frobenius and ϕ_α, ϕ_β bimodule morphisms:

The diagram illustrates the commutativity of the algebra \mathcal{A} using Frobenius and bimodule morphisms. It consists of three diagrams connected by equals signs, all with a single vertical line at the top and five lines at the bottom labeled $i, j, F, \bar{j}, \bar{i}$.

- The first diagram shows a vertical line at the top with a yellow box labeled ϕ_α . Two green lines descend from ϕ_α to a green box labeled ϕ_β . From ϕ_β , two green lines descend to the bottom labels j and \bar{j} , and a vertical line descends to the bottom label F .
- The second diagram shows a vertical line at the top with a yellow box labeled ϕ_α . A green line descends from ϕ_α to a green box labeled ϕ_β . From ϕ_β , a green line descends to the bottom label j , and a vertical line descends to the bottom label F . A curved green line starts from the top, goes left, then down, then right, crossing the vertical line from ϕ_β before descending to the bottom label \bar{j} .
- The third diagram is identical to the second.

Commutativity of \mathcal{A}

Using F Frobenius and ϕ_α, ϕ_β bimodule morphisms:

The diagram illustrates the commutativity of the algebra \mathcal{A} using the Frobenius F and bimodule morphisms ϕ_α, ϕ_β . The sequence of diagrams shows the following transformations:

- The initial diagram shows the composition of ϕ_α and ϕ_β with the Frobenius F .
- The second diagram shows the application of the Frobenius property, where the top part of the diagram is rearranged.
- The third diagram shows the application of the bimodule morphism property, where the bottom part of the diagram is rearranged.
- The final diagram shows the result of the transformations, demonstrating the commutativity of the algebra.

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- A tree structure with a root node ϕ_α and a child node ϕ_β . The bottom labels are $i, j, F, \bar{j}, \bar{i}$.
- The same tree structure, but with the ϕ_β node moved to the left, crossing over the F node.
- The tree structure with the ϕ_β node moved further left, crossing over the j node.
- The tree structure with the ϕ_β node moved to the right, crossing over the F node.
- The tree structure with the ϕ_β node moved to the right, crossing over the \bar{j} node.

The diagrams are connected by equals signs, indicating that the algebra is commutative.

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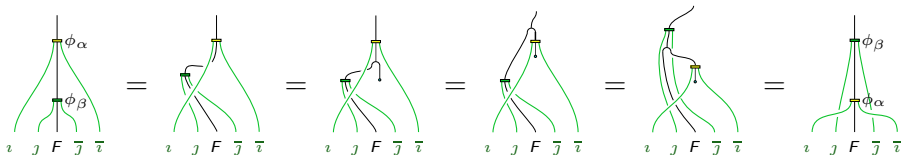
The diagram illustrates the commutativity of the algebra \mathcal{A} using the Frobenius F and bimodule morphisms ϕ_α, ϕ_β . The sequence of diagrams shows the following transformations:

- Diagram 1: A network of green and black strands. The top node is labeled ϕ_α and the bottom node is labeled ϕ_β . The strands are labeled $i, j, F, \bar{j}, \bar{i}$ at the bottom.
- Diagram 2: The same network, but the strands are rearranged to show a different configuration.
- Diagram 3: The same network, with further rearrangement of the strands.
- Diagram 4: The same network, with further rearrangement of the strands.
- Diagram 5: The same network, with further rearrangement of the strands.

The sequence of diagrams is connected by equals signs, indicating that the algebra \mathcal{A} is commutative.

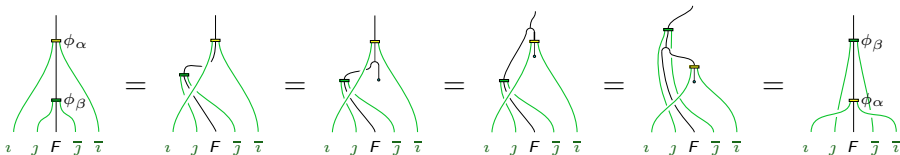
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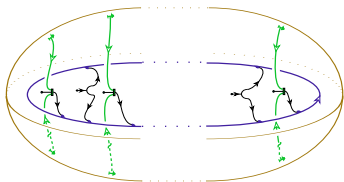


\implies

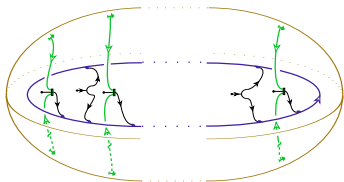
$$c_{i\alpha, j\beta}^{k\gamma} = \frac{\theta_k \dim(U_k)}{S_{0,0}} \sum_{\delta=1}^{Z_{k\bar{k}}} (c_{k\bar{k}}^{\text{bulk}})_{\delta\gamma}^{-1} \text{tft}_{\mathcal{C}} \left(\begin{array}{c} \text{Diagram of } F \text{ and } \phi_\alpha, \phi_\beta, \phi_\delta \end{array} \right) 1$$

is symmetric in the pairs $(i\alpha), (j\beta)$

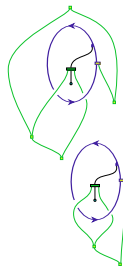
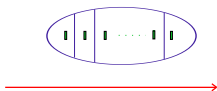
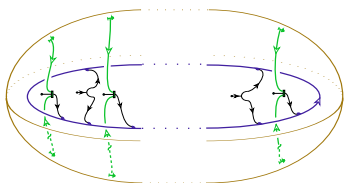
Derivation of \mathcal{A} : Overview



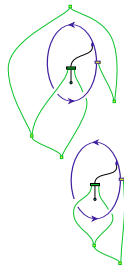
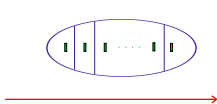
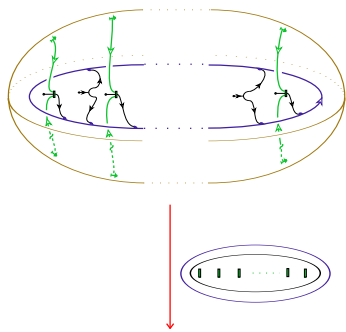
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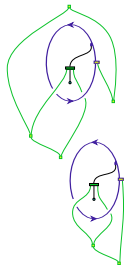
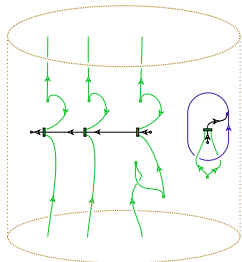
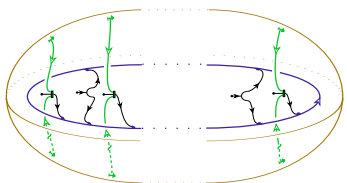
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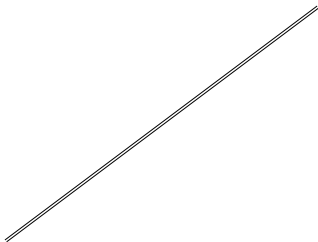
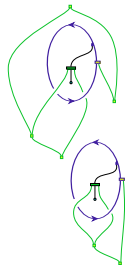
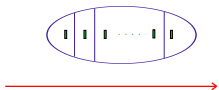
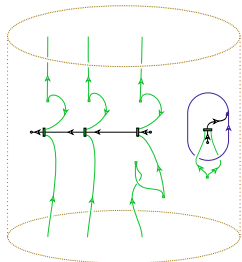
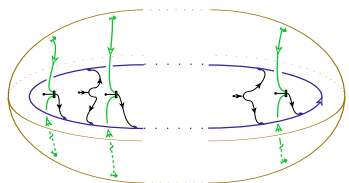
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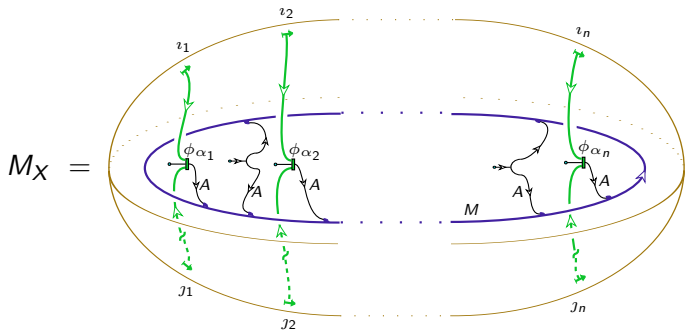
Derivation of \mathcal{A} : Overview



Boundary factorization - A movie I: the correlator

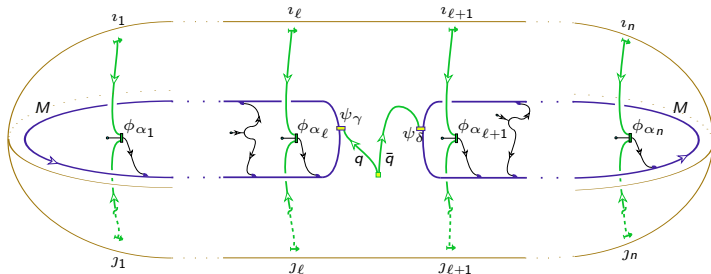
n Bulk fields on a disc with boundary condition $M \in A - \text{mod}$

$$\text{Cor}(X) = \text{tft}_C(M_X)1$$

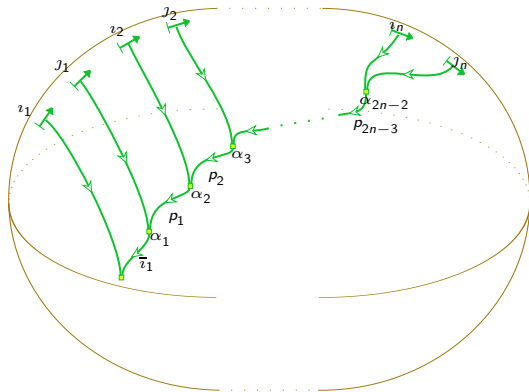


Boundary factorization - A movie II: using dominance

Sum over invariants of

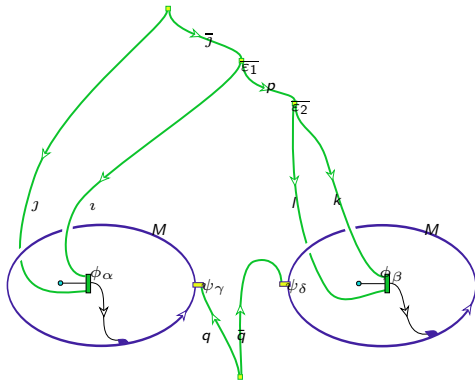


Boundary factorization - A movie III: the basis

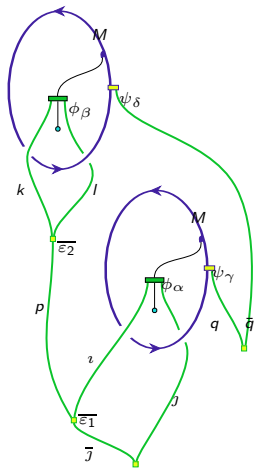


Boundary factorization - A movie IV: after evaluation on the basis

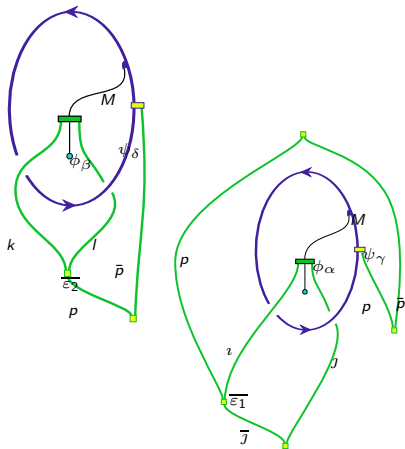
Now $n = 2$ Bulk Fields, coefficient in the basis



Boundary factorization - A movie V



Boundary factorization - A movie VI: again dominance



Projecting on $p = 0 \Rightarrow$

$$c(\Phi_{\alpha_1}, \Phi_{\alpha_2}, \dots, \Phi_{\alpha_n}; M)_0 = ((c_{M,0}^{\text{bnd}})^{-1})^{n-1} \prod_{i=1}^n c(\Phi_{\alpha_i}; M)$$

Boundary factorization - Result

Boundary factorization

$$c(\Phi_{\alpha_1}, \Phi_{\alpha_2}, \dots, \Phi_{\alpha_n}; M)_0 = ((c_{M,0}^{\text{bnd}})^{-1})^{n-1} \prod_{i=1}^n c(\Phi_{\alpha_i}; M)$$

Different normalization:

$$b(\Phi_{\alpha_1}, \dots, \Phi_{\alpha_n}; M)_0 := c(\Phi_{\alpha_1}, \dots, \Phi_{\alpha_n}; M)_0 / c_{M,0}^{\text{bnd}}$$

In particular: Reflection coefficients

$$b_M^{\lambda, \alpha} \equiv b(\Phi_{\alpha}; M) = (c_{M,0}^{\text{bnd}})^{-1} c(\Phi_{\alpha}^{\bar{\lambda}}; M)$$

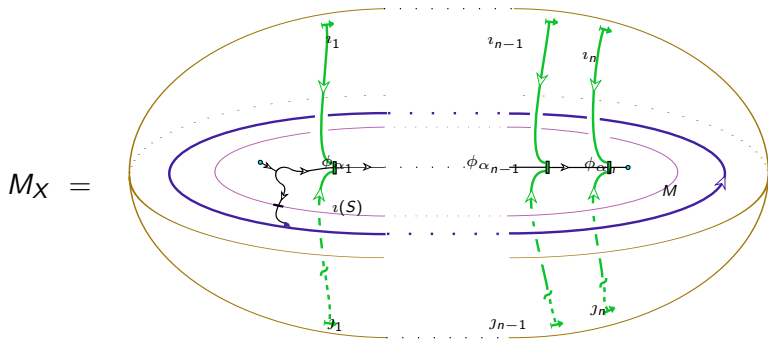
Thus

$$b(\Phi_{\alpha_1}, \Phi_{\alpha_2}, \dots, \Phi_{\alpha_n}; M)_0 = \prod_{i=1}^n b(\Phi_{\alpha_i}; M)$$

Bulk factorization - A movie I: the correlator

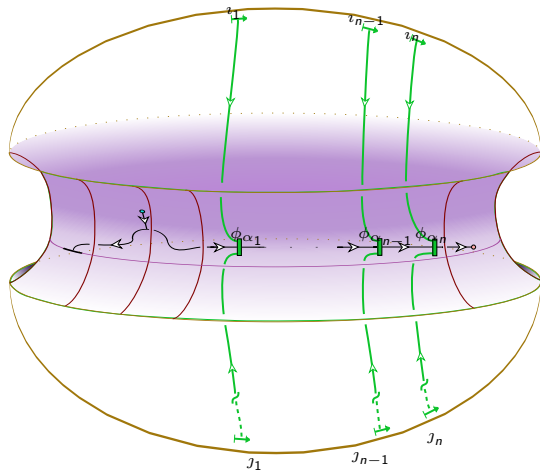
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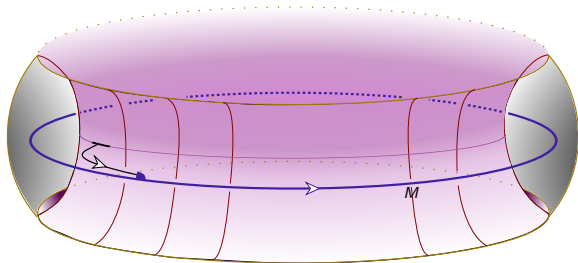
Bulk factorization - A movie II: The nibbled apple

$$M_X^{\circ,1} =$$

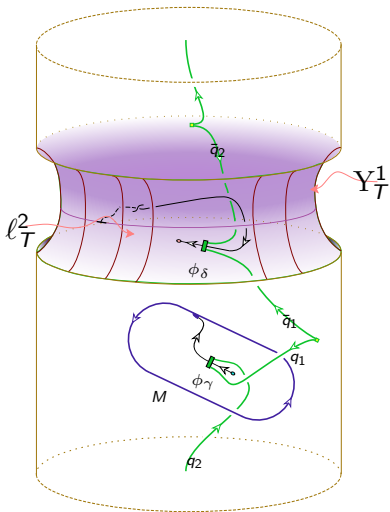


Bulk factorization - A movie III: The bigonal doughnut

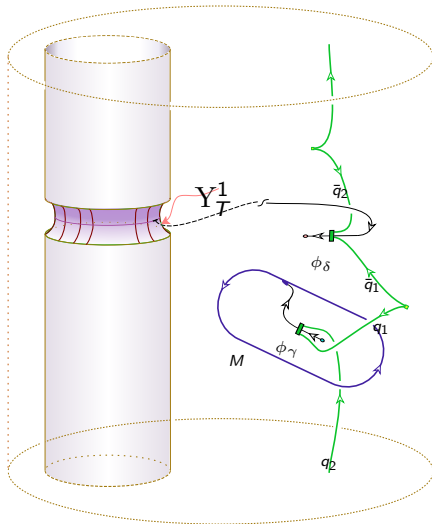
$$M_X^{o,2} =$$



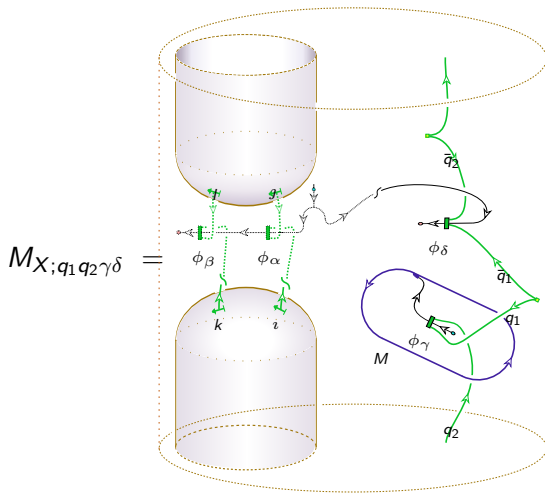
Bulk factorization - A movie V: gluing the doughnut



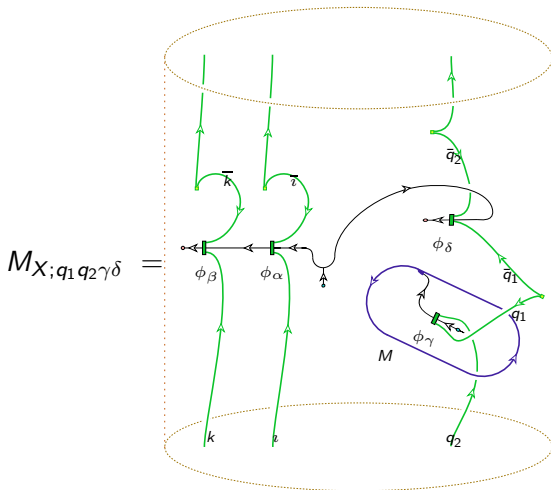
Bulk factorization - A movie VI: inside out



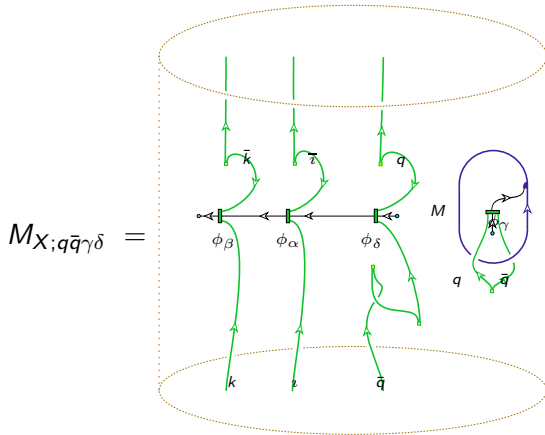
Bulk factorization - A movie VII: gluing the nibbled apple



Bulk factorization - A movie VIII: after evaluation on the basis



Bulk factorization - A movie IX: again dominance



$$c(\Phi_\alpha, \Phi_\beta; M)_0 = \frac{1}{S_{0,0}} \sum_{q \in \mathcal{I}} \sum_{\gamma, \delta} \theta_q \dim(U_q) (c_{q\bar{q}}^{\text{bulk}})^{-1} \frac{1}{\delta_\gamma} K_{\beta\alpha\delta}^{kz\bar{q}} c(\Phi_\gamma; M)$$

The classifying algebra

- The vector space: $\mathcal{A} := \bigoplus_{i \in \mathcal{I}} \text{Hom}_{F|F}(U_i \otimes^+ F \otimes^- U_i, F)$

has a natural structure of a semisimple commutative associative algebra over \mathbb{C} given by a family of intertwiners of the action of the mapping class group on the space of conformal 3-point blocks on the sphere

- Irreps of F in $\mathcal{C} \rightsquigarrow$ irreps of \mathcal{A}
- The representation functions of \mathcal{A} are reflection coefficients

Outlook

- Classifying algebra for defects
- More explicit formulas, i.e. the case F Schellekens algebra
- Classifying algebras in more general situations (rational logarithmic conformal field theories, lattice models,...)