Viacheslav Nikulin Edinburgh, 31 aug. 2009

Historical review about arithmetic hyp. refl. groups.

Discrete groups gen. by refl. in hyperplanes of hyperbolic spaces = hyp. refl. groups, W

By Coxeter, they are equivalent to f. chambers $\mathcal{M}$

$n_{ij} \in \mathbb{N} \cup \{\infty\}$
$n_{ij} \geq 2$
loc. f. convex polyh.

Coxeter classified for
spheres $\varkappa > 0$
Euclidean $\varkappa = 0$

Hyperb. $\varkappa < 0$. We assume that $vol(\mathcal{M}) < \infty$.
$\mathcal{M}$ is convex envelope of f. n. of points

2 types of hyp.

1) Arithmetic
2) General, $vol(\mathcal{M}) < \infty$

Vinberg, 1967: criterion for arithm.:

$W \subset O(S)$

ar. h.r.gr. f. index

S hyperb. form / $\mathbb{O} \subset \mathbb{K}$, $\mathbb{Q} \subset \mathbb{K}$

ring of int.   td. real al. number field $\mathcal{L}($

~~hyperb. means~~:

$S: \mathbb{M} \times \mathbb{M} \to \mathbb{O}$

$\mathbb{O}$-module   ~~symm. bil. form~~

sym. bil. form

hyperb: $\exists!\, \sigma^{(+)}: \mathbb{K} \to \mathbb{R}$, $\sigma^{(+)}(S)$ is hyp. $(1, n)$

$\sigma^{(+)} \neq \sigma: \mathbb{K} \to \mathbb{R}$, $\sigma(S)$ >0 or <0.

$O^{(+)}(S) \longrightarrow \mathcal{L}(S)$, hyp. space

$\dim \mathcal{L}(S) = n$, $O^+(S)$ is discrete in $\mathcal{L}(S)$ and has f. domain of f. volume.

$W \underset{\text{f. index}}{\subset} O^+(S)$, $W(S) \overset{\text{f. ind}}{\subset} O^+(S)$

gen. by refl.

Such S are called hyperbolic reflective:

$[O^+(S) : W(S)] < \infty$

max. r. subgroup.

Problem: Describe refl. hyp. forms $S/\mathbb{O}$.

$\bigcup$

~~...~~ descr. of maximal hyperb. ar. refl. groups $W(S)$.

Main invariants:

$n = \dim \mathcal{L}(S) = \mathrm{rk}\, S - 1$.

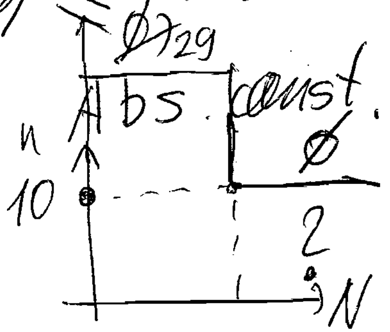
$N = [\mathbb{K} : \mathbb{Q}]$, $\mathbb{K}$ – ground field.

$S/\mathbb{O}$.

Thm 1: (N: 80) For fixed (n, N) number of reflective h. forms S is finite ~~...~~ $\Rightarrow$ number of max. ~~...~~ ar. h. r. groups is finite. Old result.

Thm 2: (N: 81).

$n \geq 10 \Rightarrow N$ ~~...~~ $\leq N(14)$



Thm 3: (Vinberg, 81).

$n \leq 29$.

Problem: Is $N \leq A$ if $n = 2, 3, \dots, 9$

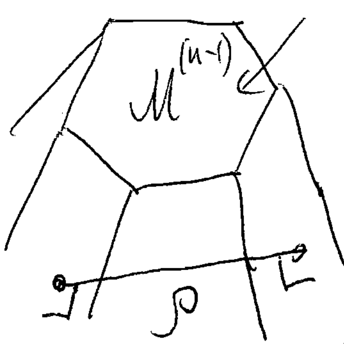
New res.

Old Methods: Find appropriate geom. properties of M of M

1. Existence of narrow face

Assume: $\varkappa = -1$

codim $= 1$ face of M



$\Longrightarrow$ finiteness for (n, N)

with some improvements.

Valid for any polyh M (f. convex).

distance between hyperp.

$2\operatorname{ch}\rho < 14$.

2. For n-dim. simple polyh. M

$$\alpha_n^{(i,k)}(M) < \alpha^{(i,k)}(\text{cube}) + O\left(\frac{1}{n}\right)$$

average number of i dimfaces in k-dim. ~~A simplex~~

$$\alpha_n^{(0,2)}(M) < 4 + O\left(\frac{1}{n}\right)$$

$0 \le i \le k \le \frac{n}{2}$ if n is big. M has a lot of non $\perp$ faces

3. Narrow face has many perpendicular neighbouring faces

if $N > N(14)$, ... (big).

some absolute constant.

Thm: M has $\mathcal{H}_1, \dots, \mathcal{H}_{n+1}$ hyp. of faces, such that:

1) generate $\mathcal{L} < 62$
2) $\rho(\mathcal{H}_i, \mathcal{H}_j)$
3) Dynkin graph is connected

Dynkin diagram of edge $r \subset M_{n-1}$

if $N > N(14)$, some absolute constant

$\mathcal{H}_i \circ \quad \circ \mathcal{H}_j$ if $\mathcal{H}_i \perp \mathcal{H}_j$

$\mathcal{H}_i \circ\!\!-\!\!\!-\!\!\!-\!\!\circ \mathcal{H}_j$ if $\mathcal{H}_i \not\perp \mathcal{H}_j$.

New results

In 2005 only (**25** years passed)
Long, Maclachlan, Reid proved
$n=2 \Longrightarrow$ # of m.a.h.r.gr. finite (number of maximal arithm. hyp. ref. groups)

Agol, 2005
$n=3 \Rightarrow$ finite.

for $n=4,5,\dots,9$ gap

In 2006, (2006, Nik):

Thm: ~~If finite for $n=2,3,$ then finite for each $n \geq 4$~~

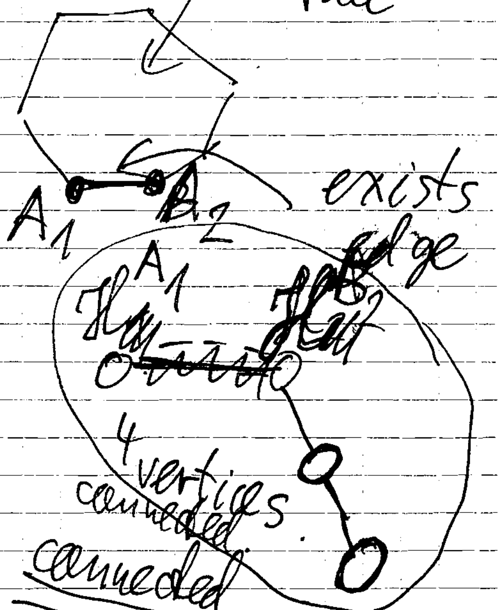
finite for $n=2,3 \Longrightarrow$ finite for each $n \geq 4$

using my methods 1980, 81

$$N^{(n\geq 4)} \leq \max\{N(\dim=2),\ N(\dim=3),\ N(14)\}$$

If N is bigger than

By induction.



with diagram

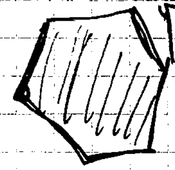
connected.

Then $N \leq N(14)$

Iz-v. Math. 2007, № 1, 4 pages.

Methods of LMR and Agol:

LMR proved finiteness for (arithm. (2-dim) max arithm.) Fuchsian groups of genus **0**. Γ : $\mathbb{H}^2/\Gamma$ has $g=0$.



$\Gamma \subset W^{\mathbb{H}^2} \subset W$, 1st kind, $g=0$

M.-F. Vigneras: (1981, 82) $\lambda_1(\Gamma) \geq \frac{3}{16}$ (using Gelbart, Jacquet, Langlands)

for Laplacian on $\mathbb{H}^2/\Gamma$ for max. ar. Fuch. groups

(1st $\neq 0$ eigen-value)

P. Zograf (1991) If Area$(\mathbb{H}^2/\Gamma) \geq 32\pi(g(\Gamma)+1)$,

then $\lambda_1(\Gamma) < \dfrac{8\pi(g(\Gamma)+1)}{\text{Area}(\mathbb{H}^2/\Gamma)} < \dfrac{128\pi}{3}$

$\Longrightarrow$ Area$(\mathbb{H}^2/\Gamma) <$ Abs. const for $g=0$

A. Borel (1981), K. Takeuchi (1983) Areas of arithm. Fuchsian groups are discrete and bounded from below.

$\Longrightarrow$ number of max. ar. Fuchsian groups of $g=0$ is finite.

Agol: Similar arguments
and conformal volumes of orbifolds due to
(Li, Yau, Yang, Hersch and Szegö) For maximal ar. h. v. groups of dim = 3 $\Rightarrow$ vol(M) < Abs. const.
also
A. Borel (1981) Volumes of arithmetic 3-dim.
orbifolds are discrete and bounded from below.
(look. Agol, Madrid).

~~Not~~

Effective finiteness:

$n \leq 29$ (Vinberg)

$N <$? enough

Fin. for (n, N) was effective.

In 2007 (preprints 2007, September–October):

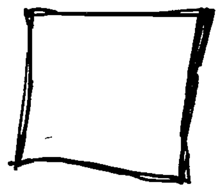
1) $N(14) \leq 56$ can be taken, then

~~M~~ $n \geq 10 \Rightarrow N \leq 56$. (1981, old methods).

methods (1980, 81. ← old result.

2) 2007; Nik. $n \geq 6 \Rightarrow N \leq 56$

My Methods 80, 81 & Borel, Takeuchi for (81, 83) only arithmetic

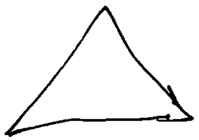


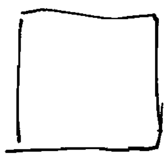
(don't need $\angle MR$, Agol)

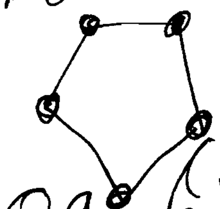
much more difficult! consider not arith

3) $n = 4, 5 \Rightarrow N \leq 138$

My methods 80, 81 & Borel, Takeuchi for







(don't need $\angle MR$, Agol).

(another proof of finiteness for $n=3$)

5) $n = 2 \Rightarrow N \leq 44$ (~~use~~ analize $\angle MR$).

4) $n = 3 \Rightarrow N \leq 909$

My methods 80, 81 & $\angle MR$

5) Later in 2007

Belolipetsky showed

$n = 3 \Longrightarrow N \leq 35$

uses Agol (estimate for volume), and very difficult

arithm. results of (by) Chinburg, Friedman (1986)

By my Theorem

$N \leq \max\{N(\dim = 2), N(\dim) = 3, N(14)\} \leq 56$

$\qquad\qquad\qquad 44 \qquad\qquad 35 \qquad\qquad 56$

Summarizing:

$\dim = n < 30$ (Vinberg, 1981)

$\deg = N \leq 56$

For fixed (n, N), the number of maximal arithmetic refl. groups is finite