On the explicit construction of arithmetic hyperbolic reflection groups

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Reflection Groups and reflective lattices

Hyperbolic n-space

- $H^n = \left\{(x_0,\dots,x_n) \in \mathbb{R}^{n+1} \mid -x_0^2+x_1^2+\cdots+x_n^2 = -1,\ x_0 \geq 1\right\}$
- $\mathrm{O}^+_{n,1}(\mathbb{R}) \subseteq \mathrm{O}_{n,1}(\mathbb{R})$ isometry group of H^n
- discrete subgroup $\Gamma \subset \mathrm{O}^+_{n,1}(\mathbb{R})$
- reflection $s_v(x) = x - \frac{2(v|x)}{(v|v)}v$, where $(x|y) := -x_0y_0 + x_1y_1 + \cdots + x_ny_n$, and $(v|v) > 0$.

Arithmetic (discrete) sugroups of $\mathrm{O}^+_{n,1}(\mathbb{R})$:

- $K \subset \mathbb{R}$ a totally real number field
- $f : K^{n+1} \to K$ a quadratic form of signature $(n,1)$
- σf pos. definite for all embeddings $\sigma : K \to \mathbb{R}$, $\sigma \neq \mathrm{id}$.
- $L \subset K^{n+1}$ a full $\mathfrak{o}_K$-lattice
- $\mathrm{O}^+(L,f) =: \mathrm{O}^+(L) \subset \mathrm{O}^+_{n,1}(\mathbb{R})$ is discrete, cofinite

Roots and reflections of L:

- $R(L) = R(L, f) := \{v \in L \mid v \text{ primitive}, s_v(L) = L\}$
- Let $v \in L$ primitive:
  $v \in R(L) \iff v \in f(v) \cdot L^{\#} \Rightarrow f(v) \mid \exp(L^{\#}/L)$
- $W(L) = W(L, f) := \langle s_v \mid v \in R(L) \rangle \subseteq \mathrm{O}^+(L)$
  the *Weyl group* of L
- If a discrete reflection group $W \subset \mathrm{O}^+_{n,1}(\mathbb{R})$ is arithmetic, i.e. commensurable to (some) $\mathrm{O}(L)$, then actually $W \subseteq W(L)$ for a canonical field $K = K_W$ and $\mathfrak{o}_K$-lattice (L_W, f_W).

A variation of the classification problem for arithmetic hyperbolic reflection groups can therefore be stated as follows:

– For which quadratic lattices L over a totally real number field K subject to the above signature conditions does the reflection subgroup $W(L) \subseteq O(L)$ have finite index?

Following Vinberg [Vi4], we shall call such a lattice (or its quadratic form) *reflective*.

In particular, all maximal arithmetic reflection groups are of the form $W(L)$, for certain lattices L.

A basic theorem by Nikulin says that the number of (similarity classes of) of reflective lattices is finite. Nikulin's proof is geometrical; starting from known combinatorial properties of the fundamental polyhedron it gives bounds on the entries of the Gram matrix and the conjugates of generating elements of K. The arithmeticity only enters at a later stage the end of the proof.

The bounds on the various parameters (dimension, field degree, field discriminant, determinant of the lattice) are weak (if worked out at all), when compared to the known examples. The finiteness results do not give an explicit classification in form of a list of groups or lattices.

The case $n = 3$, f isotropic

Theorem *The groups $W(L, f)$, for the following 49 quadratic lattices $L = \mathbb{H} \perp^s [a, b, c]$, i.e. $\mathbb{Z}^4$ with quadratic form*

$$f(x_0, x_1, x_2, x_3) = x_0 x_1 + a x_2^2 + b x_2 x_3 + c x_3^2$$

are a complete set of representatives for the maximal arithmetic reflection groups $W = W(L)$ with noncompact fundamental domain on hyperbolic 3-space
(under the assumption that $D(f) \leq 56654$):

$[1,1,1]$	$[1,0,1]$	$[1,1,2]$	$[1,0,2]$	$[1,1,3]$
${}^2[1,1,1]$	$[1,1,4]$	$[2,1,2]$	$[1,1,5]$	$[1,0,5]$
$[2,2,3]$	$[1,0,6]$	$[2,0,3]$	${}^2[1,1,2]$	$[3,1,3]$
${}^3[1,0,1]$	$[1,1,10]$	$[1,0,10]$	$[2,0,5]$	${}^2[1,1,3]$
$[1,0,13]$	$[1,0,14]$	${}^2[1,1,4]$	${}^3[1,1,2]$	$[1,0,17]$
${}^3[1,0,2]$	${}^5[1,1,1]$	$[1,0,21]$	$[3,0,7]$	$[5,4,5]$
${}^5[1,0,1]$	$[1,0,30]$	$[2,0,15]$	$[3,0,10]$	$[5,0,6]$
$[1,0,33]$	${}^2[3,1,3]$	${}^7[1,1,1]$	$[3,0,14]$	$[6,0,7]$
${}^3[1,0,5]$	${}^7[1,0,1]$	${}^5[1,0,2]$	${}^{10}[1,1,1]$	${}^3[1,0,10]$
${}^3[2,0,5]$	${}^{13}[1,1,1]$	${}^{14}[1,1,1]$	${}^{15}[1,0,1]$	

This classification has been announced in

R.S., C. Walhorn: Integral lattices and hyperbolic reflection groups, *Astérisque* **209**, 279–291 (1992).

More detailed information about the 49 groups is contained in the Report

R.S., C. Walhorn: Tables of hyperbolic reflection groups, Schriftenreihe des SFB 343 der Universität Bielefeld, Heft E 92–001 (1992)

Maximality and normal form

Proposition 2.1

If $O(L)$ is maximal, then we may assume that L has the shape $L = \mathbb{H} \perp {}^s[a,b,c]$, where $\mathbb{H}$ denotes the hyperbolic plane $(\mathbb{Z}^2, xy)$, ${}^s[a,b,c]$ the binary lattice $(\mathbb{Z}^2,\ s(ax^2+bxy+cy^2))$, the discriminant $-D_0 = -(4ac-b^2)$ of $[a,b,c]$ is a fundamental discriminant (discriminant of a quadratic field), and s is a square-free natural number relatively prime to D_0.

There is one situation where two nonisometric lattices of the abouve type give rise to conjugate groups.
Consider lattices $\mathbb{H} \perp {}^s[a,b,c]$, $\mathbb{H} \perp {}^{s'}[a',b',c']$ as above, and assume that $s = s'$ is such that $D_0 = 4ac - b^2 = 4a'c' - b'^2$, and $-D_0$ is a square mod p, resp. mod 8 if $p = 2$, for all prime divisors $p|s$.
Assume furthermore that ${}^s[a,b,c]$ is isometric to $[a',b',c']$ over the rationals, that is, ${}^s[a,b,c]$ represents rationally at least one number represented by $[a',b',c']$. We claim that $\mathbb{H} \perp {}^s[a,b,c] \cong {}^s\mathbb{H} \perp [a',b',c']$. It is sufficient to see this locally everywhere.

If we dualize the lattice $L = {}^s\mathbb{H}\perp[a', b', c']$ with respect to all $p|s$, that is, define a new lattice N by

$$N\mathbb{Z}_p = (L\mathbb{Z}_p)^{\#} \text{ if } p|s, \quad N\mathbb{Z}_p = L\mathbb{Z}_p \text{ if } p \nmid s,$$

then we have

$$N \cong {}^{s^{-1}}\mathbb{H} \perp [a', b', c'],$$

since $[a', b', c']$ is unimodular with respect to the primes p dividing s. Thus

$$\mathrm{O}(\mathbb{H}\perp{}^s[a, b, c]) \cong \mathrm{O}(L) = \mathrm{O}(N) = \mathrm{O}({}^sN) \cong \mathrm{O}(\mathbb{H}\perp{}^s[a', b', c']).$$

Necessary conditions for reflectivity

Lemma 1 (Vinberg)

If a lattice $\mathbb{H}\perp M$, where $\mathbb{H}$ is the hyperbolic plane, M a positive definite lattice, is reflective, then M is reflective (i.e. its root system has full rank).

Since a lattice of the shape $\mathbb{H}\perp M$ has only one class in its genus, Vinberg's lemma can be sharpened to the statement that every form in the genus $\mathrm{gen}\, M$ is reflective if $\mathbb{H}\perp M$ is reflective.

$[a, b, c]$ is reflective if and only if it is of one of the following shapes:

$$[a, 0, c]$$
$$[a, a, c]$$
$$[a, b, a]$$

In view of the above results, we shall now assume that the binary form has the shape ${}^s[a, b, c]$, where $-D_0 = b^2 - 4ac$ is a field discriminant, s is square-free and $\gcd(s, D_0) = 1$.

Proposition 2.2

Let $-D_0$ be the discriminant of an imaginary quadratic field, denote by $\mathcal{C}_{D_0}$ the class group of $\mathbb{Q}(\sqrt{-D_0})$. If $\mathbb{H}\perp{}^s[a, b, c]$ is reflective, for some binary form $[a, b, c]$ of discriminant $-D_0$, and some s relatively prime to D, then $\mathcal{C}_{D_0}^4 = \{1\}$.

Proving non-reflectivity

Computing fundamental roots:

– Fix a point $p_0 \in H^3$ having a stabilizer in $W(L)$ of rank 3. We take $p_0 = (-1, 1, 0, 0)$.

– Fix a basis v_1, v_2, v_3 of the root system of the positive definite lattice $p_0^\perp \cap L$. (That is, the reflections $s_{v_1}, s_{v_2}, s_{v_3}$ generate the stabilizer of p_0 in $W(L)$). We take $v_1 = (-1, -1, 0, 0)$, and v_2, v_3 of the form $(0, 0, x_2, y_2)$, $(0, 0, x_3, y_3)$, where (x_2, y_2), (x_3, y_3) are fundamental roots of ${}^s[a, b, c]$ (see above).

– If $v_1, \ldots, v_k$ are already chosen, for some $k \geq 3$, then v_{k+1} is a root v subject to the following conditions:

(1) $(v \mid p_0) \geq 0$.

(2) $(v \mid v_i) \leq 0$ for $i = 1, \ldots, k$.

(3) $(v \mid p_0)^2 / f(v)$ is minimal among all roots v satisfying (1) and (2).

The function $(v \mid p_0)^2 / f(v)$ measures the distance between the point p_0 and the plane $v^\perp$.

Conjugacy of roots under $\mathrm{O}^+(L)$

Now we must effectively distiguish $W(L)$ from $\mathrm{O}^+(L)$ in the general case.
We do this by considering the action on roots:

Proposition 2.3

If D_0 is odd or if $8 \nmid D_0$ and $f(v)$ is even, then any two roots v, v' such that $f(v) = f(v') > 0$ are conjugate under $\mathrm{O}^+(L)$. In any case, there are at most two conjugacy classes of roots v for fixed $f(v) > 0$.

Non-Conjugacy of roots under $W(L)$

Proposition 2.4

Each of the following conditions on the lattice L and the value q implies that two distinct fundamental roots v, v' s.th. $f(v) = f(v') = q$ are never conjugate under $W(L)$:

(i) *$D_0 \neq 3$, and there exists a prime number $p \neq 2$ such that $p|q, \; p|D_0$.*

(ii) *There exists a prime number $p \neq 2$ such that $p|q, \; p|s$, and $(\frac{3D_0}{p}) \neq 1$.*

(iii) $2|q, \; 2|D_0$.

(iv) $D_0 = 3, \; q \not\equiv s(3)$

(v) *$D_0 = 4$, there exists a prime number $p \neq 2$ such that $p|q, \; p \equiv \pm 5 \; (12)$.*

The following proposition, which is an easy consequence of the last two propositions, allows to prove "in practice" the infiniteness of $[O(L) : W(L)]$ in all non-reflective cases.

Proposition 2.5

Suppose that L and q satisfy one of the conditions stated in Proposition 3.2. Let $o(L, q) = 1, 2$ be the number of orbits of $\mathrm{O}^+(L)$ on roots v with $f(v) = q$. If f has more than $4 \cdot o(L, q)$ fundamental roots v with $f(v) = q$, then $W(L)$ is of infinite index in $\mathrm{O}^+(L)$.

Proof. The stabilizer $A = A(L)$ in $\mathrm{O}^+(L)$ of any fundamental domain for $W(L)$ is a complement to $W(L)$ in $\mathrm{O}^+(L)$. If A is finite, it obviously follows from Proposition 3.2 that the number of fundmental roots with $f(v) = q$ is at most $|A| \cdot o(L, q)$.
The bound $|A| \leq 4$ follows from the following Proposition 2.6, taking into account the fact that a finite A is isomorphic to a subgroup of $SL_3(\mathbb{Z})$. □

Proposition 2.6

Any rotation of order 3 or 4 in $O^+(L)$ is a product of two reflections in $\mathrm{O}^+(L)$. Here, L can be any integral lattice of signature $(3, 1)$.

The case $n \geq 4$, $K = \mathbb{Q}$

We are considering reflective $\mathbb{Z}$-lattices L of signature $(n, 1)$. Their number is finite, but they are far from being classified.

Known results:

- $n = 21$ or $n \leq 19$ (F. Esselmann, 1990); for $n = 21$ and $W(L)$ maximal, L is unique (found by Borcherds in 1985)
- For $n = 4$ and $W(L)$ maximal, the explicit classification is known, 43 lattices (C. Walhorn 1992).

Examples Lorentzian lattices of small determinant D are reflective for $n \leq n_{\max}$, where the bound is as follows:

D	3+	3−	4	4e	5	6+	6−	7	8	9+	9−	10-
$n_{\max}$	10	13	13	15	8	9	10	7	11	11	9	6

(The sign distinguishes different genera of the same determinant; the list is incomplete in several respects.)

Reflectivity is proved using an implementation of Vinberg's algorithm and checking the combinatorics of the computed poyhedron.

Non-reflectivity typically comes from Vinberg's lemma and the explicit knowledge of non-reflective Euclidean lattices in most genera of dimension $n - 1$ (often a by-product of Esselmann's work).

Problems

- Classify totally-reflective genera of euclidean lattices M (of dimension $n - 1$, with locally everywhere maximal groups).
- Work out additional conditions which imply that $\mathbb{H} \perp {}^s M$, for such M, is Lorentzian reflective. (See: R: Borcherds, *Automorphism groups of Lorentzian lattices*, 1987)

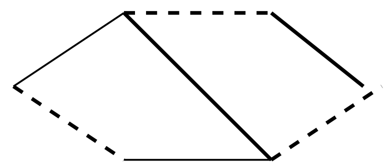
Example For $n = 16, D = 2^6$, the genus $II_{16}(2^6_{II})$ (of 2-elementary, totally even lattices) is totally reflective (Sch-Venkov). Borcherds (Duke Math. J. 104, 2000) showed that $\mathbb{H} \perp M$ is reflective.

The case $n = 3$, $K \neq \mathbb{Q}$

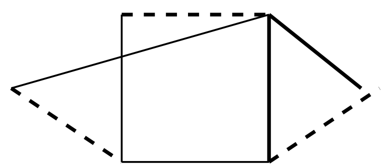
Question: Which totally real number fields do actually occur as ground fields of arithmetic reflection groups.

- The degree bound presently is 35 (Belolipetsky 2008).
- Examples are presently known only for degrees 1,2, and 3. Already in the late 1980s, V. Bugaenko gave examples over those fields with the smallest discriminants, also in certain higher dimensions.
- No good bounds on the field discriminant seem to be known (even for degree 2 or 3).

Example (S. Sauer): The following Coxeter-Vinberg diagram describes a compact combinatorial cube in H^3 which is arithmetic over $\mathbb{Q}(\sqrt{119})$:



Example (S. Sauer): The following Coxeter-Vinberg diagram describes a compact combinatorial cube in H^3 which is arithmetic over the cubic field of discriminant $1944 = 2^3 3^5$:



Summary

When constructing and possibly classifying arithmetic hyperbolic reflection groups, integral quadratic lattices are useful

- for systematically constructing lots of examples (without specifying the combinatorial structure of the fundamental domain)
- for finding embeddings and deciding maximality
- in the non-compact case: for giving a sharp bound on the dimension and on arithmetic invariants.