Mapping class group representations and conformal boundary conditions

Joint work with J Fuchs & C Stigner

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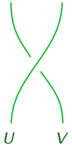
Background: Vertex algebras and modular tensor categories

- Huang Lepowsky: Tensor structure on representation category of vertex operator algebras
- Huang (2004): Conditions on a conformal vertex algebra that ensure that its representation category is a modular tensor category
- Definition of a modular tensor category
  - Abelian, semisimple, noetherian category enriched over $\mathcal{V}ect_{\text{fin}}(\mathbb{C})$
  - Ribbon category, in particular braided: $c_{U,V} \in \text{Hom}(U \otimes V, V \otimes U)$:
  - Non-degeneracy condition on braiding

$c_{U,V} =$

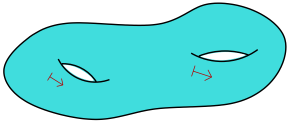


- Examples from untwisted affine Lie algebras

Fact [Reshetikhin-Turaev]:

Modular tensor category $\mathcal{C}$ gives 3d Topological Field Theory
i.e. tensor functor

$$\mathrm{tft}_{\mathcal{C}} : \text{3-cobord}(\mathcal{C}) \to \mathcal{V}ect_{\mathrm{fin}}(\mathbb{C})$$



Introduction: Topological field theory

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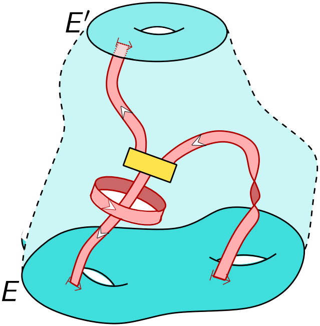

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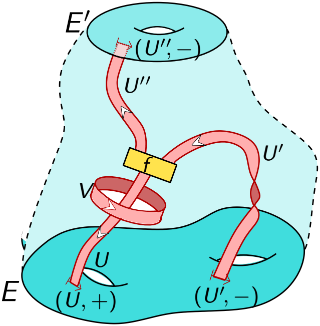

In particular:

- $\mathrm{tft}_{\mathcal{C}}(\emptyset) = \mathbb{C}$
- $\mathrm{tft}_{\mathcal{C}}(\emptyset \xrightarrow{M} \partial M)1 \in \mathrm{tft}_{\mathcal{C}}(\partial M)$
- Representations of mapping class groups of X on $\mathrm{tft}_{\mathcal{C}}(X)$

TFT-construction of Rational Conformal Field Theory $(\mathcal{C}, F)$:

- Correlators of full RCFT on X:

$$\mathrm{Cor}(X) \in \mathrm{tft}_{\mathcal{C}}(\hat{X}) = \text{space of conformal blocks on } \hat{X}$$

- TFT-construction: $\mathrm{Cor}(X) = \mathrm{tft}_{\mathcal{C}}(\emptyset \xrightarrow{M_X} \hat{X})1$ Invariant under mapping class group of X, compatible with factorization

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- Decoration data: Modular tensor category $\mathcal{C}$ with special symmetric *Frobenius algebra* F:
  - Algebra and coalgebra F in $\mathcal{C}$
  - Coproduct $\Delta \in \mathrm{Hom}(F, F \otimes F)$ morphism of F-bimodules
  - F also special and symmetric

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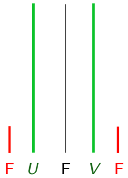
- Dictionary:

Physics	Mathematics
Boundary conditions	Left F-modules
⋮	⋮

- $\rightsquigarrow$ Study modules over F internally in $\mathcal{C}$

A classifying algebra

- $\{U_\imath \,|\, \imath \in \mathcal{I}\}$: representatives of the isomorphism classes of simple objects of $\mathcal{C}$
- F special symmetric Frobenius algebra in $\mathcal{C}$
- For any $U, V \in \mathcal{O}bj(\mathcal{C})$: F-bimodule $U \otimes^+ F \otimes^- V$:



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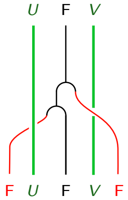

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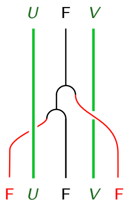
- For any pair of F-bimodules $D, \tilde{D}$:

$$\mathrm{Hom}_{F|F}(D, \tilde{D}) := \{\phi \in \mathrm{Hom}(D, \tilde{D}) \,|\, \phi \text{ intertwining actions of } F\}$$



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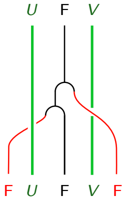

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Idea:

Endow the vector space $\mathcal{A} := \bigoplus_{\imath \in \mathcal{I}} \mathrm{Hom}_{F|F}(U_\imath \otimes^+ F \otimes^- U_{\bar{\imath}}, F)$ with the structure of a commutative semisimple associative algebra over $\mathbb{C}$ such that the irreps of $\mathcal{A}$ describe elementary boundary conditions, including reflection coefficients

A classifying algebra

- Homogeneous component $\mathcal{A}_\imath := \mathrm{Hom}_{F|F}(U_\imath \otimes^+ F \otimes^- U_{\bar\imath}, F)$ of dimension $Z_{\imath\bar\imath}$

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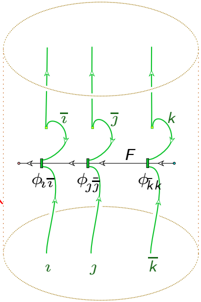
- Non-degenerate bilinear pairing

$$c^{\mathrm{bulk}}_{k\bar{k}} : \mathcal{A}_k \otimes \mathcal{A}_{\bar{k}} \to \mathbb{C}$$

- Define multiplication: $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ for $\Phi_\imath \in \mathcal{A}_\imath$ by:

$$c^{\mathrm{bulk}}(\Phi_{\imath\bar\imath} \star \Phi_{\jmath\bar\jmath}, \Phi_{\bar{k}k}) := \frac{\theta_k \dim(U_k)}{S_{0,0}} \, \mathrm{tft}_{\mathcal{C}}\Big(\quad \Big) 1$$



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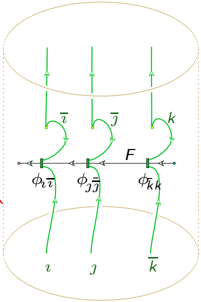
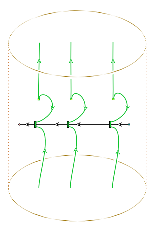


- F Morita equivalent to $\mathbf{1}$ $\Rightarrow$ the Verlinde algebra
- The invariant of the ribbon graph in $S^2 \times S^1$ is given by the trace of an endomorphism of conformal 3-point blocks on the sphere

Mapping class group representations



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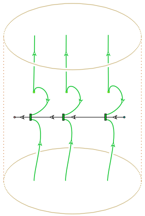
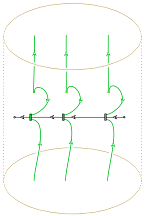


- Consequence of the TFT-axioms: space of conformal blocks carries a representation of the mapping class group
- Properties of F and $\phi_\alpha, \phi_\beta, \phi_\gamma$ morphisms of bimodules $\Rightarrow$ intertwiners of the action of the mapping class group
- Analogously: intertwiners of for conformal n-point blocks on the sphere for any $n \rightsquigarrow n$-ary product on $\mathcal{A}$

Mapping class group representations



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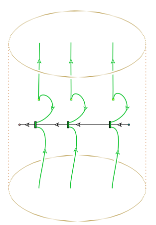
Theorem, the classifying algebra

$\mathcal{C}$ - modular tensor category
F - special symmetric Frobenius algebra in $\mathcal{C}$

- The vector space $\mathcal{A} := \bigoplus_{\imath\in\mathcal{I}} \mathrm{Hom}_{F|F}(U_\imath \otimes^+ F \otimes^- U_{\bar\imath}, F)$

  with the product from intertwiners of mapping class group action is a semisimple commutative associative algebra over $\mathbb{C}$
- The representation functions of irreps are reflection coefficients

Mapping class group representations



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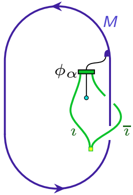
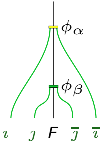
$$\rho_M(\phi_{\imath\alpha}) = \frac{1}{\dim(M)}$$



Commutativity of $\mathcal{A}$

Using F Frobenius and ϕ_α, ϕ_β bimodule morphisms:



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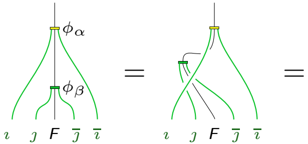

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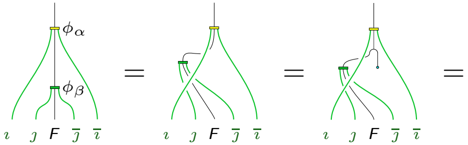

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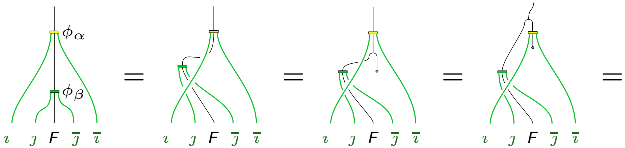

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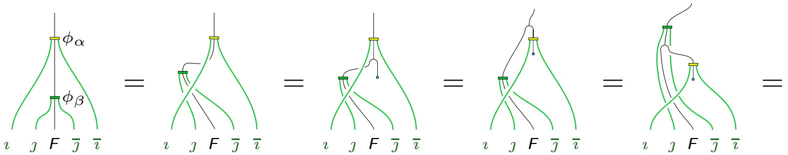

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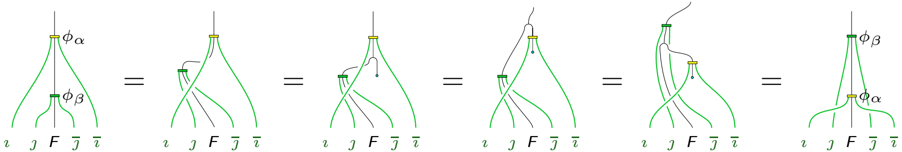

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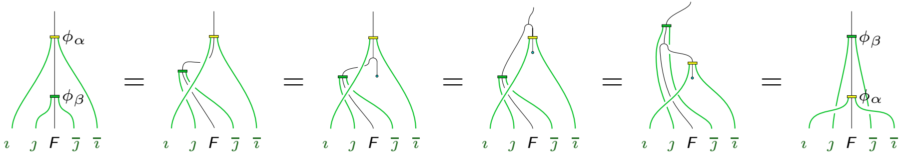
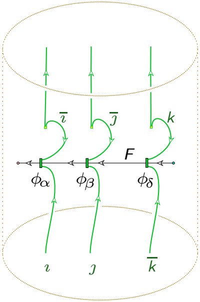


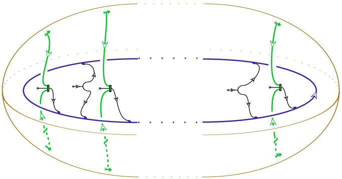
$$\Longrightarrow$$

$$C_{\imath\alpha,\,\jmath\beta}^{\;\;k\gamma} = \frac{\theta_k \dim(U_k)}{S_{0,0}} \sum_{\delta=1}^{Z_{k\bar{k}}} (c^{\text{bulk}}_{k\bar{k}})^{-1}_{\delta\gamma}\ \text{tft}_{\mathcal{C}} \Big(\quad \Big) 1$$



is symmetric in the pairs $(\imath\alpha), (\jmath\beta)$

Derivation of $\mathcal{A}$: Overview



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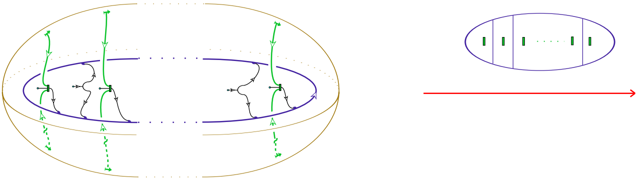

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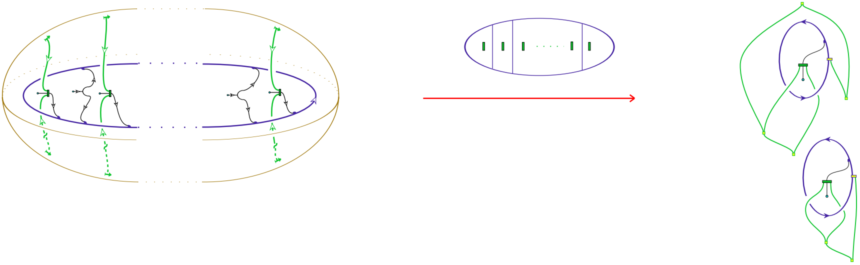

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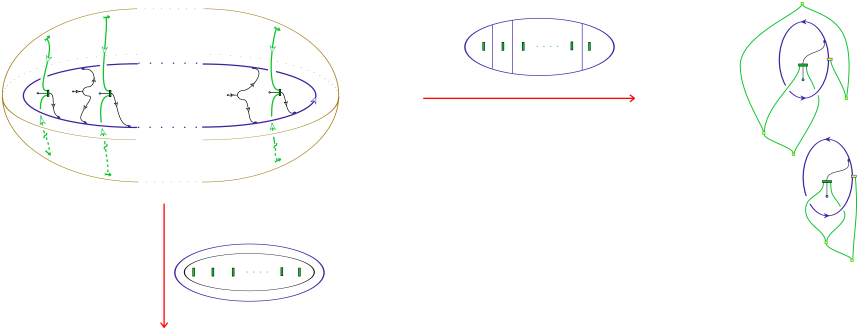

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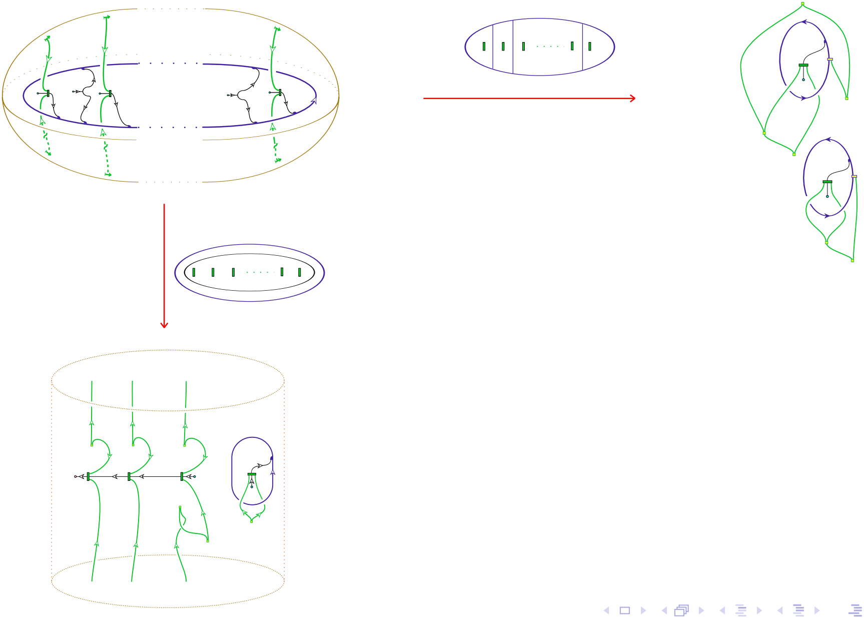

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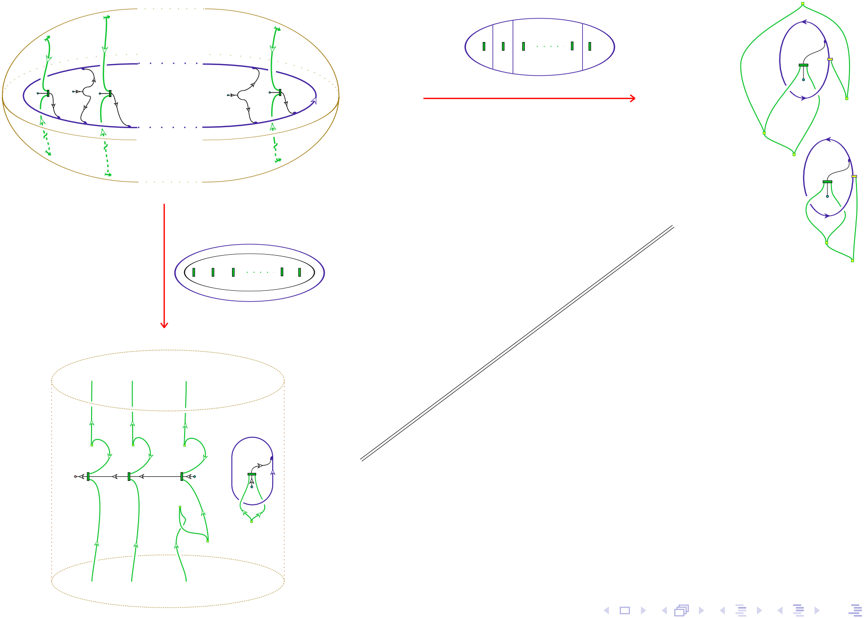
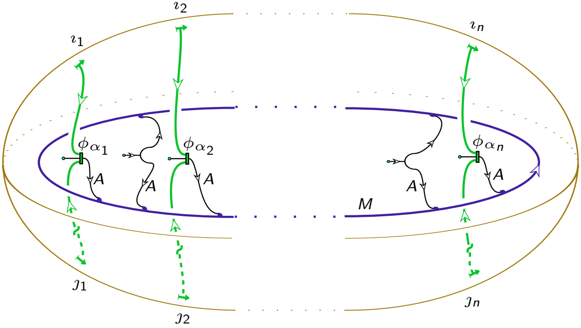


Boundary factorization - A movie I: the correlator

n Bulk fields on a disc with boundary condition $M \in A - \mathrm{mod}$

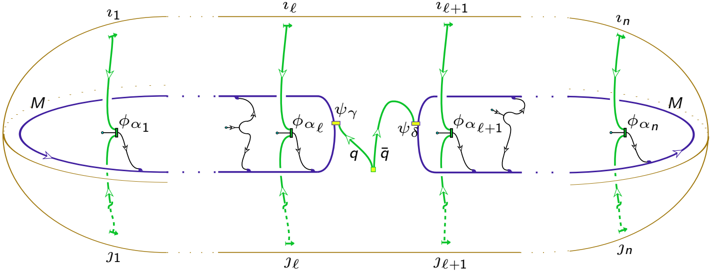
$$Cor(X) = \mathrm{tft}_{\mathcal{C}}(M_X)1$$

$M_X =$

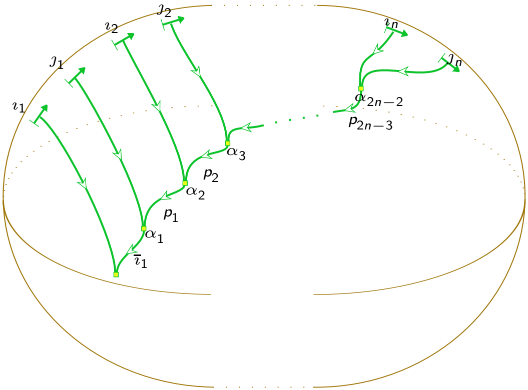


Boundary factorization - A movie II: using dominance

Sum over invariants of

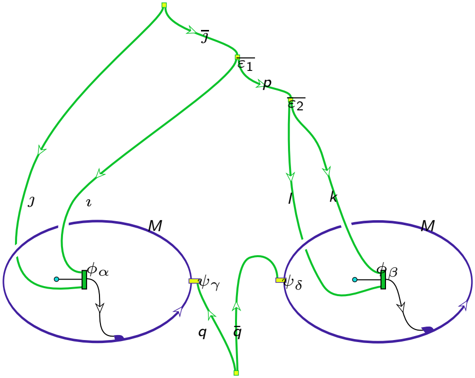


Boundary factorization - A movie III: the basis

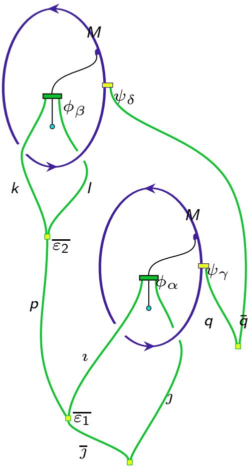


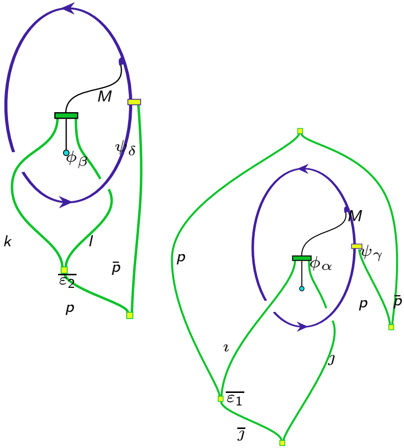
Boundary factorization - A movie IV: after evaluation on the basis

Now $n = 2$ Bulk Fields, coefficient in the basis



Boundary factorization - A movie V





Projecting on $p = 0 \Rightarrow$

$c(\Phi_{\alpha_1}, \Phi_{\alpha_2}, \ldots, \Phi_{\alpha_n}; M)_0 = ((c^{\mathrm{bnd}}_{M,0})^{-1})^{n-1} \prod_{i=1}^{n} c(\Phi_{\alpha_i}; M)$

Boundary factorization - Result

Boundary factorization

$$c(\Phi_{\alpha_1}, \Phi_{\alpha_2}, \dots, \Phi_{\alpha_n}; M)_0 = ((c_{M,0}^{\mathrm{bnd}})^{-1})^{n-1} \prod_{i=1}^{n} c(\Phi_{\alpha_i}; M)$$

Different normalization:

$$b(\Phi_{\alpha_1}, \dots, \Phi_{\alpha_n}; M)_0 := c(\Phi_{\alpha_1}, \dots, \Phi_{\alpha_n}; M)_0 / c_{M,0}^{\mathrm{bnd}}$$

In particular: Reflection coefficients

$$b_M^{\imath,\alpha} \equiv b(\Phi_\alpha; M) = (c_{M,0}^{\mathrm{bnd}})^{-1}\, c(\Phi_\alpha^{\imath\bar{\imath}}; M)$$

Thus

$$b(\Phi_{\alpha_1}, \Phi_{\alpha_2}, \dots, \Phi_{\alpha_n}; M)_0 = \prod_{i=1}^{n} b(\Phi_{\alpha_i}; M)$$

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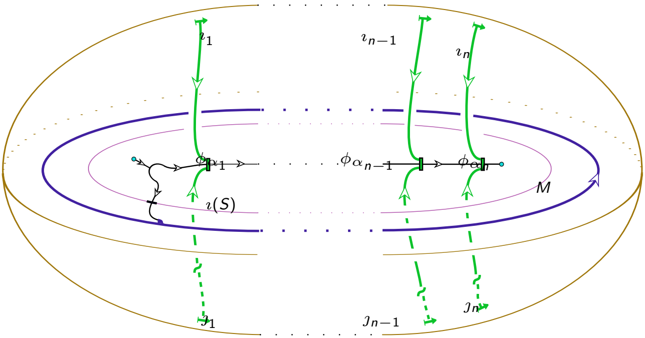
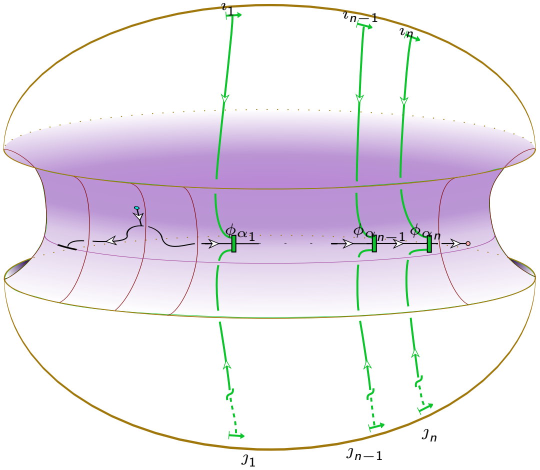
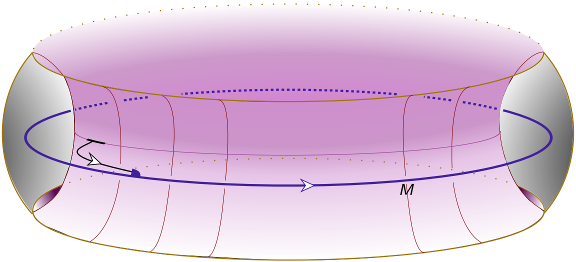


Bulk factorization - A movie II: The nibbled apple

$M_X^{\circ,1} \; =$

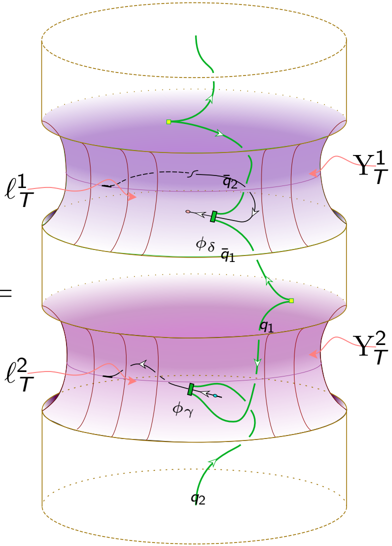


$M_X^{\circ,2} =$

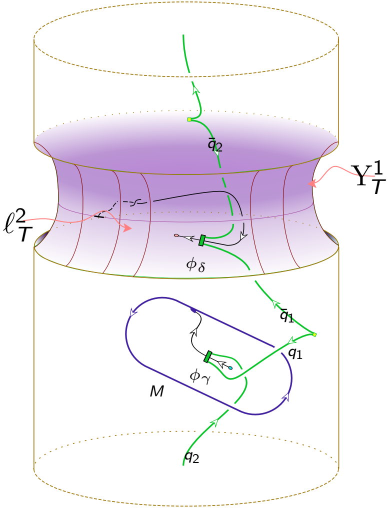


Bulk factorization - A movie IV: The gluing cylinder

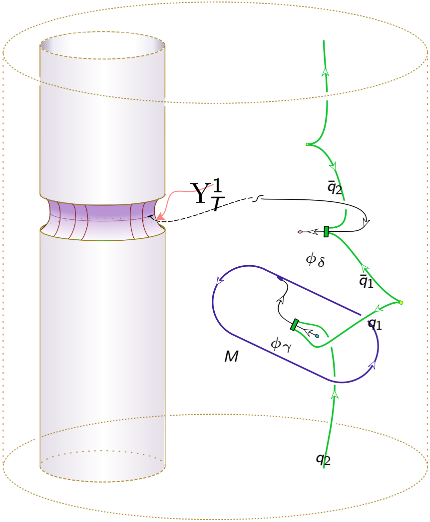
$$T_{q_1 q_2 \gamma \delta} \;=$$



Bulk factorization - A movie V: gluing the doughnut

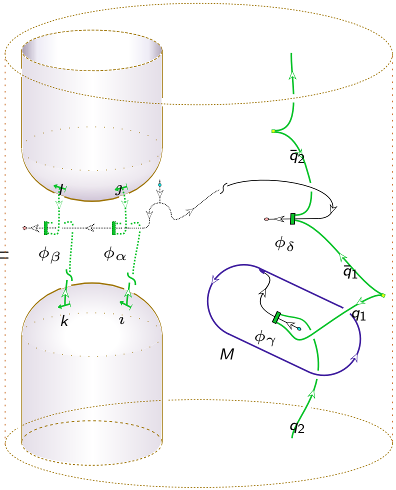


Bulk factorization - A movie VI: inside out



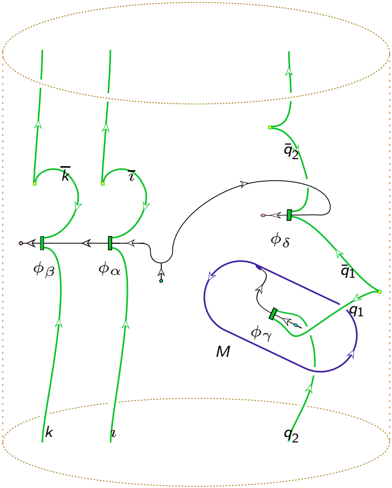
Bulk factorization - A movie VII: gluing the nibbled apple

$M_{X;q_1 q_2 \gamma\delta}$ =



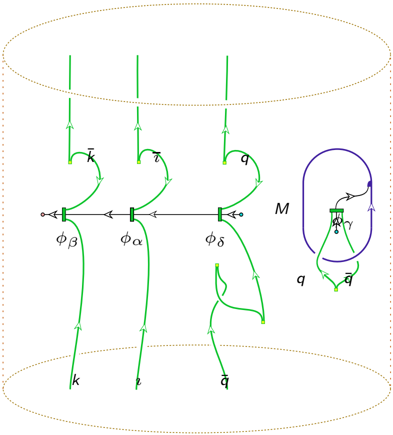
Bulk factorization - A movie VIII: after evaluation on the basis

$M_{X;q_1 q_2 \gamma \delta}$ =



Bulk factorization - A movie IX: again dominance

$M_{X;q\bar{q}\gamma\delta}$ =



$$c(\Phi_\alpha,\Phi_\beta;M)_0 = \frac{1}{S_{0,0}}\sum_{q\in\mathcal{I}}\sum_{\gamma,\delta}\theta_q\,\dim(U_q)\,(c^{\text{bulk}}_{q\bar{q}})^{-1}_{\delta\gamma}\,K^{k\imath\bar{q}}_{\beta\alpha\delta}\,c(\Phi_\gamma;M)$$

Conclusion

The classifying algebra

- The vector space: $\mathcal{A} := \bigoplus_{\imath\in\mathcal{I}} \mathrm{Hom}_{F|F}(U_\imath \otimes^+ F \otimes^- U_{\bar\imath}, F)$

  has a natural structure of a semisimple commutative associative algebra over $\mathbb{C}$ given by a family of intertwiners of the action of the mapping class group on the space of conformal 3-point blocks on the sphere
- Irreps of F in $\mathcal{C} \rightsquigarrow$ irreps of $\mathcal{A}$
- The representation functions of $\mathcal{A}$ are reflection coefficients

<u>Outlook</u>

- Classifying algebra for defects
- More explicit formulas, i.e. the case F Schellekens algebra
- Classifying algebras in more general situations (rational logarithmic conformal field theories, lattice models,...)