On the classification of automorphic products and generalized Kac-Moody algebras

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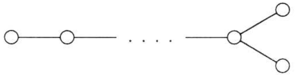
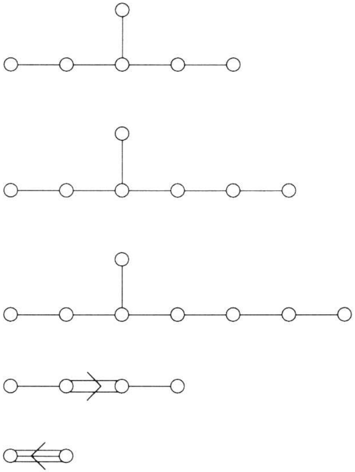
Introduction

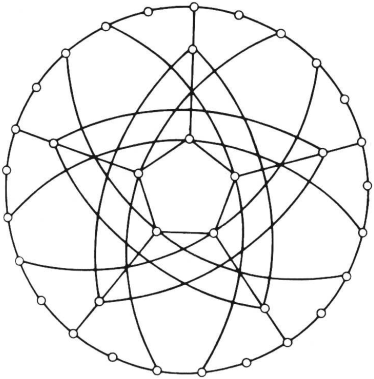
The Dynkin diagrams of the finite dimensional simple complex Lie algebras are



Introduction



The following graph shows a section of the Dynkin diagram of the fake monster algebra



Introduction

This Lie algebra is related to the Eisenstein series

$$E_{14}(\tau) = 1 - 24 \sum_{n \geq 1} \sigma_{13}(n) q^n$$

Generalized Kac-Moody algebras

Let (a_{ij}) be a real quadratic matrix with

1. $a_{ij} = a_{ji}$
2. $a_{ij} \leq 0 \quad \text{for } i \neq j$
3. $2a_{ij}/a_{ii} \in \mathbb{Z} \quad \text{if } a_{ii} > 0$

The generalized Kac-Moody algebra associated to (a_{ij}) is defined as the Lie algebra with generators $\{e_i, h_i, f_i\}$ and relations

1. $[e_i, f_j] = \delta_{ij} h_i$
2. $[h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j$
3. $\text{ad}(e_i)^{1-2a_{ij}/a_{ii}} e_j = \text{ad}(f_i)^{1-2a_{ij}/a_{ii}} f_j = 0 \quad \text{if } a_{ii} > 0,\ i \neq j$
4. $[e_i, e_j] = [f_i, f_j] = 0 \quad \text{if } a_{ij} = 0$

Generalized Kac-Moody algebras

As in the finite dimensional case there is a denominator identity

$$e^{\rho} \prod_{\alpha>0} (1 - e^{\alpha})^{\mathrm{mult}(\alpha)} = \sum_{w \in W} \det(w) w\Big(e^{\rho} \sum \varepsilon(\alpha) e^{\alpha} \Big) .$$

Outer automorphisms give twisted denominator identities. These are sometimes automorphic forms on orthogonal groups.

Lattices

Let L be an even lattice. A root of L is a primitive vector α of positive norm such that the reflection $\sigma_\alpha(x) = x - 2(x,\alpha)\alpha/\alpha^2$ is an automorphism of L. The level of L is the smallest positive integer N such that $N\lambda^2/2 \in \mathbb{Z}$ for all λ in L'.

Let L be an even lattice of level N and α a root of L with norm $\alpha^2 = 2k$. Then $k|N$ and $\alpha \in L \cap kL'$. Conversely a vector α in L with $\alpha^2 = 2k$ and $\alpha \in L \cap kL'$ where $k|N$ is a multiple of a root.

Lattices

Let L be an even lattice of level N and γ an element in the discriminant form of norm $\gamma^2/2 = 1/k \mod 1$ where $k|N$. We say that γ corresponds to roots if

1. The order of γ divides k.
2. If there is a vector $\alpha \in L \cap kL'$ of norm $\alpha^2 = 2k$ with $\alpha/k = \gamma$ then α is a root.

If N is squarefree then the roots of L are the vectors in $L \cap kL'$ of norm $2k$ with $k|N$ and the elements in the discriminant form corresponding to roots are the γ of norm $\gamma^2/2 = 1/k \mod 1$ and order k with $k|N$.

The Weil representation

Let L be an even lattice of even dimension and $D = L'/L$. The Weil representation of $SL_2(\mathbb{Z})$ on $\mathbb{C}[D]$ is defined by

$$\rho_D(T)e^\gamma = e(-\gamma^2/2)\, e^\gamma$$
$$\rho_D(S)e^\gamma = \frac{e(\mathrm{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta \in D} e((\gamma, \beta))e^\beta$$

A holomorphic function $F(\tau) = \sum_{\gamma \in D} F_\gamma(\tau)e^\gamma$ on the upper halfplane is a modular form for ρ_D of weight k if

1. $F(M\tau) = (c\tau + d)^k \rho_D(M)F(\tau)$
   for all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $SL_2(\mathbb{Z})$.
2. f is meromorphic at ∞.

The Weil representation

Suppose L is a positive definite even lattice of even dimension $2k$. For $\gamma \in D$ define $\theta_\gamma(\tau) = \sum_{\alpha \in \gamma + L} q^{\alpha^2/2}$. Then

$$\theta(\tau) = \sum_{\gamma \in D} \theta_\gamma(\tau) e^\gamma$$

is a modular form for the dual Weil representation $\overline{\rho}_D$ of weight k.

<u>Theorem</u>

Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. Then

$$\rho_D(M)e^\gamma = \xi \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\beta \in D^{c*}} e(-a\beta_c^2/2) e(-b\beta\gamma) e(-bd\gamma^2/2) e^{d\gamma + \beta}$$

where $\xi = e(\mathrm{sign}(D)/4) \prod \xi_p$.

The Weil representation

Theorem

Suppose L has level dividing N. Let f be a modular form on $\Gamma_0(N)$ with character χ_D and S_0 an isotropic subgroup of D. Then

$$F_{f,S_0} = \sum_{M \in \Gamma_0(N) \backslash SL_2(\mathbb{Z})} \sum_{\gamma \in S_0} f|_M \, \rho_D(M^{-1}) e^{\gamma}$$

is a modular form for ρ_D.

We will use this result to construct reflective modular forms and vector valued Eisenstein series.

Automorphic products

Let L be an even lattice of signature $(n, 2)$, $V = \mathbb{R} \otimes L$ and $D = \{Z \subset V \mid \dim(Z) = 2, \mathrm{Z}$ negative definite$\}$. Then $O^+(V)$ acts on D and D has a realization as tube domain H in $\mathbb{C}^n$.

A meromorphic function f on H is an automorphic form of weight k for $\Gamma \subset O^+(L)$ if

$$f(MZ) = j(M, Z)^k f(Z)$$

for all M in Γ.

Automorphic products

Let L be an even lattice of signature $(n, 2)$, $n \geq 2$ and n even. Let F be a modular form of weight $1 - n/2$ for ρ_D with a pole at ∞ and θ_L the Siegel theta function of L. Then

$$\Phi(Z) = \int_{\mathcal{F}} F(\tau)\overline{\theta}(Z,\tau)y\frac{dx\,dy}{y^2},$$

is formally invariant under $O^+(L, F)$. The function $\Psi(Z) = \exp(\Phi(Z))$ is an automorphic form on $O^+(L, F)$ with a nice product expansion. The zeros and poles of Ψ lie on divisors $D_\lambda = \{Z \in D \,|\, Z \perp \lambda\}$ where λ is a primitive vector in L of positive norm. Their orders are determined by the principal part of F.

Symmetric and reflective forms

Let L be an even lattice of level N and signature $(n, 2)$ with n even and $n > 2$. Let F be a modular form for the Weil representation of L.

We say that F is symmetric if F is invariant under the automorphisms of the discriminant form.

An automorphic product is called symmetric if it is the theta lift of a symmetric modular form.

We say that F is reflective if

1. F has weight $1 - n/2$.
2. The only singular terms of F are of the form $q^{-1/k}$ and come from components F_γ with γ corresponding to roots.

Symmetric and reflective forms

Let F be a reflective modular form on L. For elements γ in the discriminant form with $\gamma^2/2 = 1/k \mod 1$ and $k\gamma = 0$ where $k|N$ we denote the coefficient of F_γ at $q^{-1/k}$ by $c_{\gamma,k}$. Then $c_{\gamma,k}$ is 0 or 1. F is called completely reflective if $c_{\gamma,k} = 1$ for all γ corresponding to roots.

We say that an automorphic product Ψ is reflective if it is the theta lift of a reflective modular form F.

Let λ be a root of L of norm $2d$. Then the divisor D_λ has order $c_{\lambda/d,d}$.

Let λ be a primitive vector of positive norm in L. Suppose D_λ has positive order. Then λ is a root of L .

Moonshine for Conway's group

Co_0 is the automorphism group of the Leech lattice Λ. The characteristic polynomial of an element g of order n in Co_0 can be written as $\prod_{k|n}(x^k-1)^{b_k}$ and the symbol $\prod k^{b_k}$ is called the cycle shape of g. The eta product

$$\eta_g(\tau) = \prod \eta(k\tau)^{b_k}$$

is a modular form of level N. N is called the level of g.

Co_0 acts naturally on the fake monster algebra. This is a generalized Kac-Moody algebra describing the physical states of a bosonic string moving on the torus $\mathbb{R}^{25,1}/II_{25,1}$. The corresponding twisted denominator identities are automorphic forms of singular weight on orthogonal groups.

Moonshine for Conway's group

If g has squarefree level N and nontrivial fixed point lattice Λ^g this can be proved as follows

$$g \to 1/\eta_g \to F_{1/\eta_g,0} \to \Psi$$

where $F_{1/\eta_g,0}$ is a modular form for the Weil representation of

$$\Lambda^g \oplus II_{1,1} \oplus II_{1,1}(N)\,.$$

Ψ is symmetric and reflective, has singular weight and can be identified with the twisted denominator identity of g.

The nicest special case is

Theorem

Let N be a squarefree integer such that $\sigma_1(N)$ divides 24. Then there is an element g in M_{23} of cycle shape $\prod_{k|N} k^{24/\sigma_1(N)}$. The expansion of Ψ at any cusp is given by

$$e^\rho \prod_{d|N} \prod_{\alpha \in (L \cap dL')^+} (1 - e^\alpha)^{[1/\eta_g](-\alpha^2/2d)} = \sum_{w \in W} \det(w) w(\eta_g(e^\rho))$$

where $L = \Lambda^g \oplus II_{1,1}$, ρ is a primitive norm 0 vector in $II_{1,1}$ and W is the full reflection group of L. The identity is the denominator identity of a generalized Kac-Moody algebra whose real simple roots are the simple roots of W and imaginary simple roots are the positive multiples $n\rho$ of the Weyl vector with multiplicity $24\,\sigma_0((N,n))/\sigma_1(N)$.

Moonshine for Conway's group

The theorem gives 10 generalized Kac-Moody algebras similar to the fake monster algebra.

The residue theorem

Let L be an even lattice of even dimension. Let $F = \sum F_\gamma e^\gamma$ be a modular form for the Weil representation ρ of weight $2-k$ and $E = \sum E_\gamma e^\gamma$ the Eisenstein series for $\overline{\rho}$ of weight k. The unitarity of ρ implies that $\sum F_\gamma E_\gamma$ is a scalar valued modular form for $SL_2(\mathbb{Z})$ of weight 2. Hence $\sum F_\gamma E_\gamma d\tau$ defines a meromorphic 1-form on the Riemann sphere with a pole at ∞. By the residue theorem its residue has to vanish. This implies that the constant term in the Fourier expansion of $\sum F_\gamma E_\gamma$ is 0.

The residue theorem

Theorem

Suppose L has signature $(n, 2)$ with $n > 2$ and squarefree level N. Let F be a symmetric and reflective modular form on L of weight $2 - k$ where $k = 1 + n/2$ with singular coefficients c_d and constant coefficient $2(k-2)$ in F_0. Then the condition takes the following form

$$\frac{k}{k-2} \frac{1}{B_{k,\psi}} \frac{L(k,\psi)}{L(k,\chi)} \frac{m^k}{N^k} \sum_{cd|N} \varepsilon_{c,d}\, c_d\, N_d \frac{\sqrt{m_c\,|D_c|}}{\sqrt{m\,|D|}} \frac{N^k}{c^k d^{k-1}} = 1\,.$$

The residue theorem

If the Jordan components of L have even rank and all singular coefficients of F are 1 then the condition can be written

$$\frac{k}{k-2}\,\frac{1}{B_k}\prod_{p|N}\frac{1}{p^k-1}\left(\epsilon_p\left(\frac{-1}{p}\right)^{n_p/2}\left(p^{k-n_p/2}+p^{n_p/2}\right)-2\right)=1\,.$$

For $N=1$ the only solution of this equation is $k=14$ and corresponds to the fake monster algebra.

Classification results

The fast growth of Bernoulli numbers implies

Theorem

The number of automorphic products of singular weight which are symmetric and reflective on lattices of signature $(n, 2)$ with $n > 2$, squarefree level and p-ranks at most $n + 1$ is finite.

We have explicit bounds on the solutions of the necessary condition and therefore can determine them by computer search. We find

Theorem

Let L be an even lattice of signature $(n, 2)$ with $n > 2$, prime level p and p-rank at most $n + 1$. Suppose Ψ is an automorphic product of singular weight on L which is symmetric and reflective. Then L is one of the following lattices

Classification results

k	L
3	$II_{4,2}(23^{-3})\,(1,23)$
4	$II_{6,2}(5^{+3})\,(5)$, $II_{6,2}(5^{+5})\,(1)$,
	$II_{6,2}(11^{-4})\,(1,11)$
5	$II_{8,2}(3^{-3})\,(3)$, $II_{8,2}(3^{-7})\,(1)$,
	$II_{8,2}(7^{-5})\,(1,7)$
6	$II_{10,2}(2_{II}^{+2})\,(2)$, $II_{10,2}(2_{II}^{+10})\,(1)$,
	$II_{10,2}(5^{+6})\,(1,5)$
8	$II_{14,2}(3^{-8})\,(1,3)$
10	$II_{18,2}(2_{II}^{+10})\,(1,2)$

The numbers in brackets give the root lengths of the divisors of Ψ.

Classification results

The automorphic form Ψ corresponds to a unique class of order p in Co_0 and can be obtained by lifting $1/\eta_g$ or an Atkin-Lehner transformation thereof.

Theorem

Let L be an even lattice of signature $(n, 2)$ with $n > 2$ and squarefree level N. Suppose L splits $II_{1,1} \oplus II_{1,1}(N)$. Let G be a generalized Kac-Moody algebra whose denominator identity is a completely reflective automorphic product of singular weight on L. Then G can be constructed from an element of order N in M_{23}.

Further results

Gritsenko and Nikulin proved classification results for certain generalized Kac-Moody algebras of rank 3 with automorphic denominator function.

Gritsenko and Grandpierre recently derived classification results for automorphic forms obtained by restricting the denominator identity of the fake monster algebra.

Summary

Generalized Kac-Moody algebras are natural generalizations of the finite dimensional simple Lie algebras which are defined by generators and relations.

Modular forms for $\Gamma_0(N)$ can be lifted in a natural way to modular forms for the Weil representation.

The singular theta correspondence is a map from modular forms for the Weil representation to automorphic forms on orthogonal groups.

A vector valued modular form is reflective if its poles correspond to roots. The divisors of the theta lift of a reflective form come from roots and have order 0 or 1.

Summary

Conway's group Co_0 acts on the fake monster algebra. The corresponding twisted denominator identities are automorphic forms of singular weight. The elements of squarefree order in M_{23} give 10 generalized Kac-Moody algebras similar to the fake monster algebra.

Multiplying a modular form for the Weil representation of weight $2-k$ with the Eisenstein series for the dual Weil representation of weight k we obtain a modular form of weight 2. The constant coefficient of this function is 0. We calculate this condition explicitly for lattices of squarefree level and modular forms which are symmetric and reflective.

Summary

There are only finitely many automorphic products of singular weight which are symmetric and reflective on lattices of signature $(n, 2)$ with $n > 2$, squarefree level and p-ranks at most $n + 1$.

In the prime order case they all correspond to automorphisms of the Leech lattice with nontrivial fixed point lattice.

The 10 generalized Kac-Moody algebras coming from elements of squarefree order in M_{23} are the only generalized Kac-Moody algebras whose denominator identities are completely reflective automorphic products of singular weight on lattices of squarefree level splitting 2 hyperbolic planes.